Rigorous numerics for homoclinic tangencies

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Homoclinic tangencies

$I, J, Z \subset \mathbb{R}$ – intervals,
$u_\mu \colon I \to \mathbb{R}^2$, $s_\mu \colon J \to \mathbb{R}^2$ for $\mu \in Z$, smooth also wrt to μ,

$$\begin{aligned} u_{\mu_0}(t_u) &= s_{\mu_0}(t_s) = q_0, \\ u'_{\mu_0}(t_u) &= \text{const } s'_{\mu_0}(t_s) - \text{ and nonzero.} \end{aligned}$$

Definition

If there exist μ-dependent smooth coordinates in a neighborhood of q for μ close to μ_0, such that in these coordinates

$$\begin{aligned} s_\mu(\tau) &= (\tau, 0), \\ u_\mu(\tau) &= (\tau, a\tau^2 + b(\mu - \mu_0)) \end{aligned}$$

where $a \neq 0, b \neq 0$ then we say that **the quadratic tangency of u and s unfolds generically**.

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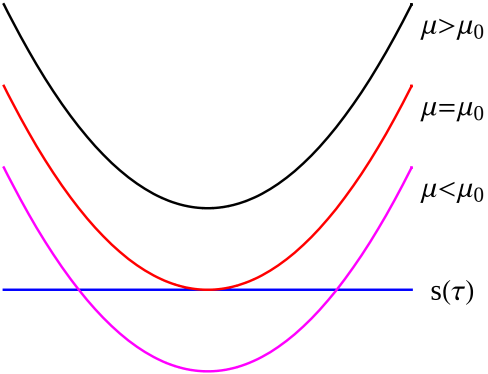

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$$

Newhouse intervals

Theorem (Newhouse)

Assume

- $\{f_\lambda\}_{\lambda\in\Lambda}$ - C^3 *diff of the plane*
- *there is a curve of fixed points* x_λ *with* $|\det Df(x_\lambda)| < 1$
- f_{λ_0} *admits* **quadratic** *homoclinic tangency*

Then for every $\varepsilon > 0$ *there is an interval* $I \subset [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$ *with a dense subset* J *such that for* $\lambda \in J$ f_λ *has generic homoclinic tangency.*

Some further consequences

Dissipative case

- Gavrilov, Shilnikov - sinks of unbounded periods accumulated to tangency
- Newhouse, Robinson - infinitely many coexisting sinks

Conservative case

- Duarte and Gonchenko, Shilnikov - infinitely many coexisting elliptic points
- Gorodetski, Kaloshin - locally maximal invariant hyperbolic sets of Hausdorff dimension arbitrary close to 2

Newhouse intervals computed numerically

Conjecture:
I. Kan, H. Koçak, J. Yorke, Physica D (1995)
Consider the Hénon map

$$f_\lambda \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda - x^2 + by \\ x \end{pmatrix}$$

The following intervals $[1.2702, 1.299]$, $[1.3087, 1.3233]$, $[1.3238, 1.42]$ are Newhouse intervals for f_λ.

Homoclinic tangencies

J. E. Fornaess and E. A. Gavosto,
Existence of Generic Homoclinic Tangencies for Hénon mappings,
Journal of Geometric Analysis, 2 (1992), 429–444.

J. E. Fornaess and E. A. Gavosto,
Tangencies for real and complex Hénon maps: an analytic method,
Experiment. Math., 8 (1999), 253–260.

- high order Taylor expansion of invariant manifolds
- hand made computations for certain parameter values of the Hénon map
- interval arithmetics with rational endpoints for other parameter values

Conservative case

A. Gorodetski, V. Kaloshin,
Conservative homoclinic bifurcations and some applications,
to appear in Steklov Institute Proceedings, volume dedicated to the 70th anniversary of Vladimir Arnold.

- homoclinic tangencies for suitable Poincaré map for the PCR3BP

Homoclinic tangencies

Z. Arai, K. Mischaikow, *Rigorous computations of homoclinic tangencies*, SIAM J. App. Dyn. Sys. **5** (2006), 280–292.

Theorem

There exist parameter values close to (estimation is given)

$$a \approx 1.392,\ \ b = 0.3$$
and
$$a \approx 1.314,\ \ b = -0.3$$

such that the Hénon map

$$H_{a,b}(x,y) = (a - x^2 + by, x)$$

has a quadratic homoclinic tangency which unfolds generically for the fixed points in first and third quadruple, respectively.

Time of computations: 240 and 100 minutes on the PowerMac G5 2GHz.

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Computational result for the Hénon map:

Theorem

There exists an open neighborhood B of the parameter value $b = -0.3$ such that for each $b \in B$ there is a parameter

$$a \in 1.3145271093265 + [-10^{-5}, 10^{-5}]$$

such that the Hénon map

$$H_{a,b}(x, y) = (a - x^2 + by, x)$$

has a quadratic homoclinic tangency unfolding generically for the fixed point

$$x_{a,b} = y_{a,b} = \frac{1}{2}\left(b - \sqrt{(b-1)^2 + 4a} - 1\right)$$

Time of computations: 0.2sec on the Intel Xeon 5160, 3GHz.

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Computational result for the forced-damped pendulum:

Pendulum equation

$$\ddot{x} + \beta\dot{x} + \sin(x) = \cos(t).$$

Poincaré map

$$T_\beta(x, \dot{x}) = (x(2\pi), \dot{x}(2\pi)).$$

Theorem

For all parameter values

$$\beta \in \mathcal{B} = 0.247133729485 + [-1, 1] \cdot 1.2 \cdot 10^{-10}$$

there exists a hyperbolic fixed point for T_β. Moreover, there exists a parameter value $\beta \in \mathcal{B}$ such that the map T_β has a quadratic homoclinic tangency unfolding generically for that fixed point.

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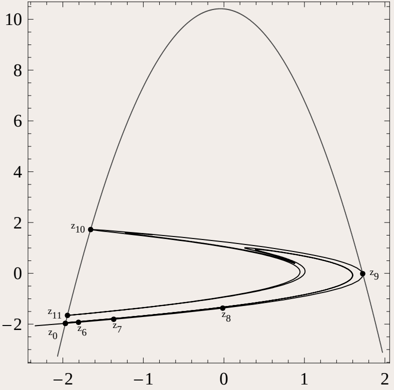
Numerical simulations.

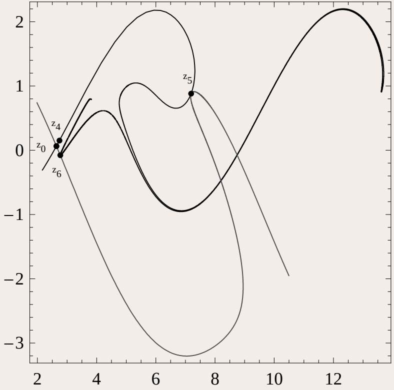
The Hénon map



Eigenvalues

$$\lambda \approx 3.858, \quad \mu \approx 0.0777$$

The pendulum equation



Eigenvalues

$$\lambda \approx 211.83, \quad \mu \approx 0.000999$$

Let $f_a : \mathbf{R}^2 \to \mathbf{R}^2$ be a C^2 wrt to both arguments and parameter.

Following the paper by Arai and Mischaikow - consider projectivization of the map $Pf : \mathbf{R}^2 \times S^1 \times \mathbf{R} \to \mathbf{R}^2 \times S^1 \times \mathbf{R}$

$$\begin{aligned} Pf(p,[u],a) &= (f_a(p),[Df_a(p)\cdot u],a) \\ Pf_a(p,[u]) &= (f_a(p),[Df_a(p)\cdot u]) \end{aligned}$$

Observations: p – hyperbolic fixed point for f_a with real eigenvalues and eigenvectors u (unstable), s (stable)

$(p,[u],a)$ is a fixed point for Pf with

- two-dimensional stable manifold
- one-dimensional unstable manifold
- one-dimensional center manifold

$(p,[s],a)$ is a fixed point for Pf with

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Main theorem

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p - hyperbolic fixed point for f_a
'unstable' and 'stable' eigenvectors u, s.

If the map Pf has **transversal heteroclinic connection** *between*

$$(p, [u], a) \quad \textit{and} \quad (p, [s], a)$$

then the map f admits quadratic homoclinic tangency for parameter a which unfolds generically.

Remark

This means that the two-dimensional surfaces:

- center-unstable manifold at $(p, [u], a)$
- center-stable manifold at $(p, [s], a)$

must intersect transversally.

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Arai-Mischaikow approach

Tools:

- the Conley index as a general method to prove connecting orbit
- apply it to the projectivization of the map to prove the existence of tangency
- apply it to the projectivization of projectivization to prove generic unfolding - **second order derivatives required, given explicitly for the Hénon map**

Computational tools:

- CHOMP - Computational Homology Project
- GAIO project

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Our approach

Tools:

- The covering relations to prove connecting orbit
- The cone conditions to estimate center-unstable and center-stable manifolds
- The cone conditions to prove their transversal intersection

Computational tools:

- CAPD library for the interval arithmetics and linear algebra package, **http://capd.ii.uj.edu.pl**
- CAPD library for the C^2-Lohner algorithm to integrate the pendulum equation and its variational equations

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Description of the method

Step 1 Parameterize the center-unstable manifold at $(p, [u], a)$ as a horizontal disc satisfying the cone conditions

Step 2 Parameterize the center-stable manifold at $(p, [s], a)$ as a vertical disc satisfying the cone conditions

Step 3 Construct a 'heteroclinic' chain of covering relations for Pf between the points $(p, [u], a)$ and $(p, [s], a)$

Step 4 Verify the cone conditions along this chain of covering relations

The above imply the existence of **transversal heteroclinic orbit** between $(p, [u], a)$ and $(p, [s], a)$.

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Covering relations and cone conditions

Definition (Gidea, Zgliczyński 2003)

***h*-set** N is an object consisting of

- $|N|$ - compact subset of $\mathbb{R}^n$ (called **support**)
- $u(N), s(N) \in \{0, 1, 2, \ldots\}$,
  such that $u(N) + s(N) = n$
- a homeomorphism $c_N : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$ such that

$$c_N(|N|) = \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1).$$

$$\dim(N) = n,$$
$$N_c = \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1),$$
$$N_c^- = \partial\overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1)$$
$$N_c^+ = \overline{B_{u(N)}}(0,1) \times \partial\overline{B_{s(N)}}(0,1)$$
$$N^- = c_N^{-1}(N_c^-), \quad N^+ = c_N^{-1}(N_c^+)$$

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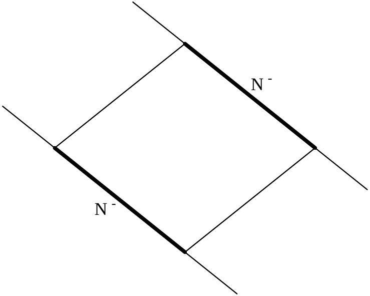

N, M h-sets with $u(N) = u(M) = u$

$f : |N| \to \mathbb{R}^n$ – continuous, $f_c = c_M \circ f \circ c_N^{-1} : N_c \to \mathbb{R}^u \times \mathbb{R}^{s(M)}$.

Let w be a nonzero integer.

Definition (Gidea, Zgliczyński 2003)

N f-covers M with degree w ($N \overset{f,w}{\Longrightarrow} M$) iff

1. There exists $h : [0,1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^{s(M)}$ such that

$$\begin{aligned} h(0,\cdot) &= f_c, \\ h([0,1], N_c^-) \cap M_c &= \emptyset, \\ h([0,1], N_c) \cap M_c^+ &= \emptyset. \end{aligned}$$

2. There exists a map $A : \mathbb{R}^u \to \mathbb{R}^u$ such that

$$\begin{aligned} h_1(p,q) = (A(p), 0), \text{for } p \in \overline{B_u}(0,1), q \in \overline{B_{s(N)}}(0,1), \\ A(\partial B_u(0,1)) \subset \mathbb{R}^u \setminus \overline{B_u}(0,1), \\ \deg(A, \overline{B_u}(0,1), 0) = w. \end{aligned}$$

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N f-covers M with degree w ($N \overset{f,w}{\Longrightarrow} M$) iff

1. There exists $h : [0,1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^{s(M)}$ such that

$$\begin{aligned} h(0,\cdot) &= f_c, \\ h([0,1], N_c^-) \cap M_c &= \emptyset, \\ h([0,1], N_c) \cap M_c^+ &= \emptyset. \end{aligned}$$

2. There exists a map $A : \mathbb{R}^u \to \mathbb{R}^u$ such that

$$\begin{aligned} h_1(p,q) = (A(p), 0), \text{for } p \in \overline{B_u}(0,1), q \in \overline{B_{s(N)}}(0,1), \\ A(\partial B_u(0,1)) \subset \mathbb{R}^u \setminus \overline{B_u}(0,1), \\ \deg(A, \overline{B_u}(0,1), 0) = w. \end{aligned}$$

N, M h-sets with $u(N) = u(M) = u$
$f : |N| \to \mathbb{R}^n$ – continuous, $f_c = c_M \circ f \circ c_N^{-1} : N_c \to \mathbb{R}^u \times \mathbb{R}^{s(M)}$.
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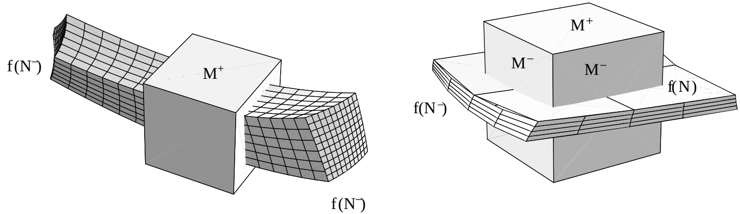
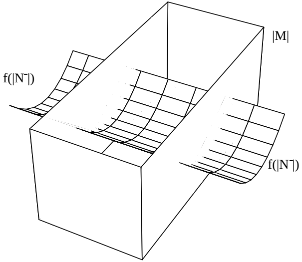


$N \overset{f,1}{\Longrightarrow} M$, where (left) $u(N) = 1$ and (right)



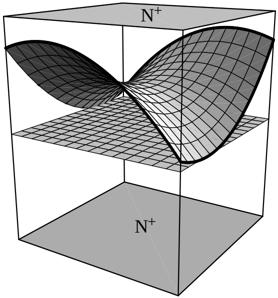
an example $N \overset{f,1}{\Longrightarrow} M$, where $s(N) = 1$, $s(M) = 2$

N h-set, $b : \overline{B_{u(N)}} \to |N|$ continuous.
Put $b_c = c_N \circ b$.

Definition

We say that b is **a horizontal disc in N** if there exists a homotopy $h : [0,1] \times \overline{B_{u(N)}} \to N_c$ such that

$$\begin{aligned} h_0 &= b_c \\ h_1(x) &= (x, 0), \quad \text{for all } x \in \overline{B_{u(N)}} \\ h(t,x) &\in N_c^-, \quad \text{for all } t \in [0,1] \text{ and } x \in \partial \overline{B_{u(N)}} \end{aligned}$$



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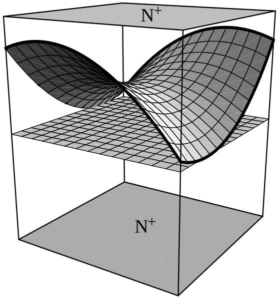
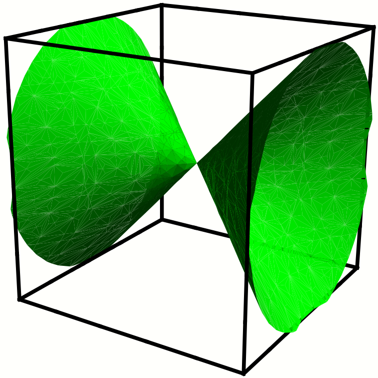


Definition

$N \subset \mathbb{R}^n$ be an h-set and $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form

$$Q(x, y) = \alpha(x) - \beta(y), \qquad (x, y) \in N_c \subset \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)},$$

where $\alpha : \mathbb{R}^{u(N)} \to \mathbb{R}$, and $\beta : \mathbb{R}^{s(N)} \to \mathbb{R}$ are positive definite quadratic forms.
The pair (N, Q) will be called an ***h*-set with cones**.

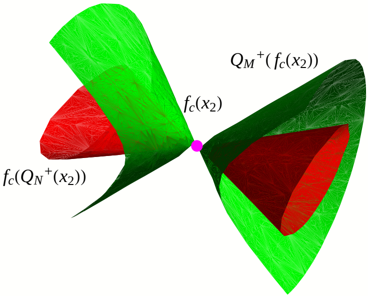


Definition

$(N, Q_N), (M, Q_M)$ are h-sets with cones, such that
$u(N) = u(M) = u$.
$f : N \to \mathbb{R}^{\dim(M)}$ and $N \overset{f}{\Longrightarrow} M$.
We say that f **satisfies the cone condition** (with respect to the pair (N, M)), if any $x_1, x_2 \in N_c$ with $x_1 \neq x_2$ satisfy

$$Q_M(f_c(x_1) - f_c(x_2)) > Q_N(x_1 - x_2).$$

Here $Q_M^+(x_2) = \{x : Q_M(x - x_2) > 0\}$.



Remark

This condition is computable in interval arithmetics. It is enough to verify if the symmetric interval matrix

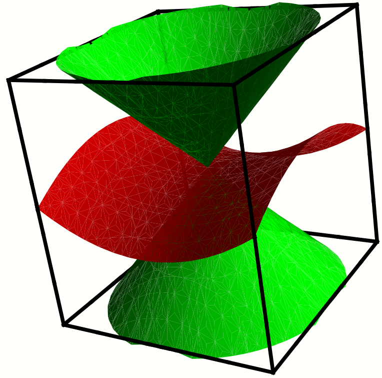
$$[Df_c(N_c)]_I^T Q_M [Df_c(N_c)]_I - Q_N$$

is positive definite.

Definition

(N, Q) - h-set with cones.
$b : \overline{B_u} \to |N|$ - a horizontal disk.
We will say that b **satisfies the cone condition**, if any $x_1, x_2 \in \overline{B_u}$ with $x_1 \neq x_2$ satisfy

$$Q(b_c(x_1) - b_c(x_2)) > 0.$$



Estimation of the center-stable manifold

- (N, Q) – h-set with cones
- Q has the form
  $Q(x, y) = \alpha(x) - \beta(y) = \sum_{i=1}^{u} a_i x_i^2 - \sum_{i=1}^{s} a_{i+u} y_i^2$.
- C - compact interval
- $f_\lambda : N \to \mathbf{R}^n$, $\lambda \in C$ smooth also wrt λ
- Define

$$
\begin{aligned}
M &= \max_{\lambda \in C, z \in N} \left(\sum_i |a_i| \left\| \frac{\partial \pi_{z_i} f_\lambda}{\partial z}(z) \right\| \cdot \left\| \frac{\partial \pi_{z_i} f_\lambda}{\partial \lambda}(z) \right\| \right), \\
L &= \|\beta\| \cdot \max_{\lambda \in C, z \in N} \left\| \frac{\partial \pi_y f_\lambda}{\partial \lambda}(z) \right\|^2 .
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Estimation of the center-stable manifold

Theorem

- $N \overset{f_\lambda}{\Longrightarrow} N$ *for* $\lambda \in C$ *and the cone conditions are satisfied,*
- *Choose* $\epsilon > 0$, $A > 0$ *such that for all* $\lambda \in C$, $z_1, z_2 \in N$

$$Q(f_\lambda(z_1) - f_\lambda(z_2)) - (1+\epsilon)Q(z_1 - z_2) \geq A(z_1 - z_2)^2.$$

- *Choose* $\Gamma > 0$ *such that*

$$A - 2M\Gamma - L\Gamma^2 > 0.$$

- *Put*

$$\delta = \frac{\Gamma^2}{\|\alpha\|}.$$

Then the set $W^s_N(p_\lambda, f_\lambda)$ *for* $\lambda \in C$ *can be parameterized as a vertical disk in* $C \times N$ *satisfying the cone condition for the quadratic form* $\tilde{Q}(\lambda, z) = \delta Q(z) - \lambda^2$.

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Transversal intersection of manifolds

Theorem

Assume that

- $$N_0 \overset{f_0}{\Longrightarrow} N_1 \overset{f_1}{\Longrightarrow} N_2 \overset{f_2}{\Longrightarrow} \cdots \overset{f_{k-1}}{\Longrightarrow} N_k,$$

  and for each covering relation the cone conditions are satisfied.
- $b : \overline{B_{s(N_k)}} \to |N_k|$ *- a vertical disc in* N_k *satisfying the cone condition.*

Then there exists a vertical disc $b_0 : \overline{B_{s(N_0)}} \to |N_0|$ *which satisfies the cone condition and such that for all* $y \in \overline{B_{s(N_0)}}$ *there holds*

$$\begin{aligned} f_{i-1} \circ f_{i-2} \circ \cdots \circ f_0(b_0(y)) &\in N_i, \qquad \text{for } i = 1, \dots, k \\ f_{k-1} \circ \cdots \circ f_0(b_0(y)) &= b_k(y_1), \qquad \text{for some } y_1 \in \overline{B_{s(N_k)}} \end{aligned}$$

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How to construct the sets?

a_0 - approximate 'tangency' parameter
p - approximate fixed point for f_{a_0}
u, s - approximate eigenvectors of $Df_{a_0}(p)$

We have to construct the chain of covering relations between $(p, [u], a)$ and $(p, [s], a)$

$$N_0 \overset{Pf}{\Longrightarrow} \cdots \overset{Pf}{\Longrightarrow} N_k \overset{\mathbf{Pf}}{\Longrightarrow} M_s \overset{Pf}{\Longrightarrow} \cdots \overset{Pf}{\Longrightarrow} M_0$$

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Key observations:

- At the beginning of the sequence the sets N_i have **two stable** directions. Therefore we must use the **parameter** as an '**unstable**' direction. This can be achieved by **decreasing** the range of parameters along the sequence of N_i's.

- At the end of the sequence the sets M_i have **two unstable** directions. Hence, the parameter must be used as a 'stable' direction. This can be achieved by **increasing** the range of parameters along the sequence of M_i's.

- In the switch between N_k and M_s we change the role of the parameter.

  parameter coord in N_k **'covers'** unstable coord in M_s

  unstable coord in N_k **'covers'** tangent coord in M_s

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i	$10^5 \cdot (d_i)_1$ unstable dir.	$10^5 \cdot (d_i)_2$ stable dir.	$10^5 \cdot (d_i)_3$ tangent dir.	$10^5 \cdot (d_i)_4$ parameter
0	7	1	2	$(1.01)^8$
1	1	1	2	$(1.01)^7$
2	1	1	2	$(1.01)^6$
3	1	1	2	$(1.01)^5$
4	1	1	2	$(1.01)^4$
5	1	1	2	$(1.01)^3$
6	1	1	2	$(1.01)^2$
7	1	1	2	1.01
8	1	1	2	1
9	0.5	1.25	0.25	1.01
10	0.75	1.25	0.25	$(1.01)^2$
11	1	1.25	0.25	$(1.01)^3$
12	1	1.25	0.25	$(1.01)^4$
13	1	1.25	0.25	$(1.01)^5$
14	1	1.25	0.25	$(1.01)^6$
15	1	2	0.25	$(1.01)^7$

i	$(p_i)_1$ unstable dir.	$(p_i)_2$ stable dir.	$(p_i)_3$ tangent dir.	$(p_i)_4$ parameter
0	$3/\lambda^2$	$-\mu^2$	$-(\mu/\lambda)^2$	$2(1.5)^{-8}$
1	$1/\lambda^2$	-0.1	-0.5	$2(1.5)^{-7}$
2	$1/\lambda^2$	-0.1	-1	$2(1.5)^{-6}$
3	$1/\lambda^2$	-0.1	-1	$2(1.5)^{-5}$
4	$1/\lambda^2$	-0.1	-1	$2(1.5)^{-4}$
5	$1/\lambda^2$	-0.1	-1	$2(1.5)^{-3}$
6	$1/\lambda^2$	-0.1	-1	$2(1.5)^{-2}$
7	$1/\lambda^2$	-0.1	-1	$2(1.5)^{-1}$
8	$0.5/\lambda^2$	-1	-1	2
9	$100/\lambda^2$	-0.1	$100(\mu/\lambda)^2$	-2
10	$40/\lambda^2$	-0.1	$(\mu/\lambda)^2$	$-2(1.5)^{-1}$
11	$10/\lambda^2$	-0.1	$(\mu/\lambda)^2$	$-2(1.5)^{-2}$
12	$1/\lambda^2$	-0.1	$(\mu/\lambda)^2$	$-2(1.5)^{-3}$
13	$1/\lambda^2$	-0.1	$(\mu/\lambda)^2$	$-2(1.5)^{-4}$
14	$1/\lambda^2$	-0.1	$(\mu/\lambda)^2$	$-2(1.5)^{-5}$
15	$0.3/\lambda^2$	-0.1	$(\mu/\lambda)^2$	$-2(1.5)^{-6}$

Details in:

D. Wilczak, P. Zgliczyński,
Computer assisted proof of the existence of homoclinic tangency for the Hénon map and for the forced-damped pendulum,
SIAM J. App. Dyn. Sys. to appear.