

---

# **Integral Equation Methods in Mathematical Finance**

John Chadam  
Mathematics Department  
University of Pittsburgh

Joint work with: Xinfu Chen & Huibin Cheng

Supported by NSF grant DMS-0504691 (XC) and DMS-0707953 (HC & JC).

---

# OUTLINE

1. Introduction / Background.
2. Near Expiry Estimates.
3. Far-From-Expiry Estimates.
4. Convexity of the boundary.
  - a) The Integro-Differential Equations.
  - b) Proof of Non-Convexity (Sketch) ( $0 < D - r \ll 1$ ).

# Introduction/Background

---

$$\frac{\partial P}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 P}{\partial S^2} + (r - D)S \frac{\partial P}{\partial S} - rP = 0, \quad S > B(t), \quad t < T$$

$$P(S, t) = E - S, \quad S = B(t), \quad t < T$$

$$\frac{\partial P}{\partial S}(S, t) = -1, \quad S = B(t), \quad t < T$$

$$P(S, T) = \max(E - S, 0), \quad B(T) = \min(E, \frac{rE}{D})$$

$$P(S, t) = E - S, \quad S < B(t), \quad t < T.$$

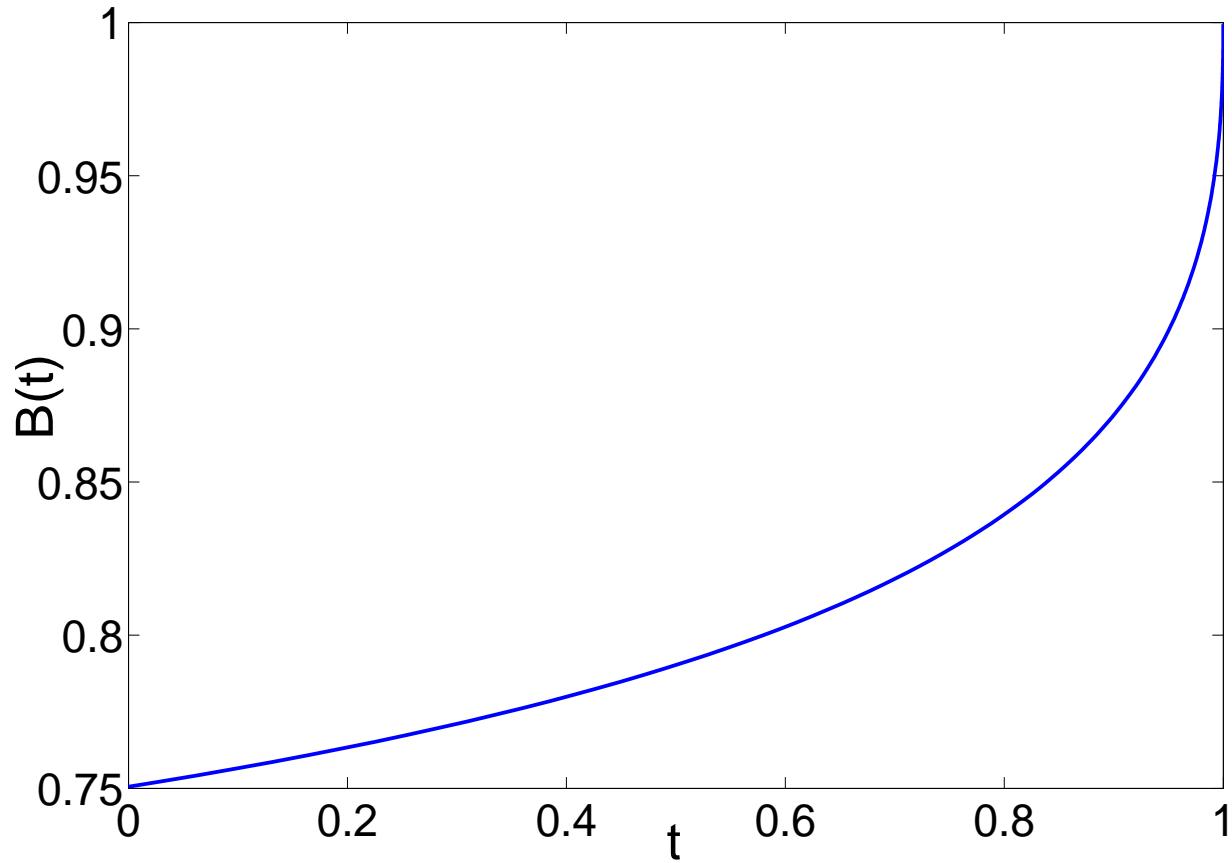


Figure 1:  $r = .05, d = 0, E = 1, \sigma = 0.25, T = 1$

---

Aitshlia & Lai

Broadie & Detemple

Carr, Jarrow & Myneni

Geske & Johnson

Huang, Subrahmanyam & Yu

Jacka

Jaillet, Lamberton & Lapeyre

Karatzas

Kim

Parkinson

Salopek

etc.

---

$$x := \ln \frac{S}{E}, \quad s := \frac{\sigma^2}{2}(T - t), \quad k := \frac{2r}{\sigma^2}, \quad \ell := \frac{2D}{\sigma^2}, \quad \alpha := k - \ell - 1,$$

$$P(S, t) = E p(x, s), \quad B(t) = E e^{b(s)}.$$

$$p_s = p_{xx} + \alpha p_x - kp, \quad b(s) < x, \quad s > 0$$

$$p(b(s), s) = 1 - e^{b(s)}, \quad s > 0$$

$$p_s(b(s), s) = -e^{b(s)}, \quad s > 0$$

$$p(x, s) \rightarrow 0, \quad x \rightarrow \infty, \quad s > 0$$

$$p(x, 0) = \max(1 - e^x, 0) = p_0(x), \quad b(0) = \min(0, \ln(k/l))$$

$$p(x, s) = p_0(x), \quad x < b(s), \quad s > 0$$

---

For  $D = l = 0$ ,

$$p_s - \{p_{xx} + (k-1)p_x - kp\} = kH(b(s) - x)$$

$$p(x, 0) = \max(1 - e^x, 0).$$

where  $H(y) = \begin{cases} 1, & y > 0 \\ 0, & y < 0 \end{cases}$

$$p(x, s) = \int_{-\infty}^{\infty} p(y, 0) \Gamma(x - y, s) dy + k \int_0^s \int_{-\infty}^{b(u)} \Gamma(x - y, s - u) dy du$$

in terms of the fundamental solution

$$\Gamma(x, s) = e^{-ks} F(x + (k-1)s, s)$$

$$F(z, s) = \frac{1}{2\sqrt{\pi s}} e^{-z^2/4s}$$

---


$$p(x, s) = \int_{-\infty}^{b(0)=0} (1 - e^y) \Gamma(x - y, s) dy + k \int_0^s \int_{-\infty}^{b(u)} \Gamma(x - y, s - u) dy du$$

$$p_s(x, s) = \Gamma(x, s) + k \int_0^s \Gamma(x - b(u), s - u) \dot{b}(u) du$$

$$p_s(b(s), s) = 0$$

$$\Gamma(b(s), s) = -k \int_0^s \Gamma(b(s) - b(u), s - u) \dot{b}(u) du.$$

Combinations of this with other derivatives,  $p_x, p_{x\tau}, p_{xx}$ , evaluated on the boundary lead to other integral equations as well as the integro-differential equation

$$\dot{b}(s) = -2 \frac{\Gamma_x(b(s), s)}{k} + 2 \int_0^s \Gamma_x(b(s) - b(u), s - u) \dot{b}(u) du.$$

---

**Theorem.** (Theorem 3.2, SIAM Math. Anal. 38, 1613 (2006)). Suppose that  $b \in C^1((0, \infty)) \cap C^0([0, \infty))$  and  $\alpha(s) = b(s)^2/4s$ . Assume that as  $s \searrow 0$ ,  $\alpha(s) = [-1 + o(1)] \ln \sqrt{s}$  and  $s\dot{\alpha}(s) = O(1)$ . Then  $b$ , together with  $p$  defined by (2), solves the well-known free boundary problem for the American put. Finally, the previous IE has a unique solution with the properties listed above.

Peskir used a variant of IEs to prove uniqueness in the class of continuous boundaries.

# Near Expiry Estimates

---

$$s = \frac{\sigma^2}{2}(T - t), \quad k = \frac{2r}{\sigma^2}, \quad B(t) = E e^{-2\sqrt{s}\sqrt{\alpha(s)}}$$

Barles, Burdeau, Romano & Samsoen (1995) - BBRS

$$B(t) \sim E \left( 1 - \sigma \sqrt{T-t} \sqrt{|\ln(T-t)|} \right), \quad t \sim T$$

$$(\Leftarrow \alpha(s) = -\ln \sqrt{cs}, \quad c \text{ arbitrary})$$

Barone-Adesi & Whaley (1987); MacMillan (1997) - BWM

$$\sqrt{\pi} h(s) = \int_{\sqrt{\alpha(s)}}^{\infty} e^{-[z - \frac{(k+1)}{2}\sqrt{s}]^2} \left\{ (1 + \eta(s)) e^{-2\sqrt{\alpha(s)}\sqrt{s}} - e^{-2z\sqrt{s}} \right\}$$

$$\text{with } h(s) = 1 - e^{-ks}, \quad \eta(s) = \sqrt{h(s)} \left[ k + \frac{(k-1)^2}{4} h(s) + \frac{(k-1)}{2} \sqrt{h(s)} \right]$$

---

Kuske & Keller (1998) - KK

$$\sqrt{s} \alpha e^\alpha = 1/\sqrt{9\pi k^2}$$

Bunch & Johnson (2000) - BJ

$$\sqrt{\alpha} e^{\alpha-(k-1)\sqrt{s}\sqrt{\alpha}} = \sqrt{d} e^{(d-1)(k+1)^2/4} (4k^2 s)^{-1/2}$$

with  $d := 1 - k^2[(1+k)^2(2 + (1+k)^2 s)]^{-1}$

---

$$s = \frac{\sigma^2}{2}(T - t), \quad k = \frac{2r}{\sigma^2}, \quad B(t) = E e^{-2\sqrt{s}\sqrt{\alpha(s)}}$$

(BBRS)  $\alpha(s) \sim -\ln \sqrt{cs}$ ,  $c$  arbitrary

(BWM)  $\sqrt{s} \sqrt{\alpha} e^\alpha \sim 1/\sqrt{4\pi k^2}$

(KK)  $\sqrt{s}\alpha e^\alpha \sim 1/\sqrt{9\pi k^2}$

(BJ)  $\sqrt{s} \sqrt{\alpha} e^\alpha \sim \left[ \left( 1 - \frac{1}{2} \left( \frac{k}{1+k} \right)^2 \right) / 4k^2 \right]^{-1/2}$

Stamicar, Sevcovic & Chadam (1999); Chen, Chadam & Stamicar (2000) - CCSS

(CCSS)  $\sqrt{s} e^\alpha \sim 1/\sqrt{4\pi k^2} \Rightarrow \alpha(s) \sim -\ln \left( \sqrt{4\pi k^2 s} \right)$

---


$$\Gamma(b(s), s) = -k \int_0^s \Gamma(b(s) - b(u), s - u) \dot{b}(u) du.$$

$$\Gamma(b(s) - b(u), s - u) = F(b(s) - b(u), s - u)[1 + O(s)], \quad 0 < u < s$$

With  $\eta = (b(s) - b(u))/2\sqrt{s-u}$ , the rhs for small  $s$

$$\sim -k \int_0^{\frac{b(s)}{2\sqrt{s}}(\rightarrow -\infty)} \left[ 1 - \frac{b(s) - b(u)}{2\dot{b}(u)(s-u)} \right]^{-1} \rightarrow \frac{1}{2} \text{ uniformly in } u \frac{e^{-\eta^2}}{\sqrt{\pi}} d\eta.$$

$$\Rightarrow \quad \Gamma(b(s), s) \simeq \frac{e^{-b(s)^2/4s}}{2\sqrt{\pi s}} \sim k$$

$$\Rightarrow \quad b(s) \sim -2\sqrt{s} \sqrt{-\ln \sqrt{4\pi k^2 s}}$$

$$\text{i.e., } \alpha(s) \sim -\ln(\sqrt{4\pi k^2 s})$$

Similar approach developed independently by Goodman & Ostrov (2002).

---

For  $D > r$

**Theorem:** The above free boundary problem has a unique solution. In addition,  $b \in C^\infty((0, \infty)) \cap C([0, \infty))$ .

[ $C^\infty$  for jump-diffusion: E. Bayraktar & H. Xing, preprint;  
Continuity for Lévy processes; D. Lamberton & M. Mikou,  
Finance Stoch., 12 (2008)]

**Theorem:** Assume that  $D > r$ . Let  $A = 0.903446597884\dots$ . Then for  $0 < s \ll 1$

$$b(s) = \ln \frac{r}{D} - [A + o(1)]\sqrt{s}, \quad \dot{b}(s) = -\frac{A + o(1)}{2\sqrt{s}}, \quad \ddot{b}(s) = \frac{A + o(1)}{4s^{3/2}}$$

[Rigorous proof of asymptotic analysis result of P. Wilmott, J. Dewynne & S. Howison, "The Mathematics of Financial Derivatives", p121]

# Far-From-Expiry Estimates

---

$(p^*, x^*)$  the Merton solution for the infinite horizon problem.

**Theorem.** (Theorem 1, X. Chen, H. Cheng & J. Chадам, to appear on PAMS) There exists a constant  $m > 0$  such that for  $s \geq 1$  and  $\beta = k + \alpha^2/4$  (recall,  $\alpha = k - l - 1$ ),

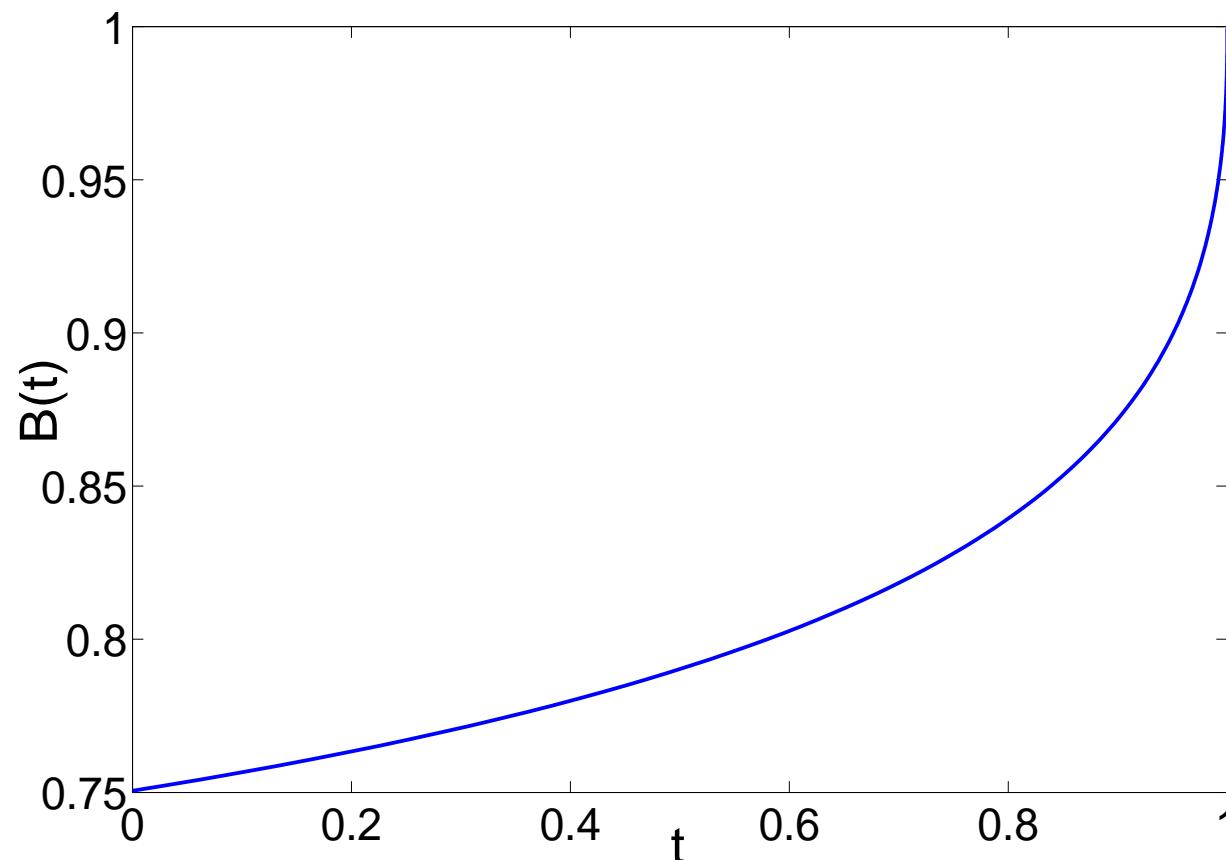
$$b(s) = x^* + [m + O(1)s^{-1/2}]s^{-3/2} e^{-\beta s},$$
$$\dot{b}(s) = -[m\beta + O(1)]s^{-1/2}s^{-3/2} e^{-\beta s}.$$

- generalizes the  $D = 0$  version sketched in [SIAM Math. Anal. 38, 1613 (2006)] and the weaker version, (still for  $D = 0$ ,) recently announced by Ahn et al, to appear on PAMS.
- have also obtained the decay rates to the price function  $p^*$  and its asymptotic profile.

# Convexity of Boundary

**Theorem:** For  $D = 0$ , the early exercise boundary is convex; i.e.,  $B''(t) > 0$ .

[X. Chen, J. Chadam, L. Jiang & W. Zheng, Math. Finance, 18, (2008); E. Ekstrom, J. Math. Anal. Appl., 299, (2004)]



Numerical evidence suggests that convexity is lost when  $D > r$ .  
(J. Detemple, P., Duck, G. Meyer, etc.)

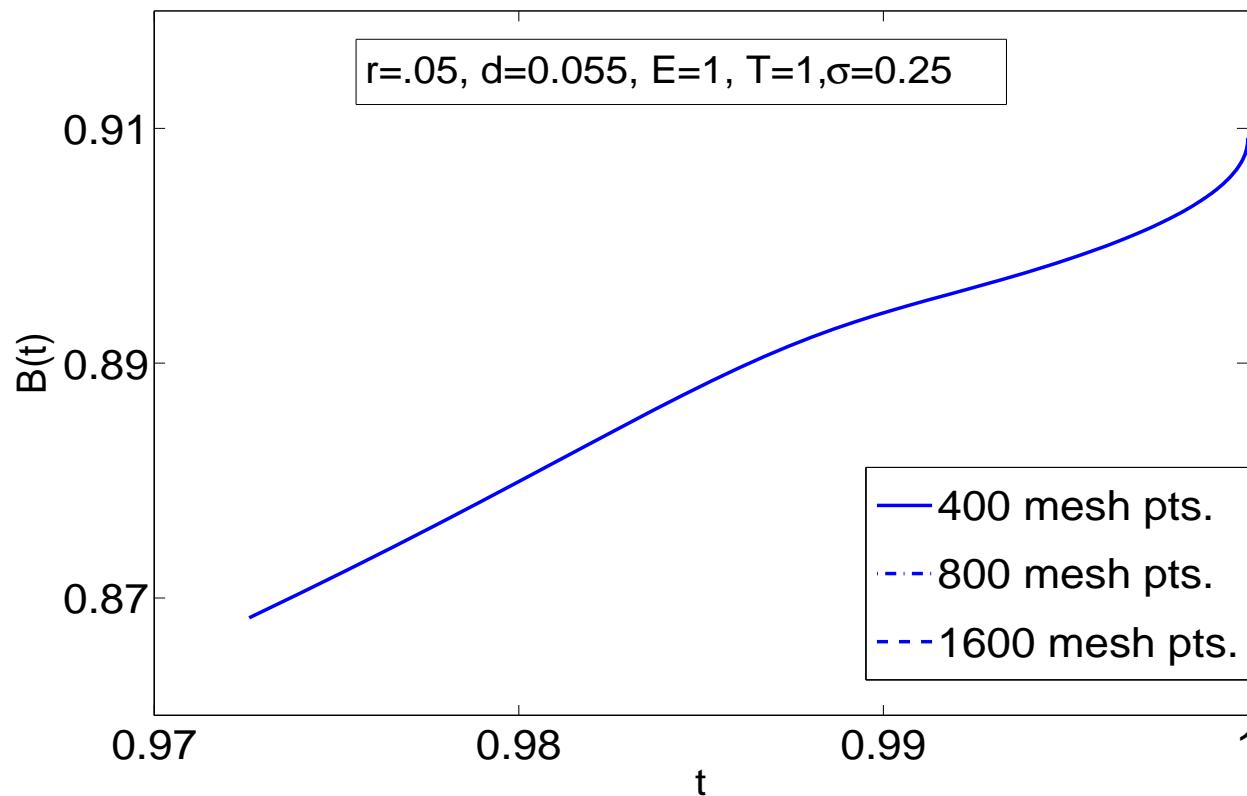


Figure 3:

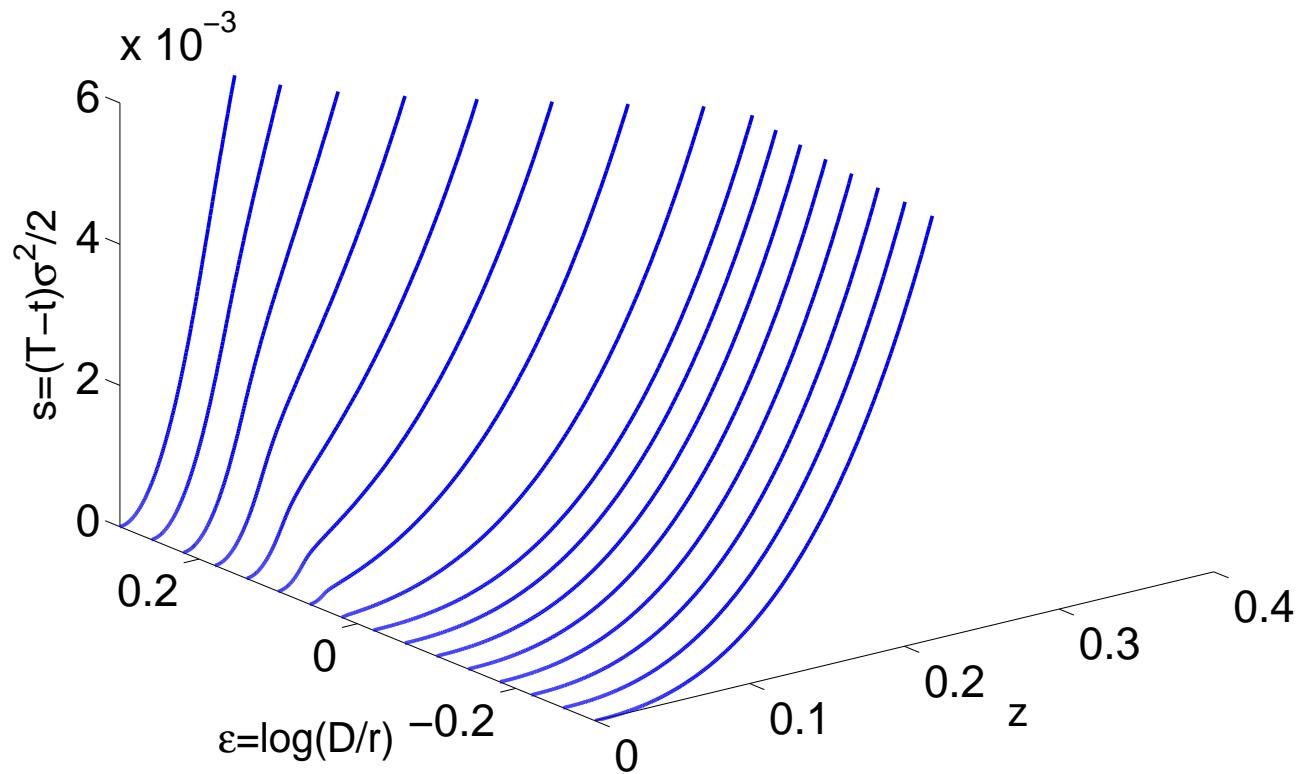


Figure 4:

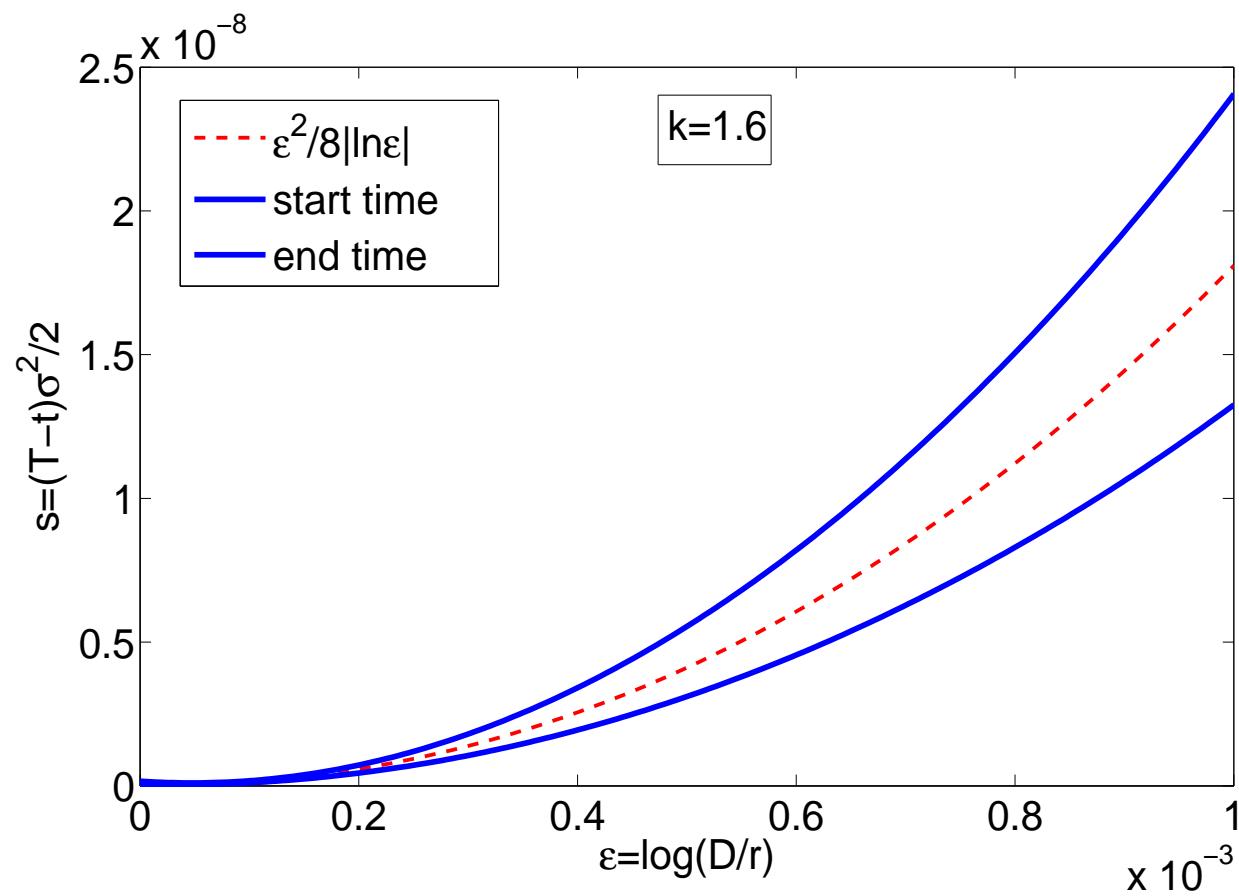


Figure 5:

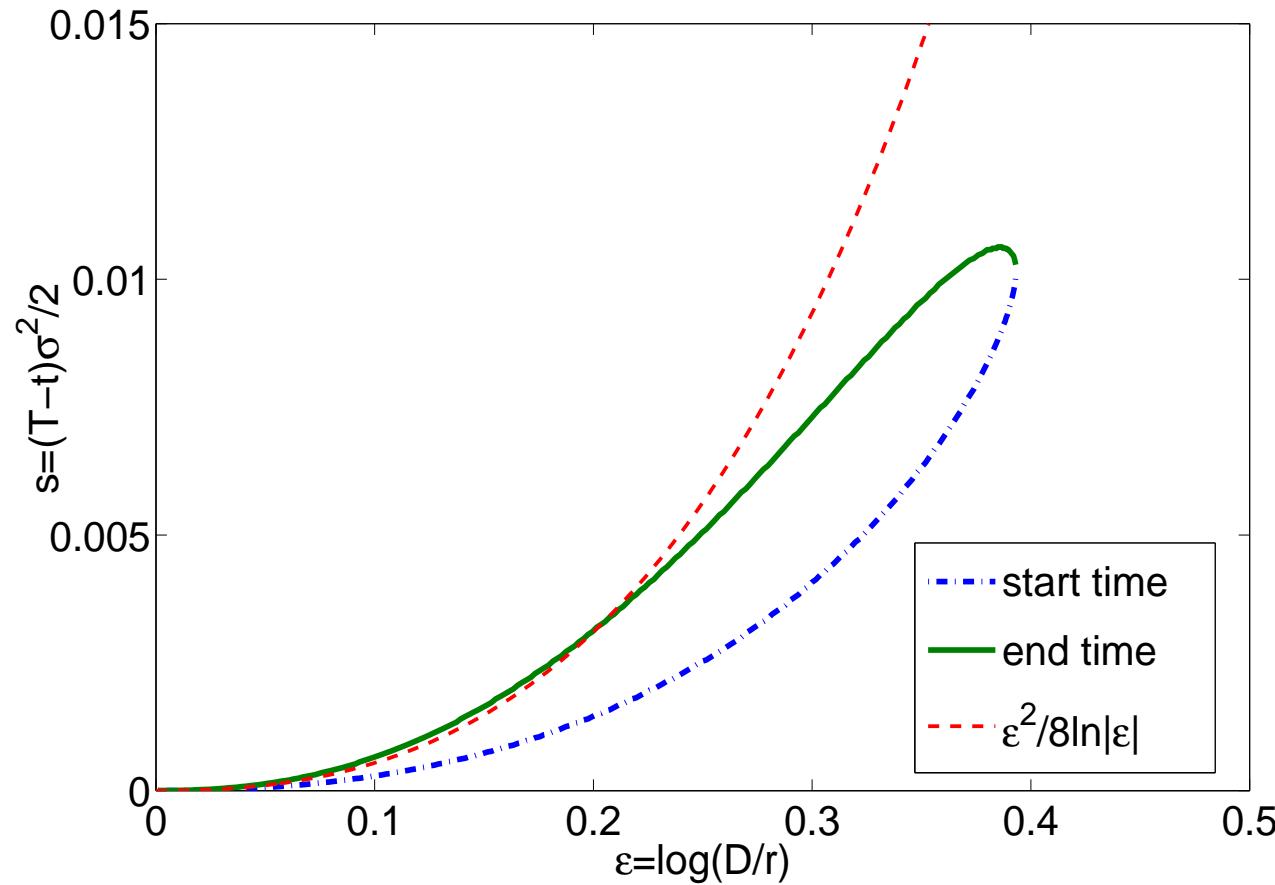


Figure 6:

---

**Main Theorem:** When  $0 < D - r \ll 1$ , the optimal exercise boundary is not convex. More precisely, when  $\varepsilon := \ln(D/r) = \ln(\ell/k)$  is positive and sufficiently small, neither  $S = B(t)$  nor  $x(:= \ln(S/E)) = b(s)$  is convex. In particular, there exist a  $\hat{t}$  for which  $B''(\hat{t}) < 0$  and (hence)  $\ddot{b}(\hat{s}) < 0$ , where

$$0 < \hat{s} \leq \frac{\varepsilon^2}{6|\ln \varepsilon|} \quad \text{and} \quad \hat{t} = T - \frac{2\hat{s}}{\sigma^2}.$$

$$B''(t) = \frac{E b^{b(s)} \sigma^4}{4} [\ddot{b}(s) + (\dot{b}(s))^2]$$

Xinfu Chen, Huibin Cheng & John Chadam, submitted for publication, Math. Finance (2009).

# The Integro-Differential Equations

---

$$p_s = p_{xx} + \alpha p_x - kp = \mathcal{L}p, \quad b(s) < x, \quad s > 0$$

$$p(x, s) = p_0(x) = \max(1 - e^x, 0), \quad x < b(s), \quad s > 0.$$

$$\phi(x, s) = p(x, s) - p_0(x), \quad -\infty < x < \infty, \quad s > 0$$

$$\begin{aligned} \phi_s - (\phi_{xx} + \alpha \phi_x - k \phi) &= 1_+(x - b(s)) \mathcal{L}p_0(x) \\ &= 1_+(x - b(s)) [\delta(x) + (\ell e^x - k) 1_+(-x)], \quad -\infty < x < \infty, \quad s > 0 \\ \phi(x, 0) &= 0 \end{aligned}$$

---

Fundamental solution of  $\mathcal{L}$  is

$$\Gamma(x, s) := K(x + \alpha s, s) e^{-ks}, \quad K(z, t) := (4\pi t)^{-1/2} e^{-z^2/4t}.$$

and solution of PDE is

$$\phi(x, s) = \int_0^s \Gamma(x, t) dt + \int_0^s \int_{b(s-t)}^0 [\ell e^y - k] \Gamma(x - y, t) dy dt.$$

---

Compute  $\phi_x, \phi_s, \phi_{xs}$  and evaluating on  $x = b(s)$  we have for any  $\theta \in \mathbb{R}$

$$\begin{aligned}\dot{b}(s)[\ell e^{b(s)} - k] &= \left(\theta - \frac{b(s)}{s}\right) \Gamma(b(s), s) \\ &+ \int_{b_0}^0 [\ell e^y - k] \left(\theta - \frac{b(s) - y}{s}\right) \Gamma(b(s) - y, s) dy \\ &- \int_0^s \dot{b}(t)[\ell e^{b(t)} - k] \left(\theta - \frac{b(s) - b(t)}{s - t}\right) \Gamma(b(s) - b(t), s - t) dt\end{aligned}$$

---

With  $\theta = 0$  and  $\theta = [b(s) - b_0]/s$  we have

$$\begin{aligned}\dot{b}(s)[\ell e^{b(s)} - k] &= \frac{-b(s)\Gamma(b(s), s)}{s} \\ &\quad + \frac{1}{s} \int_{b_0}^0 [\ell e^y - k][y - b(s)]\Gamma(b(s) - y, s)dy \\ &\quad + \int_0^s \dot{b}(t)[e^{b(t)-b_0} - 1] \frac{b(s) - b(t)}{s - t} \Gamma(b(s) - b(t), s - t) dt, \\ \dot{b}(s)[\ell e^{b(s)} - k] &= \frac{-b_0\Gamma(b(s), s)}{s} \\ &\quad + \frac{1}{s} \int_{b_0}^0 [\ell e^y - k][y - b_0]\Gamma(b(s) - y, s)dy \\ &\quad - \int_0^s \dot{b}(t)[\ell e^{b(t)} - k] \left( \frac{b(s) - b_0}{s} - \frac{b(s) - b(t)}{s - t} \right) \Gamma(b(s) - b(t), s - t) dt.\end{aligned}$$

Remark: In the numerics, Huibin Cheng takes  $\theta = (b(s) - b_0)/(2s)$  which removes singularities in integrals.

---

Now assume  $\ell > k$  ( $\Leftrightarrow D > r$ )

$$z(s) = b_0 - b(s) = \ln\left(\frac{k}{l}\right) - b(s) = -\varepsilon - b(s)$$

$\dot{z}(s) = -\dot{b}(s) > 0$  for all  $s > 0$   
 $\Rightarrow z = z(s)$  has inverse,  $s = s(z)$ .

$$\frac{4B''(t)}{E\sigma^4} = e^{b(s)}[\ddot{b}(s) + (\dot{b}(s))^2] = e^{-2z}(\dot{z}(s))^3 \frac{d}{dz}\left(e^z \frac{ds}{dz}\right)$$

Thus  $B$  not convex (i.e.,  $B''(t) \geq 0$  not true)  
 $\Leftarrow$  (sufficient to show)  $e^z \frac{ds}{dz}$  not increasing somewhere.

---

## Will use the IDEs

$$[1 - e^{-z}] \frac{dz}{ds} = I_1 + I_2 - I_3 = J_1 + J_2 + J_3$$

$$I_1 := \frac{(\varepsilon + z)\Gamma(-\varepsilon - z, s)}{k s},$$

$$I_2 := \frac{1}{s} \int_0^\varepsilon [e^y - 1][z + y]\Gamma(-y - z, s)dy,$$

$$I_3 := \int_0^z [1 - e^{-y}] \left( \frac{z - y}{s(z) - s(y)} \right) \Gamma(y - z, s(z) - s(y)) dy,$$

$$J_1 := \frac{\varepsilon \Gamma(-\varepsilon - z, s)}{k s},$$

$$J_2 := \frac{1}{s} \int_0^\varepsilon y[e^y - 1]\Gamma(-z - y, s)dy,$$

$$J_3 := \int_0^z [1 - e^{-y}] \left( \frac{z}{s(z)} - \frac{z - y}{s(z) - s(y)} \right) \Gamma(y - z, s(z) - s(y)) dy.$$

# Proof of Non-Convexity (Sketch)

---

$$s_1 = \frac{\varepsilon^2}{(8+2)|\ln \varepsilon|}, \quad s_2 = \frac{\varepsilon^2}{(8-2)|\ln \varepsilon|}, \quad z_i = z(s_i)$$

$$s^* = \sup\{s > 0 | \ddot{b}(s) + (\dot{b}(s))^2 \geq 0 \text{ in } (0, z)\}, \quad z^* = z(s^*)$$

i.e., boundary convex iff  $s^* = \infty$ .  $s_2^* = \min\{s_2, s^*\}$

$$\frac{d}{dz} \left( e^z \frac{ds}{dz} \right) \geq 0 \text{ for } z \in (0, z^*)$$

Integrating over  $[x, x+h] \subset (0, z^*) \Rightarrow$

$$s'(x+h) \geq e^{-h} s'(x) \quad \forall 0 < x < x+h \leq z^*.$$

$\Rightarrow$  (by contradiction)  $z_2^* \leq 2z_1$ .

---

## (i) Lower Bound for $ds/dz$ .

$$(1 - e^{-z}) \frac{dz}{ds} \leq I_1 + I_2, \quad \frac{ds}{dz} \geq \frac{1 - e^{-z}}{I_1 + I_2}.$$

$$I_2 \leq \frac{1}{2}, \quad \forall s > 0$$

$$I_1 \leq \frac{\varepsilon e^{-\varepsilon^2/(4s)}}{\sqrt{4\pi k^2 s^3}}, \quad s \in (0, \varepsilon^2/2]$$

In  $(0, s_1] \subset (0, \varepsilon^2/2]$ ,  $I_1 = o(1)$  so that

$$\frac{ds}{dz} \geq \frac{ze^{-z}}{o(1) + \frac{1}{2}} \geq \frac{z}{2}, \text{ for } z \in (0, z_1]$$

---

Then for  $z \in [z_1, z_2^*] \subset [z_1, 2z_1]$ ,

$$\frac{ds(z)}{dz} \geq e^{-z_2^*} \frac{ds(z_1)}{dz} \geq \frac{e^{-z_2^*} z_1 e^{-z_1}}{o(1) + \frac{1}{2}} \geq \frac{z}{2}.$$

$$\Rightarrow \frac{ds(z)}{dz} \geq \frac{z}{2}, \quad s(z) \geq \frac{z^2}{4},$$

$$\text{and } s(z) - s(y) \geq \frac{z^2 - y^2}{4} \quad \forall 0 \leq y \leq z \leq z_2^*.$$

---

## (ii) Upper Bounds for $ds/dz$

$$J_1 \geq \varepsilon^{1/4} \quad \forall z \in (0, z_2^*]$$

$$J_2 \geq \frac{1}{4} \int_2^3 e^{-\eta^2/4} d\eta := c > 0.$$

$$J_3 \geq -\frac{\sqrt{\pi} z^3 e^z}{8 s(z)} \geq -z \quad \forall z \in (0, z_2^*].$$

$$\Rightarrow J_2 + J_3 \geq c - z > 0 \quad \forall s \in (0, s_2^*] \quad .$$

---

### (iii) Completion of proof

Suppose  $z^* > z_2$ . Then  $z_2^* = z_2$  and  $s_2^* = s_2$ .

$$\frac{1}{z} \frac{ds(z)}{dz} \Big|_{z=z_2} = \frac{1 - e^{-z}}{z} \frac{1}{J_1 + J_2 + J_3} \leq \frac{1}{J_1} \leq \varepsilon^{1/4}.$$

---

$$\begin{aligned} e^{z_2} \frac{ds(z_2)}{dz} - e^{z_1} \frac{ds(z_1)}{dz} &\leq z_2 e^{z_2} \varepsilon^{1/4} - \frac{1}{2} e^{z_1} z_1 \\ &\leq 2z_1 e^{2z_1} \varepsilon^{1/4} - \frac{e^{z_1} z_1}{2} < 0. \end{aligned}$$

i.e.  $e^z ds/dz$  is not an increasing function on  $[z_1, z_2] \Rightarrow z^* < z_2$ .

**OPEN PROBLEM:** Convexity for  $0 < D \leq r \ \& \ D \geq r^*(\sigma)$ .