Hilbert transform approach to options valuation

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Options pricing

- BIS: **global OTC option market size** (notional amounts outstanding) more than $65 trillion by the end of June 09
- 13,920,865 S&P 500 index options traded on **CBOE** in January 2010
- GBM model Black Scholes 73, Merton 73: contradicts **volatility smile** effects observed in option markets
- Jump diffusion/Lévy: Merton 76, Kou 02, Madan Carr Chang 98, Barndorff-Nielsen 98, Eberlein Keller Prause 98, Carr Geman Madan Yor 02, Carr Wu 03; local volatility: Dupire 94, Derman Kani 94; stochastic volatility: Hull and White 87, Heston 93; stochastic volatility jump diffusion: Bates 96, Duffie Pan Singleton 00

Transform methods

- **Numerical methods** often needed for computing option prices in alternative models: numerical solution to PIDEs, monte carlo simulation, transform methods
- **Fourier transform** based methods for models with known characteristic functions
- Fourier inverse representation for European vanilla options Carr Madan 99, Lee 04
- Coupled with the **fast Fourier transform** (computing convolution integrals using FFT for options pricing Eydeland 94)
- **Hilbert transform** approach to options valuation Feng Linetsky 08, 09 (illustrated with Bermudan options in Lévy models, Feng Lin 09)

Hilbert transform

- **Hilbert transform** of $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$

$$\mathcal{H}f(x) = \frac{1}{\pi}\, p.v. \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy$$

- For any $f \in L^1(\mathbb{R})$ with $\hat{f} \in L^1(\mathbb{R})$, and $l \in \mathbb{R}$,

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}} e^{i\xi x} f(x) dx$$

$$\mathcal{F}(1_{(l,\infty)} \cdot f)(\xi) = \frac{1}{2}\hat{f}(\xi) + \frac{i}{2} e^{i\xi l} \mathcal{H}(e^{-i\eta l}\hat{f}(\eta))(\xi)$$

European vanilla options

- European **vanilla** options ($S_t = S_0 e^{X_t}$):

$$\mathbb{E}[(K - S_T)^+] = K\mathbb{E}[\mathbf{1}_{\{S_T<K\}}] - \mathbb{E}[S_T \mathbf{1}_{\{S_T<K\}}]$$

$$\mathbb{P}(X \leq x) = \int_{\mathbb{R}} \mathbf{1}_{(-\infty,x)}(y)p(y)dy = \frac{1}{2} - \frac{i}{2}\mathcal{H}(e^{-i\xi x}\phi(\xi))(0)$$

where ϕ is the c.f. of X

- Inverting c.f. of a distribution to obtain the cdf; monte carlo simulation Feng Lin 10

Barriers and lookbacks

- European discrete **barrier** options/defaultable bonds in Lévy models: Feng Linetsky 08

$$f^j(x) = \mathbf{1}_{(l,u)} \cdot \mathbb{E}_{t_j,x}[f^{j+1}(X_{t_{j+1}})]$$

- Exponential moments of the discrete maximum of a Lévy process, European discrete **lookbacks** in Lévy models: Feng Linetsky 09

$$M_j - X_j = \max(M_{j-1}, X_j) - X_j = \max(0, M_{j-1} - X_{j-1} - (X_j - X_{j-1}))$$

Discrete Hilbert transform

- **Discrete Hilbert transform** with step size $h > 0$

$$\mathcal{H}_h f(x) = \sum_{m=-\infty}^{\infty} f(mh) \frac{1 - \cos[\pi(x - mh)/h]}{\pi(x - mh)/h}, \quad x \in \mathbb{R}$$

- For f analytic in a horizontal strip $\{z \in \mathbb{C} : |\Im(z)| < d\}$

$$||\mathcal{H}f - \mathcal{H}_h f||_{L^\infty(\mathbb{R})} \leq \frac{Ce^{-\pi d/h}}{\pi d(1 - e^{-\pi d/h})}$$

- Related to **Whittaker cardinal series**

Whittaker cardinal series

- **Whittaker cardinal series** (sinc expansion) Whittaker 1915

$$c(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh) \frac{\sin(\pi(x-kh)/h)}{\pi(x-kh)/h}$$

- For entire functions of exponential type π/h, sinc expansion is exact: $c(f,h) = f$
- For functions analytic in a strip $\{z \in \mathbb{C} : |\Im(z)| < d\}$, Stenger 93

$$||f - c(f,h)||_{L^\infty} \leq \frac{Ce^{-\pi d/h}}{\pi d(1 - e^{-2\pi d/h})}$$

Trapezoidal rule

- Take Hilbert transform on $c(f, h)$, obtain discrete Hilbert transform
- **Trapezoidal rule** very accurate for f analytic in a strip $\{z \in \mathbb{C} : |\Im(z)| < d\}$

$$\left| \int_{\mathbb{R}} f(x)dx - \sum_{m=-\infty}^{\infty} f(kh)h \right| \leq \frac{Ce^{-2\pi d/h}}{1 - e^{-2\pi d/h}}$$

- We use trapezoidal rule to compute Fourier inverse integral

Bermudan vanilla options

- **Bermudan put**: payoff $G(S) = (K - S)^+$, discrete monitoring

$$\mathbb{T} = \{t_0, t_1, \cdots, t_N\} = \{0, \Delta, 2\Delta, \cdots, N\Delta = T\}$$

- **Exponential Lévy model**: X_t a Lévy process in $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, start from 0, equivalent martingale measure $\mathbb{P}$ is given

$$S_t = S_0 e^{X_t}$$

- **Optimal stopping**

$$V^0(S_0) = \sup_{\tau} \mathbb{E}_0[e^{-r\tau} G(S_\tau)]$$

Backward induction in state space

- Variable change $x = \ln(S/K)$,

$$g(x) = G(Ke^x) = K(1 - e^x)^+, \quad f^0(x) = V^0(Ke^x)$$

- Backward induction

$$f^N(x) = g(x)$$

$$f^j(x) = \max\left(g(x), e^{-r\Delta}\mathbb{E}_{j\Delta,x}[f^{j+1}(X_{(j+1)\Delta})]\right), \quad 0 \leq j < N$$

$$V^0(S_0) = f^0(\ln(S_0/K))$$

Implementation

- For each time step, need to compute

$$\mathbb{E}_{j\Delta,x}[f^{j+1}(X_{(j+1)\Delta})]$$

- Monte carlo simulation: Longstaff Schwartz 01, Glasserman 04
- Double exponential fast Gauss transform: Broadie Yamamoto 05
- Lattice approximation of the transition density Kellezi Webber 04
- Conditional expectation is a convolution, its Fourier transform is a product; **FT of f^{j+1}** $\rightarrow$ multiply by c.f. $\rightarrow$ FI representation of the conditional expectation $\rightarrow$ take max $\rightarrow$ **FT of f^j** Jackson Jaimungal Surkov 08:

Dampening for integrability

- For $\alpha > 0$ (for puts), define

$$f_\alpha^j(x) = e^{\alpha x} f^j(x), \quad g_\alpha(x) = e^{\alpha x} g(x)$$

- **Esscher transform**: Radon-Nikodým derivative

$$\frac{d\mathbb{P}^\alpha}{d\mathbb{P}}|_{\mathcal{F}_t} = e^{-\alpha X_t} / \phi_t(i\alpha)$$

$$e^{\alpha x}\mathbb{E}_{j\Delta,x}[f^{j+1}(X_{(j+1)\Delta})] = \phi_\Delta(i\alpha)\mathbb{E}^\alpha_{j\Delta,x}[f_\alpha^{j+1}(X_{(j+1)\Delta})]$$

- Esscher transformed Lévy process is still a Lévy process

$$\phi_t^\alpha(\xi) = \phi_t(\xi + i\alpha)/\phi_t(i\alpha)$$

Dampened backward induction

- Dampened backward induction

$$f_\alpha^N(x) = g_\alpha(x)$$

For $0 \leq j < N$, let x_j^* be the **early exercise boundary** at $j\Delta$

$$\begin{aligned} f_\alpha^j(x) &= \max\left(g_\alpha(x), e^{-r\Delta}\phi_\Delta(i\alpha)\mathbb{E}^\alpha_{j\Delta,x}[f_\alpha^{j+1}(X_{(j+1)\Delta})]\right) \\ &= g_\alpha(x)\cdot \mathbf{1}_{(-\infty,x_j^*)}(x) \\ &\quad + e^{-r\Delta}\phi_\Delta(i\alpha)\mathbb{E}^\alpha_{j\Delta,x}\left[f_\alpha^{j+1}(X_{(j+1)\Delta})\right]\cdot \mathbf{1}_{[x_j^*,\infty)}(x) \end{aligned}$$

Backward induction in Fourier space

- By convolution theorem,

$$\mathcal{F}\left(\mathbb{E}^{\alpha}_{j\Delta,x}\left[f^{j+1}_{\alpha}(X_{(j+1)\Delta})\right]\right)(\xi) = \hat{f}^{j+1}_{\alpha}(\xi)\phi^{\alpha}_{\Delta}(-\xi)$$

- **Backward induction in Fourier space**

$$\hat{f}^{N}_{\alpha}(\xi) = \hat{g}_{\alpha}(\xi)$$

$$\hat{f}^{j}_{\alpha}(\xi) = \mathcal{F}(g_{\alpha}\mathbf{1}_{(-\infty,x^*_j)})(\xi) + e^{-r\Delta}\phi_{\Delta}(i\alpha)\left(\frac{1}{2}\hat{f}^{j+1}_{\alpha}(\xi)\phi^{\alpha}_{\Delta}(-\xi) + \frac{i}{2}e^{i\xi x^*_j}\mathcal{H}\left(e^{-i\eta x^*_j}\hat{f}^{j+1}_{\alpha}(\eta)\phi^{\alpha}_{\Delta}(-\eta)\right)(\xi)\right)$$

Early exercise boundary

- **Early exercise boundary** solves

$$g_\alpha(x) = e^{-r\Delta}\phi_\Delta(i\alpha)\mathbb{E}^{\alpha}_{j\Delta,x}[f_\alpha^{j+1}(X_{(j+1)\Delta})]$$

- Using **Fourier inverse** representation

$$g_\alpha(x) = \frac{1}{2\pi}e^{-r\Delta}\phi_\Delta(i\alpha)\int_{\mathbb{R}} e^{-i\xi x}\hat{f}_\alpha^{j+1}(\xi)\phi_\Delta^\alpha(\xi)d\xi$$

- $x_N^* = K$. To solve for x_j^*, use Newton-Raphson, with starting point x_{j+1}^*

Algorithm summarized

- Start with Fourier transform of dampened payoff $\hat{f}_\alpha^N = \hat{g}_\alpha$
- At time $j\Delta$, with $\hat{f}_\alpha^{j+1}$, compute early exercise boundary x_j^* using Newton Raphson (**Fourier inverse**)
- Compute $\hat{f}_\alpha^j$ from $\hat{f}_\alpha^{j+1}$ and x_j^* (**Hilbert transform**)
- With $\hat{f}_\alpha^1$, option value at time 0 (**Fourier inverse**)

$$f_\alpha^0(x) = \max\left(g_\alpha(x), \frac{1}{2\pi} e^{-r\Delta} \phi_\Delta(i\alpha) \int_{\mathbb{R}} e^{-i\xi x} \hat{f}_\alpha^1(\xi) \phi_\Delta^\alpha(\xi) d\xi \right)$$

Discrete approximation

- Need to repeatedly evaluate Fourier inverse integrals and $\mathcal{H}\psi(\xi)$
- **Trapezoidal rule** for Fourier inverse integral, truncate infinite series with truncation level M, computational cost $O(M)$
- Replace $\mathcal{H}\psi$ by **discrete Hilbert transform**, truncate resulting infinite series

$$\mathcal{H}\psi(\xi) \Leftarrow \sum_{m=-M}^{M} \psi(mh) \frac{1-\cos[\pi(\xi - mh)/h]}{\pi(\xi - mh)/h}$$

Toeplitz matrix vector multiplication

- Evaluate

$$\mathcal{H}\psi(\xi) \Leftarrow \sum_{m=-M}^{M} \psi(mh) \frac{1-\cos[\pi(\xi - mh)/h]}{\pi(\xi - mh)/h}$$

  for $\xi = -Mh, \cdots, Mh$
- Correspond to **Toeplitz matrix vector multiplication**
- FFT based method for such multiplications: $O(M\log(M))$
- Total computational cost of the method: $O(NM\log(M))$

Error estimate

- **Discretization error** $O(\exp(-\pi d/h)$
- With $\phi_t(\xi) \sim \exp(-ct|\xi|^\nu)$, **truncation error** is essentially

$$O(\exp(-\Delta c(Mh)^\nu))$$

- Select $h = h(M)$ according to

$$h(M) = \left(\frac{\pi d}{\Delta c}\right)^{\frac{1}{1+\nu}} M^{-\frac{\nu}{1+\nu}}$$

- Total error: $O(\exp(-CM^{\frac{\nu}{1+\nu}}))$

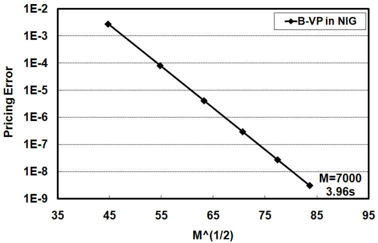
NIG model

- Pricing Bermudan put option in NIG
- Characteristic exponent

$$-i\mu\xi + \delta_{NIG}\left(\sqrt{\alpha_{NIG}^2 - (\beta_{NIG} + i\xi)^2} - \sqrt{\alpha_{NIG}^2 - \beta_{NIG}^2}\right)$$

- $\phi_t(\xi)$ has exponential tails with $\nu = 1$; error estimate in M: $O(e^{-C\sqrt{M}})$

Bermudan put in the NIG model



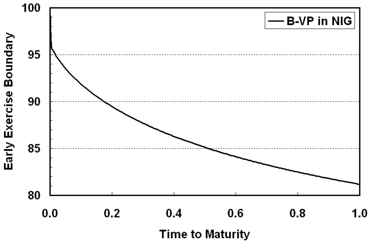


Figure: $T = 1$, **N=252**, $S_0 = 100$, $K = 100$, $r = 5\%$, $q = 2\%$, $\alpha_{NIG} = 15$, $\beta_{NIG} = -5$, $\delta_{NIG} = 0.5$, Matlab R2009a, Lenovo T400 Laptop with 2.53GHz CPU, 2G RAM; average number of NR iterations per time step 4.08

Bermudan barriers/lookbacks

- **Bermudan barrier** options

$$f^j(x) = \mathbf{1}_{(l,u)}(x) \cdot \Big(g(x) \cdot \mathbf{1}_{(-\infty, x_j^*]}(x) + e^{-r\Delta}\mathbb{E}_{j\Delta,x}[f^{j+1}(X_{(j+1)\Delta})] \cdot \mathbf{1}_{(x_j^*,\infty)}(x) \Big)$$

- **Bermudan** floating strike **lookback** options: standard backward induction involves two state variables: asset price, maximum asset price
- Can be reduced to one state variable, **maximum asset price/asset price**; double exponential fast Gauss transform method for BSM and Merton's models Yamamoto 05

Bermudan down-and-out put in Kou's model

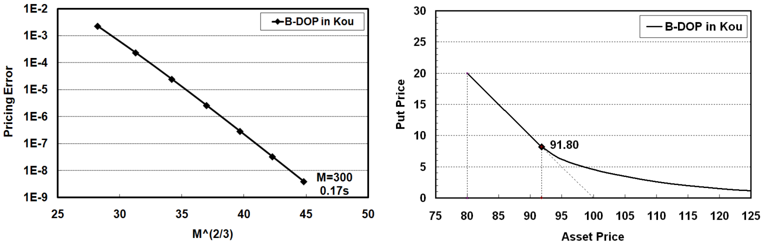


Figure: $T = 1$, **N=252**, $S_0 = 100$, $K = 100$, $L = 80$, $r = 5\%$, $q = 2\%$, $\sigma = 0.1$, $\lambda = 3$, $p = 0.3$, $\eta_1 = 40$, $\eta_2 = 12$, Matlab R2009a, Lenovo T400 Laptop with 2.53GHz CPU, 2G RAM

Bermudan floating strike lookback put in CGMY

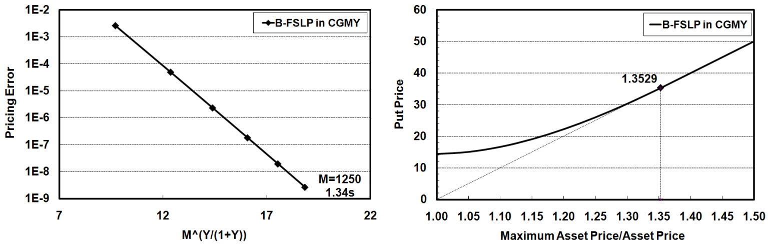


Figure: $T = 1$, **N=252**, $S_0 = 100$, $r = 5\%$, $q = 2\%$, $C = 4$, $G = 50$, $M_{cgmy} = 60$, $Y = 0.7$, Matlab R2009a, Lenovo T400 Laptop with 2.53GHz CPU, 2G RAM

American options

- In BSM convergence of Bermudan options to American options $O(1/N)$ Howison 07
- **Richardson extrapolation**: from two approximations P_1 with N_1 and P_2 with N_2

$$P_\infty \approx \frac{N_1 P_1 - N_2 P_2}{N_1 - N_2}$$

N	B-VP in BSM	Extrap
5	6.58462398	
10	6.62146556	6.65831
20	6.64073760	6.66001
40	6.65061811	6.66050
80	6.65562807	6.66064

Table: American vanilla put in the Black-Scholes-Merton model.

Summary

- Hilbert transform method for pricing Bermudan style options in Lévy process models
- Very accurate with exponentially decaying errors
- Fast with computational cost $O(NM\log(M))$
- European vanilla, barrier, lookback, defaultable bonds, Bermudan vanilla, barrier, floating strike lookback, inverting c.f., monte carlo simulation etc.