

# A rigorous lower bound for the stability regions of the quadratic map

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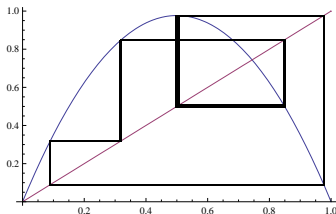
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This map is arguably the most studied object in the theory of dynamical systems, beginning with the famous article by Robert May: *Simple mathematical models with very complicated dynamics*. *Nature* **261:459**, 1976.

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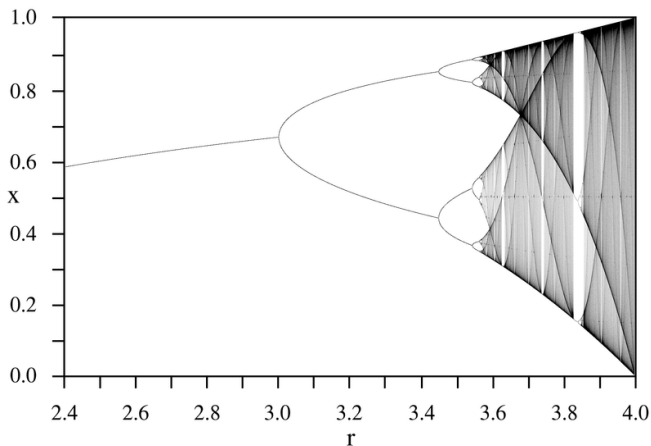
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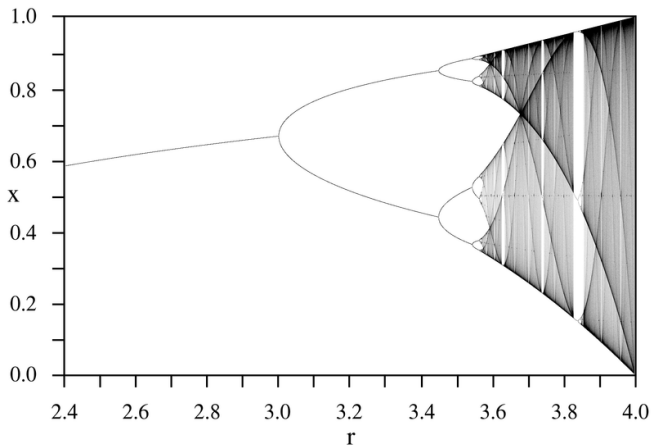
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But how large portion of the parameter space is *really* chaotic?



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In this case, the measure is unique, and almost all orbits in  $[0, 1]$  are asymptotically equidistributed with respect to it.

For convenience, let us denote the set of regular parameters by  $\mathcal{R}$ , and the stochastic parameters by  $\mathcal{S}$ .

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## Quantative questions

What are the sizes of  $\mathcal{R}$  and  $\mathcal{S}$ ?

# The measure of $\mathcal{R}$ and $\mathcal{S}$

## The stochastic parameters

The only non-trivial result is the (partial) lower bound by Luzzatto and Takahashi from 2006 (for  $1 - bx^2$ ):

$$|\mathcal{S} \cap [2 - \epsilon, 2]|/\epsilon > 0.97 \quad \text{with} \quad \epsilon = 10^{-4990}.$$

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## Theorem (Main Theorem)

*The set of regular parameters for the quadratic map satisfies the lower bound:  $|\mathcal{R} \cap [2, 4]| \geq 1.61394210853560604222$ .*



# The measure of $\mathcal{R}$ and $\mathcal{S}$

## An upper bound for $|\mathcal{S}|$

We know that also  $(0, 1) \cup (1, 2] \subset \mathcal{R}$ , so we get the upper bound

$$|\mathcal{S}| < 4 - 3.61394210853560604222 < 0.38605789146439395778$$

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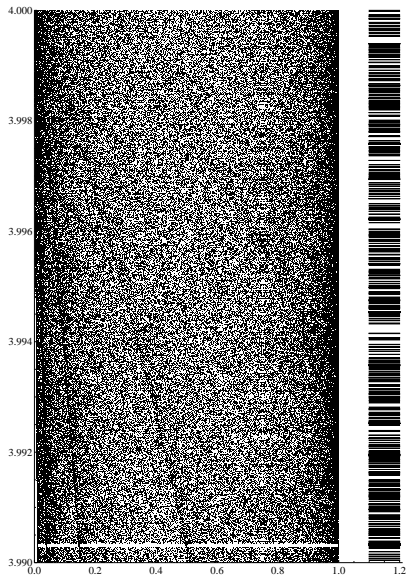
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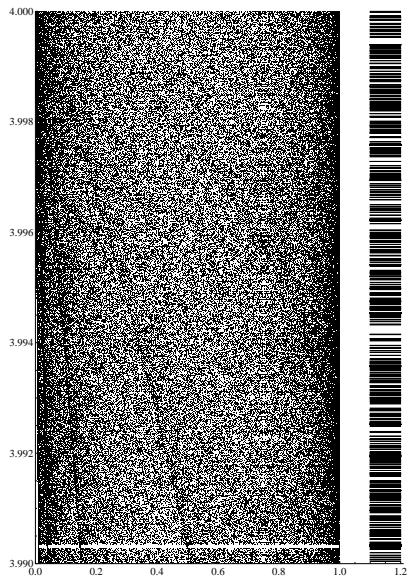
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According to non-rigorous numerical experiments (Simó and Tatjer 1991) the regular parameters in  $[a^*, 4]$  make up no more than 10.66%. The comparison is non-trivial since they use a different map:  $x \rightarrow 1 - bx^2$ .

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We also prove the existence of period doubling bifurcations.

Although  $|PD| = 0$ , we spend effort on this step.

This produces larger connected parameter sets within each period doubling cascade, and thus adds measure to our final bound.

# Strategy of the proof

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The three methods are increasing in computational complexity, and therefore, when prove the existence of a stable orbit, we first use the Brouwer theorem. If this method fails, we apply the method of backward shooting. If we still have no success, we switch to the modified interval Krawczyk method, provided the assumptions of this method are satisfied.

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- Brouwer's method is very fast, but has some disadvantages: the condition  $Q_{\mathbb{A}}^p(\mathbb{X}_1) \subset \mathbb{X}_1$  might fail due to the width of  $\mathbb{A}$ . This also forces  $\mathbb{X}_1$  to be “large”, which makes stability hard to establish close to a bifurcation parameter.

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- In general, Krawczyk's method has a cubic (in period) complexity, which is prohibitive. But using special structures from the superstable setting, we can obtain quadratic complexity only.
- Period doublings can be established/isolated by verifying some inequalities involving the derivatives of  $Q_a(x)$  and  $x - Q_a^2(x)$  (Zgliczynski and Wilczak).

# Set-valued computations

## Interval analysis

All our computations are set-valued, and are based on the *inclusion principle*:

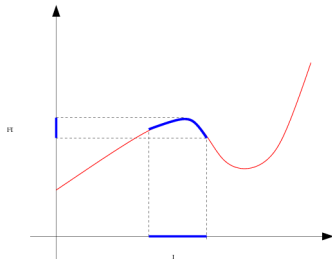
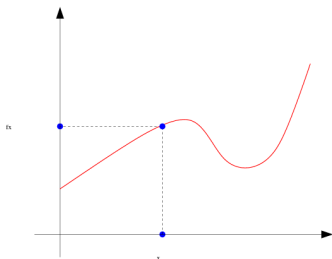
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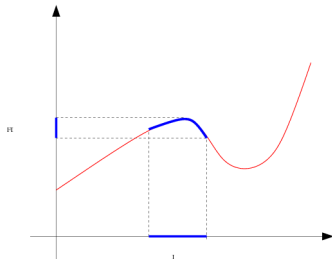
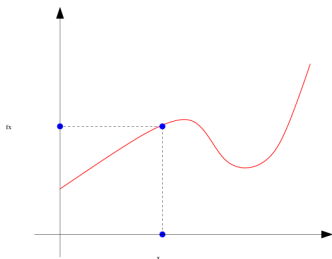


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Interval Computations Web Page

<http://www.cs.utep.edu/interval-comp>

# Backward shooting

Given a smooth function  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , an interval vector  $\mathbb{X}$ , and a point  $x \in \mathbb{X}$ , we define the *interval Newton operator* by

$$N(F, x, \mathbb{X}) = x - [DF]^{-1}(\mathbb{X})F(x). \quad (1)$$

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## Theorem

*Let  $\mathbb{X}$  be an interval vector,  $x \in \mathbb{X}$ , and  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be smooth. Assume that  $DF(\mathbb{X})$  is invertible as an interval matrix. If the interval Newton operator satisfies*

$$N(F, x, \mathbb{X}) \subset \mathbb{X}$$

*then the map  $F$  has a unique zero  $x^*$  in the box  $\mathbb{X}$ . Moreover,  $x^* \in N(F, x, \mathbb{X})$ .*

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Main idea: Trade iterations for dimension.

Instead of solving the scalar problem  $f^p(x) = x$ , we solve the  $p$ -dimensional problem  $F(x) = 0$ , where

$$F(x_1, \dots, x_p) = (x_2 - f(x_1), \dots, x_p - f(x_{p-1}), x_1 - f(x_p)). \quad (2)$$

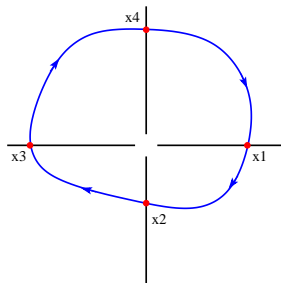
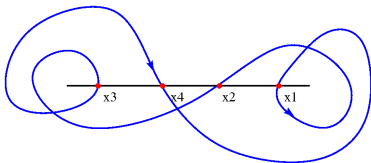


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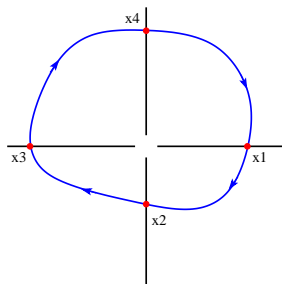
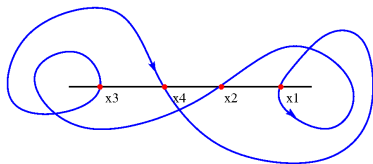


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A period- $p$  orbit of  $f$  corresponds to a zero of  $F$ . We use  $f = Q_a$ .

## Theorem (Galias'02)

Let  $\mathbb{X} = (\mathbb{X}_1, \dots, \mathbb{X}_p)$  be an interval vector,  $x = (x_1, \dots, x_p) \in \mathbb{X}$ , and let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth map. Define  $F$  as in (2). Assume that (for  $k = 1, \dots, p$ )  $\mathbb{S}_k$ ,  $\mathbb{G}_k$ , and  $\mathbb{H}_k$  are intervals such that

$$f'(\mathbb{X}_k) \subset \mathbb{S}_k \text{ and } 0 \notin \mathbb{S}_k$$

$$f(x_k) - x_{(k \bmod p)+1} \in \mathbb{G}_k$$

$$(1 - \mathbb{S}_1^{-1} \dots \mathbb{S}_p^{-1}) \sum_{i=1}^p \mathbb{S}_1^{-1} \dots \mathbb{S}_i^{-1} \mathbb{G}_i \subset \mathbb{H}_1$$

$$\mathbb{S}_k^{-1} (\mathbb{H}_{(k \bmod p)+1} + \mathbb{G}_k) \subset \mathbb{H}_k, \quad (k = 2, \dots, p)$$

Then  $[DF]^{-1}(\mathbb{X})F(x) \subset \mathbb{H}$ , and  $N(F, x, \mathbb{X}) \subset x - \mathbb{H}$ .



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$$DF(x_1, \dots, x_p) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & -Q'_a(x_2) & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -Q'_a(x_{p-1}) & 1 \\ 1 & 0 & \dots & 0 & -Q'_a(x_p) \end{bmatrix}.$$

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This means that the linear equation  $DF(x) \cdot y = z$  has the solution

$$\begin{cases} y_2 = z_1 \\ y_3 = z_2 + Q'_a(x_2)y_2 \\ \vdots \\ y_p = z_{p-1} + Q'_a(x_{p-1})y_{p-1} \\ y_1 = z_p + Q'_a(x_p)y_p \end{cases} \quad (3)$$

# Superstable orbits - Krawczyk's method

But for intervals  $\mathbb{A}$  and  $\mathbb{X}_1$ , we only have inclusion:  $0 \in Q'_{\mathbb{A}}(\mathbb{X}_1)$ .  
Thus we cannot solve explicitly for the Newton correction term,  
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For superstable orbits, we use Galias' approach applied to the interval Krawczyk operator.



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The interval Krawczyk operator is defined by

$$K(F, x, \mathbb{X}, C) = x - C \cdot F(x) + (\text{Id} - C \cdot DF(\mathbb{X}))(\mathbb{X} - x). \quad (4)$$

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## Theorem

*Let  $C$  be an invertible matrix, and let  $x \in \mathbb{X} \in \mathbb{R}^p$ . If the interval Krawczyk operator (4) satisfies*

$$K(F, x, \mathbb{X}, C) \subset \text{int } \mathbb{X}$$

*then  $F$  has a unique zero  $x^* \in \mathbb{X}$ . Moreover,  $x^* \in K(F, x, \mathbb{X}, C)$ .*

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What remains for us to show is the algorithm for the computation of  $(\text{Id} - C \cdot DF(\mathbb{X}))(\mathbb{X} - x)$ .

## Lemma

*Assume that Algorithm 1 is called with its arguments, and assume that  $s_1 = 0$ . Then the algorithm always stops and returns an interval vector  $\mathbb{Y}$  which is an enclosure for the interval vector  $(\text{Id} - C \cdot D)(\mathbb{X} - x)$ , where*

$$D = \begin{bmatrix} -s_1 & 1 & 0 & \cdots & 0 \\ 0 & -s_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -s_{p-1} & 1 \\ 1 & 0 & \cdots & 0 & -s_p \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & -s_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -s_{p-1} & 1 \\ 1 & 0 & \cdots & 0 & -s_p \end{bmatrix}^{-1}$$

---

**Algorithm 1:** Modified interval Krawczyk method ( $\mathcal{O}(p^2)$ )

---

**Data:** double  $s_1, \dots, s_p$ ;  
interval  $\mathbb{S}_1, \dots, \mathbb{S}_p$ ;  
vector  $x = (x_1, \dots, x_p)$ ;  
box  $X = (\mathbb{X}_1, \dots, \mathbb{X}_p)$ ;

```
1 begin
2   box  $\mathbb{Y} = (\mathbb{Y}_1, \dots, \mathbb{Y}_p)$ ;
    $\mathbb{Y} \leftarrow 0$ ;
   for  $i \leftarrow 1$  to  $p$  do
3     interval  $\sigma \leftarrow \mathbb{S}_i - s_i$ ;
      $\mathbb{Y}_{(i \bmod p)+1} \leftarrow \mathbb{Y}_{(i \bmod p)+1} + \sigma \times (\mathbb{X}_i - x_i)$ ;
     for  $j \leftarrow i + 1$  to  $p$  do
4        $\sigma \leftarrow \sigma \times s_j$ ;
        $\mathbb{Y}_{(j \bmod p)+1} \leftarrow \mathbb{Y}_{(j \bmod p)+1} + \sigma \times (\mathbb{X}_j - x_j)$ ;
5   return  $\mathbb{Y}$ ;
6 end
```

# Putting it all together

Our method results in a non-uniform partition of the original search domain  $\mathbb{A} = \cup_{i \in \mathcal{I}} \mathbb{A}_i$ .

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- Try to fill in the entire window.
- Scan for possible period doublings.
- Merge compatible parameter domains.



## Implementation

The algorithm was coded in C++ using the CAPD library. The program was run on *the machine* - a 32 core HP DL785 G5 (8 AMD Opteron 8354 processors) equipped with 32 GB DDR2 RAM.

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level	$N$	measure	wall time (h:m:s)	per. doublings
1	256	1.60620127942955014935	2 : 05 : 05	14
2	256	1.61118596303518551971	3 : 05 : 08	78
3	256	1.61287812977207803823	1 : 43 : 51	335
4	256	1.61349027421762439933	2 : 39 : 44	1381
5	256	1.61372268390076635341	5 : 18 : 22	4019
6	256	1.61381346643292467144	8 : 47 : 48	9075
7	512	1.61389940246044893686	67 : 26 : 04	20128
8	512	1.61391413966146151119	95 : 46 : 06	41692

**Table:** Here, the reported wall time corresponds to the current subdivision level only – not the accumulated time. The listed values of the measure and period doublings, however, correspond to the accumulated amount from all previous levels.

# Computational results

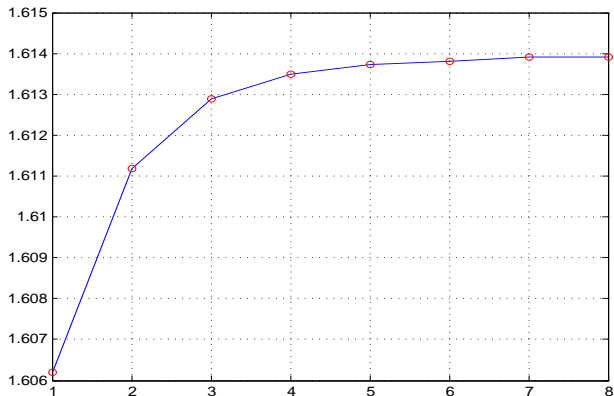
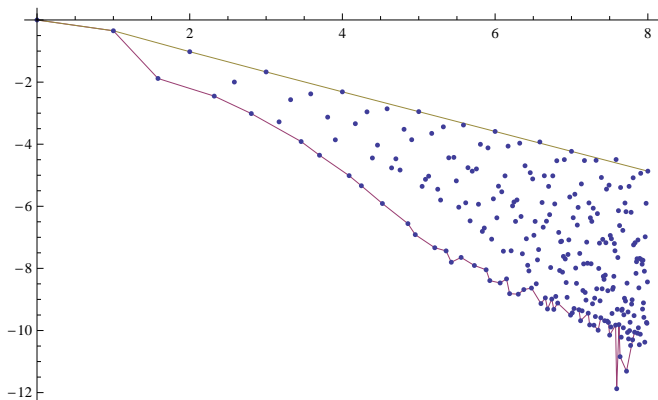


Figure: A plot of the search level versus the verified measure.

# Computational results - verified periods

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37  
38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71  
72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104  
105 106 107 108 109 110 111 112 113 114 115 116 117 118 119 120 121 122 123 124 125 126 127 128 129 130  
131 132 133 134 135 136 137 138 139 140 141 142 143 144 145 146 147 148 149 150 151 152 153 154 155 156  
157 158 159 160 161 162 163 164 165 166 167 168 169 170 171 172 173 174 175 176 177 178 179 180 181 182  
183 184 185 186 187 188 189 190 191 192 193 194 195 196 197 198 199 200 201 202 203 204 205 206 207 208  
209 210 211 212 213 214 215 216 217 218 219 220 221 222 224 225 226 227 228 230 231 232 233 234 235 236  
237 238 239 240 242 243 244 245 246 247 248 249 250 252 253 254 255 256 258 259 260 261 262 263 264 265  
266 267 268 270 272 273 274 275 276 279 280 282 284 285 286 287 288 289 290 291 292 294 295 296 297 298  
299 300 301 303 304 305 306 308 309 310 312 315 316 318 319 320 322 323 324 325 326 327 328 329 330 332  
333 334 336 338 340 341 342 343 344 345 348 350 351 352 354 356 357 358 360 361 362 363 364 366 368 369  
370 371 372 374 375 376 377 378 380 384 385 387 388 390 391 392 393 395 396 399 400 402 403 404 405 406  
407 408 410 412 413 414 416 418 420 423 424 425 426 427 428 429 430 432 434 435 436 437 438 440 441 442  
444 448 450 451 452 455 456 459 460 462 464 465 468 469 470 472 473 474 475 476 477 480 481 483 484 486  
488 490 492 493 494 495 496 497 498 500 502 504 506 507 510 512 513 516 518 520 522 525 528 532 539 540  
544 546 550 552 556 558 560 561 564 567 570 572 574 575 576 578 580 585 588 592 594 595 598 600 605 608  
609 612 616 620 621 624 625 627 630 637 638 640 644 646 648 650 656 660 665 666 672 675 676 680 682 684  
686 688 690 693 696 700 702 704 710 714 715 720 722 726 728 729 730 735 736 740 744 748 750 752 754 756  
759 760 765 768 770 780 782 784 792 798 800 805 810 812 816 819 820 825 828 832 833 836 840 850 855 858  
864 868 870 875 880 882 884 891 896 900 910 912 918 920 924 928 930 931 935 936 938 940 945 950 952 960  
966 968 972 975 980 984 988 990 992 1000 1001 1008 1012 1014 1020 1024 1040 1050 1056 1072 1080 1088  
1092 1100 1104 1120 1125 1134 1140 1144 1152 1170 1176 1184 1188 1200 1215 1216 1224 1232 1248 1260 1280  
1296 1300 1320 1344 1350 1352 1360 1368 1372 1386 1392 1400 1408 1428 1440 1456 1458 1500 1512 1520 1536  
1540 1560 1568 1584 1600 1620 1632 1664 1680 1728 1744 1760 1764 1792 1800 1824 1848 1872 1904 1920 1944  
1960 1980 2000 2016 2040 2048 2080 2100 2112 2156 2160 2176 2184 2200 2240 2268 2304 2340 2352 2376 2400  
2464 2496 2520 2560 2592 2640 2688 2700 2720 2800 2816 2880 2912 2940 3000 3024 3040 3072 3120 3136 3168  
3200 3240 3328 3360 3456 3520 3528 3584 3600 3640 3696 3780 3840 3888 3920 3960 4000 4032 4096 4160 4200  
4224 4320 4416 4480 4500 4608 4704 4752 4800 4928 4992 5000 5040 5120 5184 5280 5376 5400 5600 5632 5760  
6000 6048 6144 6272 6300 6336 6400 6480 6720 6912 7040 7168 7200 7680 7776 7840 8000 8064 8192 8400 8448  
8640 8960 9216 9408 9600 10080 10240 10368 10752 11200 11264 11520 12288 12544 12800 13440 13824 14336  
15360 16000 16128 16384 17280 17920 18432 19200 20480 21504 23040 24576 25600 27648 28672

# Computational results



**Figure:** A plot of  $\log_2$  of the period versus  $\log_{10}$  of the verified measure. Note how the prime periods form the lower line, whereas periods of the form  $2^k$  form the upper line. Also note the lack of some primes after period 222.

# Computational results

## The density point at $a = 4$

In the table below, we illustrate the fact that the parameter  $a = 4$  is a (one-sided) Lebesgue density point of  $\mathcal{S}$ . As such, the relative measure of the regular parameters should tend to zero as the density point is approached.

$\delta$	$ \mathcal{R} \cap \mathbb{A}_\delta / \mathbb{A}_\delta $	$-\log_{10}( \mathcal{R} \cap \mathbb{A}_\delta / \mathbb{A}_\delta )$
$10^{-1}$	$2.542 \times 10^{-2}$	1.595
$10^{-2}$	$7.596 \times 10^{-3}$	2.119
$10^{-3}$	$3.148 \times 10^{-4}$	3.502
$10^{-4}$	$1.555 \times 10^{-5}$	4.808
$10^{-5}$	$4.145 \times 10^{-6}$	5.382
$10^{-6}$	$1.466 \times 10^{-7}$	6.834

**Table:** The relative measure of the stable set, localised to increasingly small sets  $\mathbb{A}_\delta = [4 - \delta, 4]$ .

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- The gain is not impressive:  $\approx 3 \times 10^{-5}$ .

## As of today:

- Total measure: 1.61395259973507788887
- Number of distinct windows: 24 495 993
- Number of period doublings: 97 873



## Low periods only

If we only look for periods  $p \leq 33$ , we get

- Total measure: 1.611598655726128416455 (99.85%)
- Number of distinct windows: 20 316 378 (82.94%)
- Number of period doublings: 3 477 (3.55%)

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## CONJECTURE:

The set  $\mathcal{S}$  has measure near 0.3860474.

