Solving Consumption and Portfolio Choice Problems: The State Variable Decomposition Method

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Workshop on Computational Methods in Finance
Fields Institute, Toronto
March 22, 2010

Motivation

- Majority of problems in economics and finance are **dynamic** in nature
- **Portfolio problems** have a long and rich tradition in finance
- Most portfolio choice problems do **not** admit closed-form solutions
  - Frictions: taxes, transaction costs
  - Market incompleteness: return predictability, stochastic volatility
- **Theoretical approximations** have been developed, i.e., log-linear approximations
- **Numerical methods** still a necessity, especially for realistic problems

Brief overview of numerical methods

- Numerical solution of PDE [Brennan, Schwartz and Lagnado (1997)]
- Log-linearization of FOC/budget constr. [Campbell and Viceira (1999)]
- Perturbation of closed-form solutions [Kogan and Uppal (2001)]
- State-space discretization and linear interpolation of value function (Quadrature integration [Balduzzi and Lynch (1999)]; Simulations [Barberis (2000)]; Binomial discretization [Dammon, Spatt, and Zhang (2001)]; Non-parametric regression [Brandt (1999)])
- Malliavin calculus based methods [Detemple et. al (2003)]
- Policy function iteration and simulation-based methods for computing expectations [Brandt, Goyal, Santa-Clara, and Stroud (2005), **BGSS**]

The State Variable Decomposition (SVD) method: A simple illustration

- Static **one-period**, **one-asset** problem with **power utility**
- Utility: $u(W) = \frac{W^{1-\gamma}}{1-\gamma}$, $W = W_0(R_f + \omega R)$
- Asset return **decomposition**: $R = \mu_R + \varepsilon_R$
  where $\mu_R = E[R]$ and $E[\varepsilon_R] = 0$
- Wealth **decomposition**: $W(\omega) = \mu_W(\omega) + \varepsilon_W(\omega)$
  where $\mu_W(\omega) = W_0(R_f + \omega\mu_R)$, $\varepsilon_W(\omega) = W_0\,\omega\,\varepsilon_R$

Taylor approximation of $u(W)$:

$$\begin{aligned} W(\omega)^{1-\gamma} &= (\mu_W(\omega) + \varepsilon_W(\omega))^{1-\gamma} \\ &\approx \sum_{m=0}^{M} \frac{1}{m!}(1-\gamma)_m\, \mu_W(\omega)^{1-\gamma-m}\, \varepsilon_W(\omega)^m \end{aligned}$$

Approximate optimization problem:

$$\max_{\omega} E[u(W)] \approx \max_{\omega} \frac{W_0^{1-\gamma}}{1-\gamma} \sum_{m=0}^{M} \frac{1}{m!} (1-\gamma)_m \left(R_f + \omega \mu_R\right)^{1-\gamma-m} \omega^m E[\varepsilon_R^m]$$

- Choice variable ω is **separated** from **zero-mean** shock ε_R
- Return shock moments $E[\varepsilon_R^m]$ need to be computed **only once**
- For standard distributions (i.e., normal and lognormal) $E[\varepsilon_R^m]$ available in **closed-form**; alternatively, **simulation** can be used
- Easy to generalize to **multiple assets** (multinomial formula)

Choice of center of expansion for Taylor series

In a **static** problem, **SVD coincides with BGSS** with one **exception**:

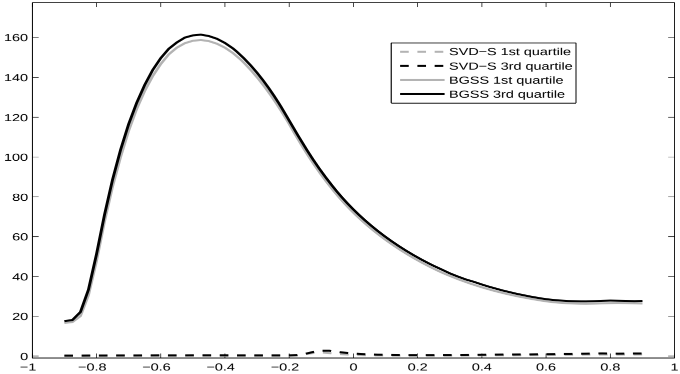
- **SVD**: expand future wealth W_1 around $\mu_W = W_0(R_f + \omega' \mu_R)$
- **BGSS**: expand future wealth W_1 around $W_0 R_f$.

Choice can be crucial in **multi-asset** problems.

Example: CE Losses from choice of expansion point

Two-asset static CRRA problem, annual data:
$\mu_1 = 7\%, \mu_2 = 12\%, \sigma_1 = 14\%, \sigma_2 = 18\%$, annual $R_f = 1.05$, $\gamma = 10$

CE losses in **annualized bps** w.r.t. quadrature



Correlation

General SVD methodology

General recursive structure of a dynamic problem

$$J_t(\mathbf{s}_t) = \max_{\mathbf{x}_t \in \mathbf{X}_t} \{\mathcal{H}\left(u(F(\mathbf{s}_t, \mathbf{x}_t)), E_t[J_{t+1}(\mathbf{s}_{t+1})]\right)\},$$

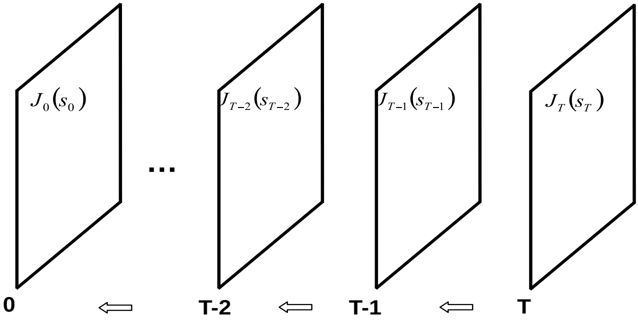
where $\mathbf{s}_{t+1} = \mathbf{\Gamma}(\mathbf{s}_t, \mathbf{x}_t, \boldsymbol{\delta}_{t+1})$: law of motion of state variables $\mathbf{s}_t$,

$$\begin{aligned} J_t(\cdot) &= \text{value function} \\ \mathbf{x}_t &= \text{choice variables} \\ \boldsymbol{\delta}_{t+1} &= \text{innovations to state variables } \mathbf{s}_t \\ \mathcal{H}(\cdot,\cdot) &= \text{“aggregator” of immediate and continuation utility} \end{aligned}$$

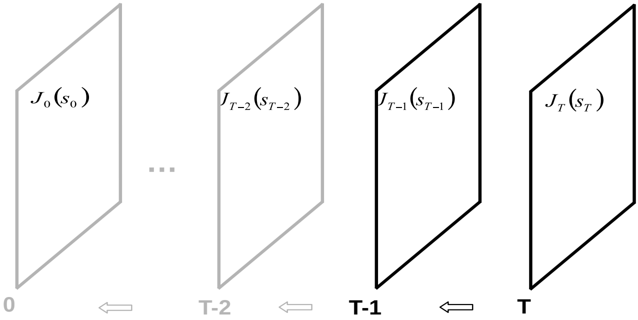
Special cases

- $\mathcal{H}(u, v) = u + \beta v$, $\beta \in (0,1) \Rightarrow$ **Time-separable utility**
- $\mathcal{H}(u, v) = \left[(1-\beta)u^{\frac{1}{\theta}} + \beta v^{\frac{1}{\theta}}\right]^{\theta}$, $\theta \neq 0 \Longrightarrow$ **Recursive utility**

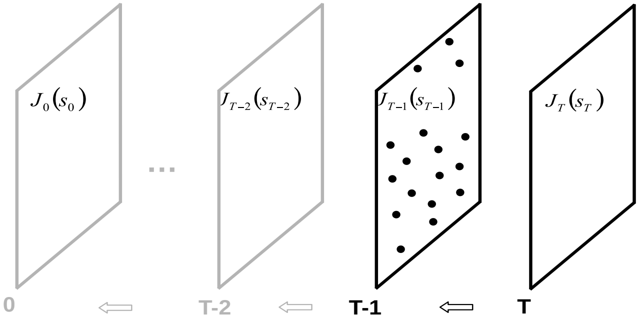
Algorithm

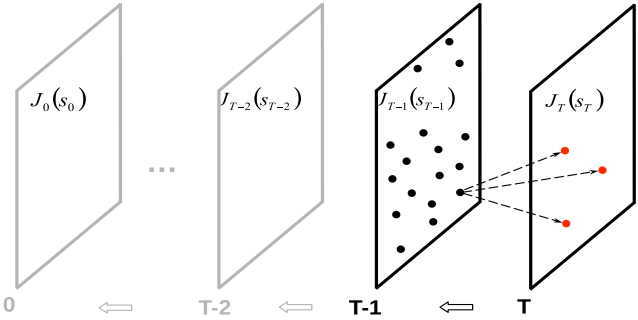


Algorithm



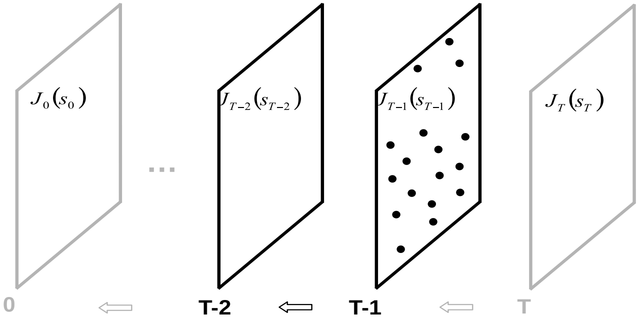
Algorithm



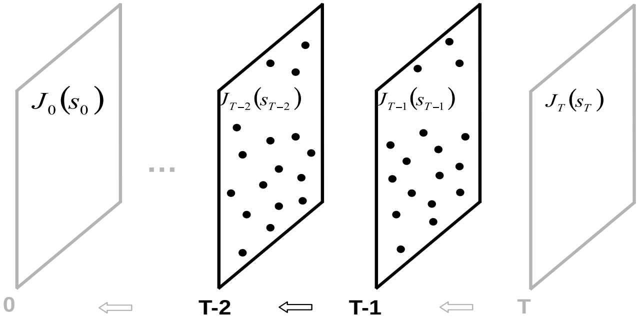


$J_0(s_0)$
$J_{T-2}(s_{T-2})$
$J_{T-1}(s_{T-1})$
$J_T(s_T)$
...
0
T-2
T-1
T

Algorithm



Algorithm



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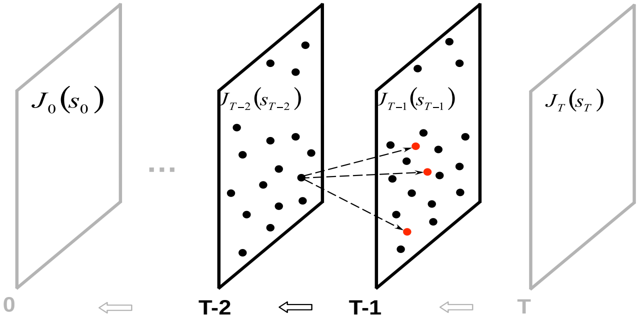

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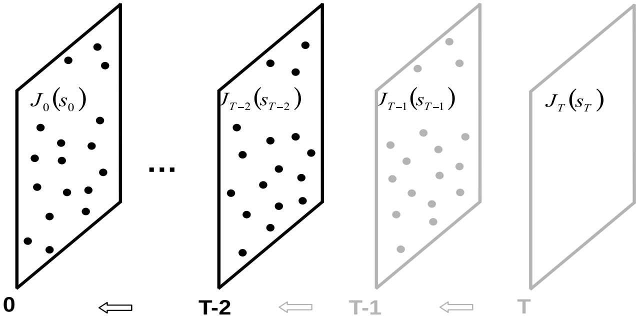


General SVD method

Preliminary step: Use a suitable **transformation** V_{t+1} of the value function J_{t+1} (e.g., **certainty equivalent**).

Backward recursion: Suppose V_{t+1} is known on a **grid** of $\mathbf{s}_{t+1}$:

- **Step A. Projection step.**
  Project V_{t+1} over the **entire state space**
- **Step B. SVD step.** Obtain V_t on a **grid** of $\mathbf{s}_t$:
  B-1. **Decomposition** of state variables;
  B-2. **Separation** of choice variables from shocks;
  B-3. **Computation** of conditional expectations.
  $\Longrightarrow$ obtain V_t on a **grid** of $\mathbf{s}_t$ $\Longrightarrow$ step A

$\Longrightarrow$ stop when $t = 0$.

Step A: Projection Step

Monotonic transformation $V_t(\mathbf{s}_t)$ instead of the value function $J_t(\mathbf{s}_t)$

$$J_t(\mathbf{s}_t) = \mathcal{U}(V_t(\mathbf{s}_t))$$

Transformed general recursion

$$\mathcal{U}(V_t(\mathbf{s}_t)) = \max_{\mathbf{x}_t \in \mathbf{X}_t} \{\mathcal{H}\left(u(F(\mathbf{s}_t, \mathbf{x}_t)), E_t[\mathcal{U}(V_{t+1}(\mathbf{s}_{t+1}))]\right)\},$$

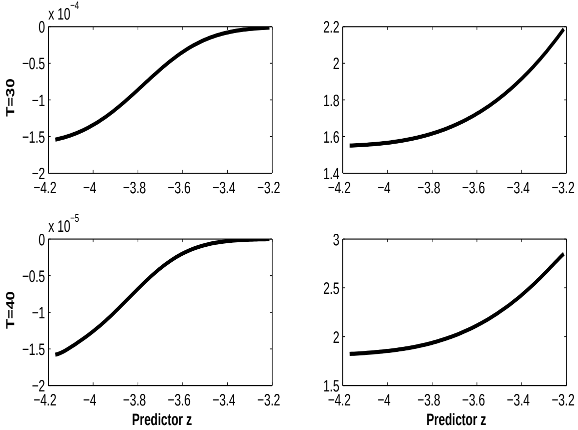
where $\mathbf{s}_{t+1} = \mathbf{\Gamma}(\mathbf{s}_t, \mathbf{x}_t, \boldsymbol{\delta}_{t+1})$.

- Example: if $\mathcal{U}(\cdot) = u(\cdot)$, $V_t(\mathbf{s}_t)$ is the **certainty equivalent** of $J_t(\mathbf{s}_t)$
- Usually $V_t(\mathbf{s}_t)$ **easier to approximate** over the state space for $\mathbf{s}_t$ (e.g., polynomials, radial basis functions).

Example: Value function vs. Certainty equivalent

- CRRA utility, index return predictable by dividend yield.
- **State variable s** is the **predictor** z.

Value Function J_t **Certainty Equivalent V_t**



Step B: SVD Step

- At time t (a projection of) V_{t+1} is known over the entire state space.
- Goal: solve for $V_t(\mathbf{s}_t)$ on a **grid** for $\mathbf{s}_t$.

Three substeps:

B-1. **Decomposition** of state variables;

B-2. **Separation** of choice variables from shocks;

B-3. **Computation** of conditional expectations.

SVD step B.1: Decomposition of state variables

B-1. Decomposition of (innovations to) state variables

$$\boldsymbol{\delta}_{t+1} = \mathbf{c}_{\delta,t}(\mathbf{s}_t) + \varepsilon_{\delta,t+1}$$

where

$$\begin{aligned} \mathbf{c}_{\delta,t}(\mathbf{s}_t) &= \text{“center” of expansion (known at time } t) \\ \varepsilon_{\delta,t+1} &= \text{stochastic deviation} \end{aligned}$$

Law of motion:

$$\mathbf{s}_{t+1} = \boldsymbol{\Gamma}\left(\mathbf{s}_t, \mathbf{x}_t, \mathbf{c}_{\delta,t} + \varepsilon_{\delta,t+1}\right).$$

SVD step B.2: Separation of choice variables from shocks

Taylor expansions of $\mathcal{U}(V_{t+1}(\underbrace{\mathbf{\Gamma}(\mathbf{s}_t, \mathbf{x}_t, \mathbf{c}_{\delta,t} + \varepsilon_{\delta,t+1})}_{\equiv \mathbf{s}_{t+1}}))$, centered at $\mathbf{c}_{\delta,t}$.

B-2. Separation of choice variables $\mathbf{x}_t$ from shocks $\varepsilon_{\delta,t+1}$

$$\mathcal{U}(V_{t+1}(\mathbf{s}_{t+1})) \approx \sum_{m=1}^{M} \underbrace{A_{t+1,m}(\mathbf{s}_t, \mathbf{x}_t)}_{\textbf{Independent of } \varepsilon_\delta} \cdot \underbrace{B_{t+1,m}(\varepsilon_{\delta,t+1})}_{\textbf{Independent of } \mathbf{x}_t},$$

where

$$\begin{aligned} A_{t+1,m}(\mathbf{s}_t, \mathbf{x}_t) &= \text{partial derivatives of } \mathcal{U}(V) \text{ w.r.t. } \varepsilon_{\delta,t+1} \\ B_{t+1,m}(\varepsilon_{\delta,t+1}) &= \text{products of powers of } \varepsilon_{\delta,t+1} \end{aligned}$$

SVD step B.3: Computation of conditional expectations

B-3. Computation of conditional expectations

Need to compute

$$E_t[\mathcal{U}(V_{t+1}(\mathbf{s}_{t+1}))] \approx \sum_{m=1}^{M} A_{t+1,m}(\mathbf{s}_t, \mathbf{x}_t) \cdot E_t\left[B_{t+1,m}(\varepsilon_{\delta,t+1})\right].$$

- $E_t\left[B_{t+1,m}(\varepsilon_{\delta,t+1})\right]$ needs to be computed **only once**(!) for each grid point
- If shocks are homoschedastic $E_t\left[B_{t+1,m}(\varepsilon_{\delta,t+1})\right]$ can be computed only **once and for all**(!)
- Computationally **very efficient** (compared to, e.g., quadrature).

Applications

1. **CRRA** utility and **stochastic** investment opportunity set
2. **CARA** utility and **constant** investment opportunity set
3. **Recursive** utility **stochastic** investment opportunity set

Requirements for SVD Applicability:

- **Smooth** utility function [need to take derivatives]
- **Compact support** of shocks [**convergence** of Taylor series]

CRRA utility and predictable returns

Maximize expected utility of terminal wealth

$$J_0(W_0, \mathbf{s}_0) = \max_{\{\mathbf{x}_t\}_{t=0}^{T-1}} E_0[u(W_T)],$$

where

$$u(W_T) = \frac{W_T^{1-\gamma}}{1-\gamma}$$

$$W_{t+1} = W_t(R_f + \mathbf{x}_t' \mathbf{R}_{t+1}) \quad \text{[endogenous s.v. } (N \text{ assets})]$$

$$\mathbf{s}_{t+1} = \Gamma(\mathbf{s}_t, \delta_{t+1}) \quad \text{[K exogenous s.v.]}$$

Bellman Equation

$$J_t(W_t, \mathbf{s}_t) = \max_{\mathbf{x}_t} E_t[J_{t+1}(W_t(R_f + \mathbf{x}_t' \mathbf{R}_{t+1}), \mathbf{s}_{t+1})].$$

Step A: Projection step

- **Certainty equivalent**: $J_t(W_t, \mathbf{s}_t) = u(V_t(W_t, \mathbf{s}_t))$.
- **Homotheticity of CRRA** $\Rightarrow$ $V_t(W_t, \mathbf{s}_t) = W_t^{1-\gamma} \frac{\mathcal{V}_t(\mathbf{s}_t)}{1-\gamma}$

Reduced Bellman Equation

$$\frac{\mathcal{V}_t(\mathbf{s}_t)^{1-\gamma}}{1-\gamma} = \max_{\mathbf{x}_t} E_t \left[R_{p,t+1}(\mathbf{x}_t)^{1-\gamma} \frac{\mathcal{V}_{t+1}(\mathbf{s}_{t+1})^{1-\gamma}}{1-\gamma} \right], \quad \mathcal{V}_T(\mathbf{s}_T) = 1$$

$$R_{p,t+1}(\mathbf{x}_t) \equiv R_f + \mathbf{x}_t' \mathbf{R}_{t+1}$$

- Solve backwards from T
- At time $t+1$ obtain a **projection** of $\mathcal{V}_{t+1}(\mathbf{s}_{t+1})$ on the state space;

Step B: SVD step

The goal is to solve for $\mathcal{V}_t(\mathbf{s}_t)$, given a known projection of $\mathcal{V}_{t+1}(\mathbf{s}_{t+1})$

B-1. **Decompose (innovations to) state variables**

$$\begin{aligned}
\mathbf{R}_{t+1} &= \mathbf{c}_{R,t} + \boldsymbol{\varepsilon}_{R,t+1} \Longrightarrow R_{p,t+1}(\mathbf{x}_t) = c_{p,t}(\mathbf{x}_t) + \varepsilon_{p,t+1}(\mathbf{x}_t) \\
\boldsymbol{\delta}_{t+1} &= \mathbf{c}_{\delta,t} + \boldsymbol{\varepsilon}_{\delta,t+1} \Longrightarrow \mathbf{s}_{t+1} = \Gamma(\mathbf{s}_t, c_{\delta,t} + \varepsilon_{\delta,t+1})
\end{aligned}$$

where $\varepsilon_{p,t+1}(\mathbf{x}_t) = \mathbf{x}_t' \boldsymbol{\varepsilon}_{R,t+1}$

Step B: SVD step (cont.)

B-2. **Separate choice variables x_t from shocks $\varepsilon_{\cdot,t+1}$**

Taylor expansion of $R_{p,t+1}(\mathbf{x}_t)^{1-\gamma}\mathcal{V}(\mathbf{s}_{t+1})^{1-\gamma}$ around $(\mathbf{c}_{R,t}, \mathbf{c}_{\delta,t})$

Separation

$$R_{p,t+1}(\mathbf{x}_t)^{1-\gamma}\mathcal{V}(\mathbf{s}_{t+1})^{1-\gamma} \approx \sum_{|\mathbf{n}|+|\mathbf{k}|\leq M} \frac{1}{\mathbf{n}!}\frac{1}{\mathbf{k}!} f_{\mathbf{n}}(\mathbf{x}_t) g_{\mathbf{k}} \prod_{i=1}^{N} \varepsilon_{R_i,t+1}^{n_i} \prod_{j=1}^{K} \varepsilon_{\delta_j,t+1}^{k_j}$$

where $\mathbf{n} = (n_1, \ldots, n_N)$, $\mathbf{k} = (k_1, \ldots, k_K)$,

$$f_{\mathbf{n}}(\mathbf{x}_t) = \left.\frac{\partial^{|\mathbf{n}|} R_{p,t+1}(\mathbf{x}_t)^{1-\gamma}}{\partial\varepsilon_{R_1}^{n_1}\cdots\partial\varepsilon_{R_N}^{n_N}}\right|_{\boldsymbol{\varepsilon}_R=\mathbf{0}_N}, \quad g_{\mathbf{k}} = \left.\frac{\partial^{|\mathbf{k}|}\mathcal{V}(\mathbf{s}_{t+1})^{1-\gamma}}{\partial\varepsilon_{\delta_1}^{k_1}\cdots\partial\varepsilon_{\delta_K}^{k_K}}\right|_{\boldsymbol{\varepsilon}_\delta=\mathbf{0}_\mathbf{K}}.$$

Use Savits (2006) generalization of **Faà di Bruno formula (1855)** for efficient computation of derivatives of composite functions.

B-3. **Compute conditional expectations**

$$E_t\left(\prod_{i=1}^{N}\varepsilon_{R_i,t+1}^{n_i}\prod_{j=1}^{K}\varepsilon_{\delta_j,t+1}^{k_j}\right)$$

- Does **not** depend on choice variable $\mathbf{x}_t$.
- Need to be computed only **once** at each point in the state space
- Expectations can be computed (i) **analytically**, when possible, (ii) by **quadrature** [Judd (1998)] or (iii) by **simulation-based** parameterized expectations [Longstaff-Schwartz (2001), BGSS (2005)]
- Once optimal $\mathbf{x}_t$ is found, $\mathcal{V}_t(\mathbf{s}_t)$ can be computed on a grid of $\mathbf{s}_t$ and then projected (back to step A.)

Numerical implementation

Example from VanBinsbergen and Brandt (2007)

- **One risky** and one risk-free asset
- **One state variable**: dividend yield (predictor)
- (log) risky asset return and (log) dividend yield follow a VAR(1) process
- **Projection** of certainty equivalent function $V_t(s_t)$: **polynomial of degree 12** in s_t
- Gauss-Hermite **quadrature**: **6 nodes** in each dimension.

Comparison with discretized state space using quadrature

Certainty Equivalent (annualized % points)

		$\gamma = 5$					$\gamma = 15$				
		z_{10}	z_{30}	z_{50}	z_{70}	z_{90}	z_{10}	z_{30}	z_{50}	z_{70}	z_{90}
						T = 30					
DSS-Q		**6.65**	**7.34**	**8.26**	**9.57**	**11.91**	**6.25**	**6.57**	**7.01**	**7.70**	**9.17**
SVD	$M = 4$	6.65	7.34	8.26	9.57	11.92	6.26	6.56	7.02	7.69	9.18
	$M = 6$	6.65	7.34	8.26	9.57	11.91	6.26	6.56	7.01	7.70	9.17
	$M = 8$	6.65	7.34	8.26	9.57	11.91	6.26	6.57	7.01	7.71	9.18
						T = 40					
DSS-Q		**6.95**	**7.67**	**8.53**	**9.69**	**11.67**	**6.40**	**6.78**	**7.26**	**7.98**	**9.43**
SVD	$M = 4$	6.95	7.67	8.53	9.69	11.67	6.41	6.78	7.26	7.99	**9.45**
	$M = 6$	6.95	7.67	8.53	9.69	11.67	6.41	6.78	7.27	7.99	**9.41**
	$M = 8$	6.95	7.67	8.53	9.69	11.67	6.40	6.78	7.25	7.98	9.43

Red: CE differ by more than than **2 bps**.

CARA utility and IID normal returns

- Objective: $\max E_0 [u(W_T)]$, $u(W_T) = -\exp(-\alpha W_T)$
- Bellman equation:

$$J_t(W_t) = \max_{\omega_t} E_t \left[J_{t+1} \left(W_t(R_f + \omega_t' \mathbf{R}_{t+1})\right], \quad J_T(W_T) = -\exp(-\alpha W_T)\right.$$

- R_f: risk-free rate, $\mathbf{R}_t \sim N(\mu, \Sigma)$: excess risky asset return.

Closed-form solution

$$J_t(W_t) = -\exp\left(-\alpha W_t R_f^{T-t} - \frac{T-t}{2}\mu'\Sigma^{-1}\mu\right), \quad t = 0, \ldots, T$$

$$\omega_t = \frac{1}{\alpha W_t R_f^{(T-1)-t}}\Sigma^{-1}\mu, \quad t = 0, \ldots, T-1$$

Applying the SVD approach

Prelim step. Use the **certainty equivalent** V_t of J_t: $J_t(W) = u(V_t(W))$

Step A. Projection step.

Modified Bellman equation:

$$-e^{-\alpha V_t(W_t)} = \max_{\omega_t} E_t\left[-e^{-\alpha V_{t+1}(W_{t+1})}\right], \quad V_T(W_T) = W_T$$

- Approximate $V(W)$ as a **polynomial** of order K in wealth W:

$$V(W) \approx V_K(W) = \sum_{k=0}^{K} c_k W^k$$

Applying the SVD approach (con't)

Step B. SVD step

B-1. **Decompose** W_{t+1} into $\mu_W + \varepsilon_W$, $\mu_W = W_t(R_f + \omega'\mu_R)$, $\varepsilon_W = W_t(\omega'\varepsilon_R)$.

$$\mathcal{U}(V_t(W_t)) = -e^{-\alpha V(W_{t+1})} \approx -e^{-\alpha \sum_{k=0}^{K} c_k(\mu_W+\varepsilon_W)^k} \equiv g_K(\varepsilon_W)$$

B-2. **Separate choice variables from shocks**

Taylor **approximate** $g_K(\varepsilon_W)$ around $\varepsilon_W = 0$

$$g_K(\varepsilon_W) \approx \sum_{m=0}^{M} \frac{1}{m!} g_K^{(m)}(0)\varepsilon_W^m$$

Use **binomial formula** to compute ε_W^m.

Approximate Maximization Problem (2-asset example)

$$-e^{-\alpha V(W_t)} = \max_{\omega_t} \sum_{m=0}^{M} W_t^m g_K^{(m)}(0) \sum_{m_1+m_2=m} \frac{1}{m_1!m_2!} [\omega_1^{m_1}\omega_2^{m_2}] E\left[\varepsilon_{R,1}^{m_1}\varepsilon_{R,2}^{m_2}\right]$$

Faà di Bruno (1885) formula for efficient computation of $g_K^{(m)}(0)$

Applying the SVD approach (con't)

B-3. **Compute** cross-moments $E\left[\varepsilon_{R,1}^{m_1}\cdots\varepsilon_{R,N}^{m_N}\right]$

- **Independent** of allocations ω,
- Computed only **once**.

Step B $\Longrightarrow$ **optimal portfolio** $\boldsymbol{\omega}_t \Longrightarrow V_t$ on a grid for $W_t \Longrightarrow$ Step A.

Numerical example:

- Projection of V_t: **Polynomial of degree** $K = 2$
- Taylor expansions with **order** $M = 4$.

Comparing SVD and exact solution

Data: 3 MSCI-Barra international indexes (annualized), $R_f = 1.05$

Certainty Equivalent (annualized % points)

		$W_0 = 1$	$W_0 = 1.25$	$W_0 = 1.5$	$W_0 = 1.75$	$W_0 = 2$
				Exact		
$\alpha = 2$	$T = 10$	8.078	7.523	7.138	6.856	6.639
	$T = 20$	6.825	6.504	6.280	6.114	5.987
	$T = 30$	6.130	5.931	5.791	5.688	5.609
$\alpha = 4$	$T = 10$	6.639	6.329	6.118	5.964	5.848
	$T = 20$	5.987	5.803	5.677	5.585	5.515
	$T = 30$	5.609	5.495	5.417	5.360	5.317
$\alpha = 6$	$T = 10$	6.118	5.902	5.757	5.652	5.572
	$T = 20$	5.677	5.548	5.460	5.397	5.349
	$T = 30$	5.417	5.337	5.283	5.244	5.215
				SVD (CE obtained via Monte Carlo simulation)		
$\alpha = 2$	$T = 10$	8.080	**7.510**	7.140	6.853	6.642
	$T = 20$	6.827	6.507	6.279	**6.123**	5.988
	$T = 30$	6.135	**5.938**	**5.793**	5.696	5.605
$\alpha = 4$	$T = 10$	6.638	6.334	6.118	5.964	5.850
	$T = 20$	5.981	5.800	5.677	5.583	5.513
	$T = 30$	5.614	5.497	5.421	5.361	5.321
$\alpha = 6$	$T = 10$	6.116	5.905	5.756	5.653	5.572
	$T = 20$	5.679	5.547	5.461	5.397	5.350
	$T = 30$	5.420	5.332	5.283	5.245	5.215

Red: **CE difference $> 1/2$ bp.**

SVD vs. Brandt et al. (2005, BGSS)

- Choice of **centers of expansion** for Taylor approximation
  - BGSS: $\mu_W = WR_f \Rightarrow$ Expansion is w.r.t. a **non-zero-mean** random shock
  - SVD: $\mu_W = W(R_f + \omega\mu_R) \Rightarrow$ Expansion is w.r.t. a **zero-mean** random shock
- **Solution technique**
  - BGSS: **Policy** Function Iteration + Taylor expansion
    - **Cannot** handle **dependence** of **future** allocation on **current** wealth
  - SVD: **Value** Function Iteration + Taylor expansion
    - **Can** handle **dependence** of **future** allocation on **current** wealth

BGSS is **OK only if** preferences are **homothetic**.
BGSS **center of expansion** still an issue even in the homothetic case.

Comparing SVD and BGSS

- **Two-asset**, **two-period** problem
- **Four** different methods considered

Center of expansion	**Dependence of ω_1 on W_0**	
	No	*Yes*
$\mu_W = W_0 R_f$	**BGSS**	**M2**
$\mu_W = W_0(R_f + \omega_0 \mu_R)$	**M1**	**SVD**

Certainty Equivalent Loss (annualized bp)

Parameters: $R_f = 1.05$, $\mu_1 = 3\%$, $\mu_2 = 9\%$, $\sigma_1 = 15\%$, $\sigma_2 = 18\%$, $\alpha = 4$

Correlation	-0.4	-0.2	0.0	0.2	0.4
BGSS					
$M = 6$	95.4	23.3	9.87	5.97	5.0
$M = 8$	32.8	11.3	5.8	3.9	3.3
$M = 10$	27.5	10.5	5.6	3.8	3.3
M1 (ω_1 **independent** of W_0)					
$M = 6$	**48.1**	**17.6**	**9.0**	**5.9**	**5.1**
$M = 8$	**53.7**	**18.9**	**9.5**	**6.2**	**5.3**
$M = 10$	**54.1**	**18.9**	**9.5**	**6.2**	**5.3**
M2 (ω_1 **depends** on W_0)					
$M = 6$	13.9	1.8	0.4	0.2	0.1
$M = 8$	0.1	0.0	0.0	0.0	0.0
$M = 10$	0.0	0.0	0.0	0.0	0.0
SVD					
$M = 6$	**0.1**	**0.0**	**0.0**	**0.0**	**0.0**
$M = 8$	**0.0**	**0.0**	**0.0**	**0.0**	**0.0**
$M = 10$	**0.0**	**0.0**	**0.0**	**0.0**	**0.0**

Life-time portfolio and consumption choice problem [Campbell, Chan and Viceira (2003, CCV)]

- **3 Assets**: nominal T-bills, nominal T-bonds, stocks
- **6 State variables**: lagged asset returns plus 90-day nominal T-bill yield, dividend-price ratio, spread b/w 5-year bond yield and the T-bill rate.
- State variables follow a **VAR dynamics**
- **Recursive Preferences (Epstein-Zin)**. Bellman equation:

$$V_t(W_t, \mathbf{y}_t) = \max_{C_t, \boldsymbol{\omega}_t} \left\{ (1-\beta) C_t^\rho + \beta \left(E_t \left(V_{t+1}^{1-\gamma}(W_{t+1}, \mathbf{y}_{t+1}) \right) \right)^{\frac{\rho}{1-\gamma}} \right\}^{1/\rho}$$

Comparison with CCV

Differences from CCV

- **Finite-horizon** [CCV solves infinite horizon]
- Short-selling **constraints** [CCV consider only unconstrained policies]
- SVD instead of **log-linearization** of budget constraint [CCV]
- EIS parameter ρ **unrestricted** [CCV works for EIS ≈ 1]

CCV and BGSS methodology **cannot** solve this problem:

- CCV cannot handle **constraints**
- BGSS cannot handle **recursive preferences**

Using SVD to solve CCV problem

Modified Bellman equation

$$\mathcal{V}_t(\mathbf{y}_t) = \left\{ 1 + \left[\beta \left(\min_{\boldsymbol{\omega}_t} E_t \left[(R_{p,t+1}(\boldsymbol{\omega}_t))^{1-\gamma} \, \mathcal{V}_{t+1}(\mathbf{y}_{t+1})^{1-\gamma} \right] \right)^{\frac{\rho}{1-\gamma}} \right]^{\frac{1}{1-\rho}} \right\}^{\frac{1-\rho}{\rho}}$$

$\mathcal{V}_T(\mathbf{y}_T) = 1$, consumption-to-wealth ratio $c_t = \mathcal{V}_t(\mathbf{y}_t)^{-\frac{\rho}{1-\rho}}$

A. **Project** $\mathcal{V}_{t+1}(\mathbf{y}_{t+1})$ over the state space for $\mathbf{y}_{t+1}$ by using radial basis function with 500 Gaussian kernels.

B-1. **Decompose** $y_{i,t+1} = \mu_{i,t} + \varepsilon_{i,t+1}$

B-2. **Separate.** Taylor expansions of $R_{p,t+1}(\boldsymbol{\omega}_t)^{1-\gamma}\mathcal{V}_{t+1}(\mathbf{y}_{t+1})^{1-\gamma}$ around $\mu_{i,t}$ (We use $M = 4$ in Taylor expansions)

B-3. **Analytically compute** $E_t \left[\prod_{i=1}^{3} \varepsilon_{i,t+1}^{n_i} \prod_{j=1}^{6} \varepsilon_{j,t+1}^{k_j} \right]$

Execution time for 30-year problem: **3.46 hours** vs. **5.3 days** for quadrature!

Comparing SVD to Quadrature

Data from Campbell et al (2003), $T = 30$, $\gamma = 5$, $EIS = 0.5$, $\beta = 0.92$.

	Q	SVD	Q	SVD	Q	SVD
	p_{25}		p_{50}		p_{75}	
			Short term nominal interest rate (z_1)			
Bond	46.48	**46.92**	46.93	47.21	47.44	47.54
Stock	53.52	**53.08**	53.07	52.79	52.56	52.46
Cons.	6.70	6.69	6.93	6.92	7.20	7.18
			Dividend yield (z_2)			
Bond	65.90	66.10	46.93	47.21	28.06	**28.38**
Stock	34.10	33.90	53.07	52.79	71.94	**71.62**
Cons.	6.83	6.82	6.93	6.92	7.11	7.10
			Yield spread (z_3)			
Bond	0.00	0.00	46.93	47.21	53.92	54.11
Stock	52.32	**51.98**	53.07	52.79	46.08	45.89
Cons.	6.88	6.88	6.93	6.92	7.01	7.00

Red: Allocation/consumption differ by **more than 0.3%**.

Conclusion

- Develop a new **approximation methodology** for portfolio based on
  - **Decomposition** of state variables
  - **Taylor** approximations
  - **Separation** between **shocks** to state variables and **choice variables**
- **Reduce** the problem of computing **conditional expectation of value function** to the problem of computing **conditional moments of shocks to state variables.**
- Shift focus from **integrals** to **derivatives**
- **Conceptually simple**, **computationally efficient**, and **accurate**
- **Broad applicability** to dynamic problems in economics and finance.