The role of Dynamical Systems in Celestial Mechanics. Applications to Astronomy and Astrodynamics

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Workshop on

Computational Differential Geometry,

Topology and Dynamics

The Fields Institute, Toronto

200911181600

Introduction

The goal of **Dynamical Systems**: To study the dynamics of a system in evolution. **Everything that moves** can be considered as a DS.

Mathematical models:

$$x' = \frac{dx}{dt} = f(t, x, \mu) \qquad \text{ODE}$$

$$\frac{\partial u}{\partial t} = F(t, x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots) \qquad \text{evolution PDE}$$

$$x \mapsto T(x) \qquad \text{discrete maps, diffeomorphisms}$$

and one can include the effect of **noise, history, etc**.

x moves in a **continuous or discrete** way in a **phase space or state space** $\mathscr{E}$ and **parameters** are in a space $\mathscr{P}$. The best way to study systems is to consider the product space $\mathscr{E} \times \mathscr{P}$.

One can include also many **numerical algorithms**.

Some **topics to study** in general:

- **topology** of the $\mathscr{E} \times \mathscr{P}$ space, **singularities**,
- identify **simple invariant objects** in $\mathscr{E}$, like fixed points (in a fixed or moving frame), periodic and quasi-periodic solutions ... ,
- **stability/instability** of these objects and **stable/unstable manifolds**,
- **connections** between unstable/stable manifolds, **homo/heteroclinic** orbits, complicated dynamics, **diffusion**, strange attractors, **chaos**,
- **dependence on parameters**, bifurcations, structural stability, persistence in measure
- **statistical properties**, etc

When applied to **Celestial Mechanics** one can be more specific.

Some topics in Celestial Mechanics

1) **Topology of $\mathscr{E}$**, of the **energy-momentum manifolds**, **central configurations** and **relative equilibria**,
2) **Regularity** of the solutions and the role of **collisions**,
3) **Stability of simple solutions** and **Normal Forms** around them,
4) **Periodic orbits**: how they appear/disappear, stability, some exceptional classes like **choreographies**, the problem of the **density** of p.o.,
5) **Invariant tori** of different dimensions, bifurcations, **creation/ /destruction**, local behaviour around them, **reducibility** properties,
6) **Invariant Cantorian sets**, like Aubry-Mather sets, the dynamics around them, their role in the rate of diffusion,
7) **Checking conditions** for the existence of bifurcations, tori, etc, **away from simple solutions**. The **jet transport** and applications,
8) **Splitting phenomena** between the invariant manifolds of quite different **invariant objects** (perhaps partially weakly hyperbolic), responsible of creation of **chaotic dynamics**,
9) **Escape/Capture boundaries** and the **mechanisms** creating them: **non-analytic** invariant manifolds of **invariant objects at infinity**,

10) **Normally Hyperbolic Invariant Manifolds**, like **centre manifolds** which include p.o., q-p.o. and chaotic zones, related **codimension 1** manifolds, "practical" stability,

11) **Return Maps** to the vicinity of some (perhaps partially, weakly) **hyperbolic invariant object**, like **separatrix maps** and extensions (multiseparatrix maps) or projections (scattering maps),

12) **The regular and chaotic solutions** taking into account the role of **resonances** and different **temporal scales**,

13) **Statistical Properties**, like rates of diffusion, rates of escape, mass transport, ergodicity in most of the phase space,

14) **Applications to Astronomy**, like motion of **comets and asteroids**, asteroids coming close to the Earth **(NEO)**, detection and analysis of orbits of **exoplanets**,

15) **Applications to Astrodynamics**, like orbits of **AES, space debris, etc** and **missions** far away from the Earth, close to **libration points**, close to **binary or larger systems**, formation flights, etc.

Relative equilibria solutions (RES) and topology

Equations of the N-body problem (Newton)

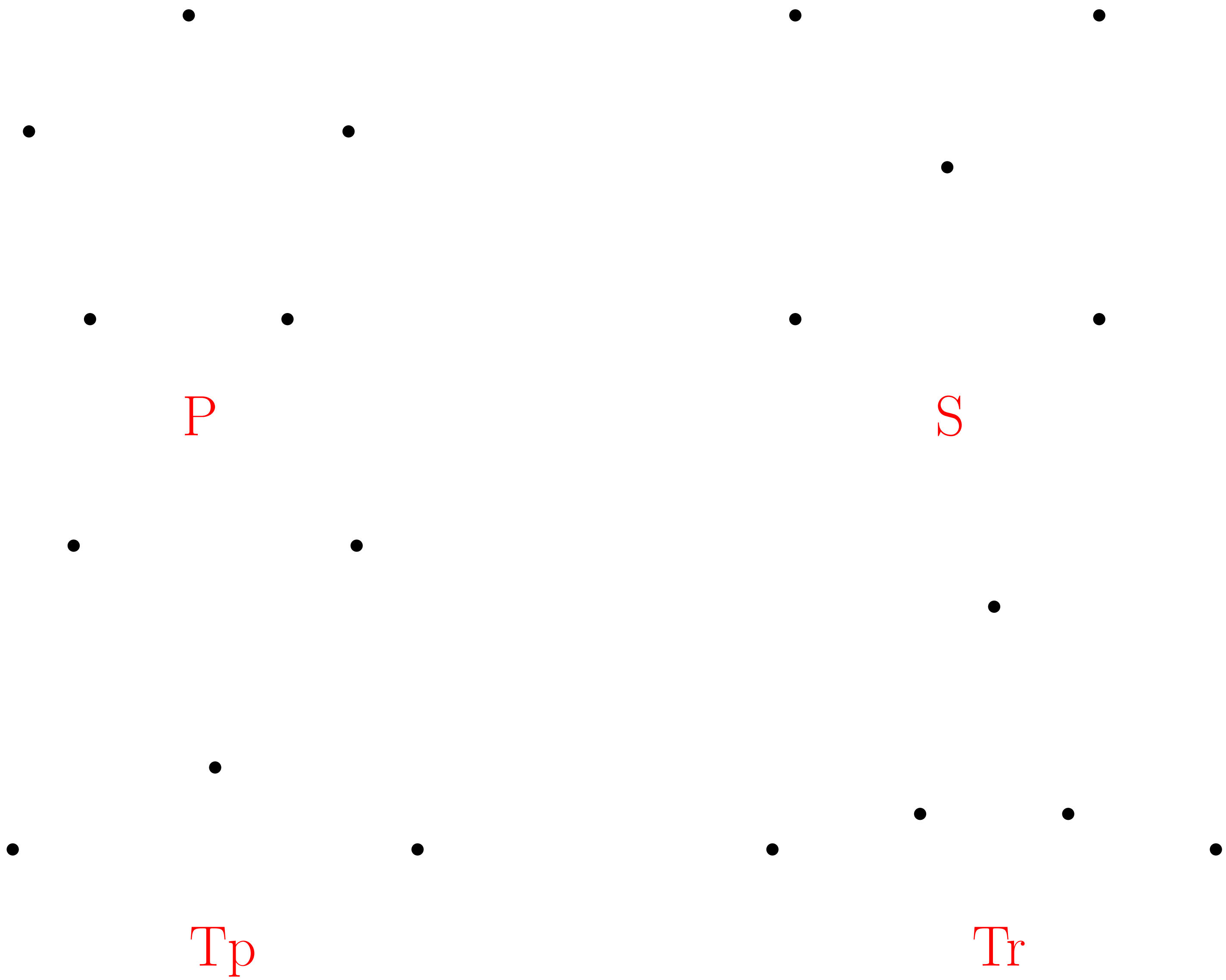
$$\frac{d^2\mathbf{q}_i}{dt^2} = \sum_{j=1, j\neq i}^{N} G m_j \frac{\mathbf{q}_j - \mathbf{q}_i}{r_{i,j}^3}, \quad r_{i,j} = ||\mathbf{q}_j - \mathbf{q}_i||_2, \quad \mathbf{p}_j = m_j d\mathbf{q}_j/dt\,.$$

Due to the **homogeneity** the total energy can be **normalised** to $0, \pm 1$.

The system has no fixed points at finite distance, but it has **relative equilibria**, e.g., when the bodies move in circles around the c.o.m. and the **centrifugal force cancels the gravitational attraction**.

For N**=3** Lagrange found that there are **5 non-equivalent RES**, independently of the values of the masses m_j.

For N**=4** there are **50 solutions** if the masses are equal (Albouy-Chenciner). But #(RES) **depends on the masses**, with a **minimum of 34 solutions**. The study of #(RES) as a function of m_j has only been done numerically. It is $\leq$ **50**. Rough theoretical bounds exist (≈ 8000, Moeckel).



RES for N =5 equal masses. P denotes a regular pentagon, S a square with a central mass, Tp a trapezoid with a central mass and Tr a triangle with 2 symmetric inner masses. The **60 collinear RES** are not displayed.

For $N=5$ results of a numerical exploration show that the total number of non-equivalent RES for equal masses is **354**. For other sets of masses the number of RES has been found to range between **294 and 450**.

N	3	4	5	6	7	8	9	10
$\mathrm{RES}(N)$	5	50	354	4104	53640	676080	13905360	250185600
$\frac{\mathrm{RES}(N)}{\Gamma(1.2N+1)}$	0.37	0.58	0.49	0.54	0.56	0.47	0.57	0.52

Estimated $\mathrm{RES}(N)$ for equal masses. The number seems to largely exceed the factorial.

Up to $N = 7$ all RES seem to have **some symmetry**. For $N = 8, 9, 10$ one has found, respectively, 2,3 and 12 **geometric configurations without any symmetry**.

It is not know how the **number of RES changes as a function of N** and even if it is **finite for all N**.

A related problem is the structure of the **energy-momentum manifolds for the N-body problem**, that is

$$I_{hc} = \{(\mathbf{q}_j, \mathbf{p}_j), j = 1, \ldots, N, \mid \quad \text{energy} = h, \quad \text{angular momentum} = c\}.$$

In fact I_{hc} are **stratified objects** instead of differentiable or topological manifolds.

Due to **homogeneity** the relevant parameter to study I_{hc} is hc^2. Changes in the **topology** can be related to the values of hc^2 **at the RES**.

For N**=3** the number of **connected components** of I_{hc} (**Hill's regions**) depends on hc^2. It can be **1,2,3**. Unfortunately:

Theorem If $N > 3$ then I_{hc} has a unique connected component for all values of hc^2.

Collisions and regularisation

Solutions of the N-body equations are not defined on $\Delta = \cup_{1\leq i<j\leq N}\Delta_{ij}$, $\Delta_{ij} = \{\mathbf{q}_i = \mathbf{q}_j\} = \{r_{i,j}\}$**=0**, the **set of collisions**.

Some special cases: a) **Binary collision** (BC), not a problem for the Newtonian potential; b) **Total collision** (GC). And between these extreme cases we can find intermediate ones: c) **Triple collision** (TC); d) **Simultaneous binary collision** (SBC).

Relevant questions: **Is it possible to regularise collisions?**

First one should understand **what means to regularise**. There are different approaches:
1) **Analytic or Siegel's** regularisation, for solutions analytic in $t^{1/m}$, m odd,
2) **Surgery or Easton's** regularisation, using **isolating blocks** and a suitable homeomorphism,
3) **Geometric regularisation**, by recovering (at least) continuous dependence w.r.t. initial conditions.

Interesting problems appear for other **homogeneous potentials**, even for the two-body problem.

Let $w(t) = (\mathbf{q}(t), \mathbf{p}(t))$ be a solution of the N-body problem (or any other having singularities), defined in $(t_0, 0)$, $t_0 < t_1 < 0$ and ending in collision when $t \to 0_-$. Let $w_c^{(i)} = w(t_1)$.

Assume that there are **initial conditions** (i.c.), $w_i^{(i)}$, in **any neighbourhood** of $w_c^{(i)}$, **not leading** to any singularity and such that $w(t; t_1, w_i^{(i)})$ is **defined until** some fixed t_3 >**0**. Let $w_i^{(f)} = w(t_2; t_1, w_i^{(i)})$ for some fixed $0 < t_2 < t_3$.

Definition: For **any sequence** $\{w_i^{(i)}\}$ of i.c. not leading to collision (except, perhaps, simple BC) with $\lim_{i\to\infty} w_i^{(i)} = w_c^{(i)}$ assume that $\lim_{i\to\infty} w_i^{(f)}$ **exists**. Then we **define** $w_c^{(f)}$ **as that limit** and look at it as $w(t_2; t_1, w_c^{(i)})$. We denote this extension as the **natural or geometric regularisation**.

For regularisable problems one can ask about the **regularity** of the map $w_i^{(i)} \mapsto w_i^{(f)}$ extended to $w_c^{(i)}$.

Some results:

a) The general **triple collision** is **not geometrically regularisable** except, at most, for a zero measure set of masses.

b) All **SBC problems** are at least $\mathcal{C}^0$ geometrically regularisable.

c) Consider four bodies and let $m_1 - m_2$ and $m_3 - m_4$ masses **colliding simultaneously** at $t = 0$ and $\mathbf{q}_j, j = 1, \ldots, 4$ their positions. Let

$$\mathbf{Q}_1 = \mathbf{q}_2 - \mathbf{q}_1, \quad \mathbf{Q}_2 = \mathbf{q}_4 - \mathbf{q}_3, \quad \mathbf{Q} = \mathbf{q}_{34} - \mathbf{q}_{12},$$

$\mathbf{q}_{12}$ and $\mathbf{q}_{34}$ the c.o.m. of $\mathbf{q}_1, \mathbf{q}_2$ and $\mathbf{q}_3, \mathbf{q}_4$.

Definition The problem is said to be **1D–reducible** if $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}$ have constant direction along the motion. This happens in some subproblems.

Theorem: In the 1D–reducible 4-body problems the SBC is $C^{8/3-\epsilon}$ **regularisable** for any $\epsilon > 0$, but it is not $C^{8/3}$ regularisable.

Conjecture: Assume the limit directions of $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}$, say $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}$ exist. Then, for all $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}$ and all masses $m_j > 0$, except by sets of zero measure, the **SBC is** $C^{8/3-\epsilon}$ **regularisable** in the 4–body problem. The exceptional cases have a higher regularity.

The TC in the planar 3-body problem

To learn about **passages near TC** it is useful to study the flow on the 4D **non-rotating TC manifold** $\mathcal{N}$. Suitable changes: a) **Blow up of variables** and then appear **10 critical points** $L_{\pm}^{i,s}, E_j^{i,s}, j = 1, 2, 3$ (5 for collision, 5 for ejection); b) **compactification** by adding "hard" binaries $B_j^{i,s}, j = 1, 2, 3$ to get $\bar{\mathcal{N}}$. In total **16 critical points** (8 for collision, X^i, 8 for ejection, X^s) all of them hyperbolic and the flow is **gradient-like**.

$$\dim W^{u,s}(L_{+,-}^{i,s}) = 2, \quad \dim W^u(B_j^i) = \dim W^s(B_j^s) = 4, j = 1, 2, 3,$$

$$\dim W^u(E_j^i) = \dim W^s(E_j^s) = 3, \dim W^s(E_j^i) = \dim W^u(E_j^s) = 1.$$

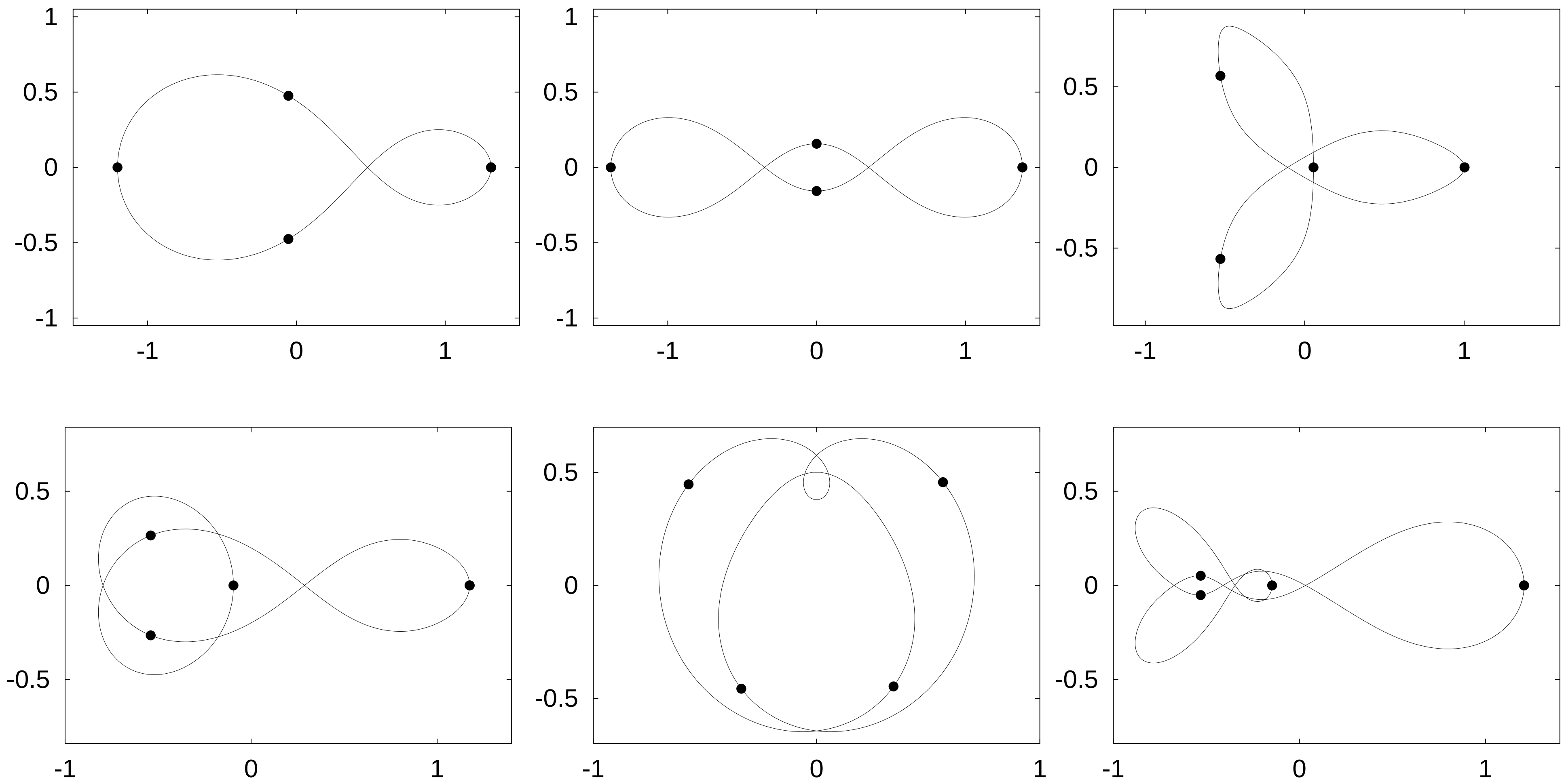
Theorem If two of the values of the masses are close enough and there are not connections $L_{+,-}^i \to L_{+,-}^s$ then points $B_j^i, E_j^i, j = 1, 2, 3$ connect to all points X^s, and $L_{+,-}^i$ connect to $B_j^s, E_j^s, j = 1, 2, 3$.
Conjecture This holds for all positive masses.

Theorem If $m_3 = \varepsilon, m_1 = m_2 = (1 - \varepsilon)/2$ the set of ε for which $L_{+,-}^i \to L_{+,-}^s$ occurs is countable, and $\varepsilon(n) = \pi^2/n^2 + \mathcal{O}(n^{-3})$ for n large.
Conjecture In the mass triangle $L^i \to L^s$ occur only in a countable set of lines.

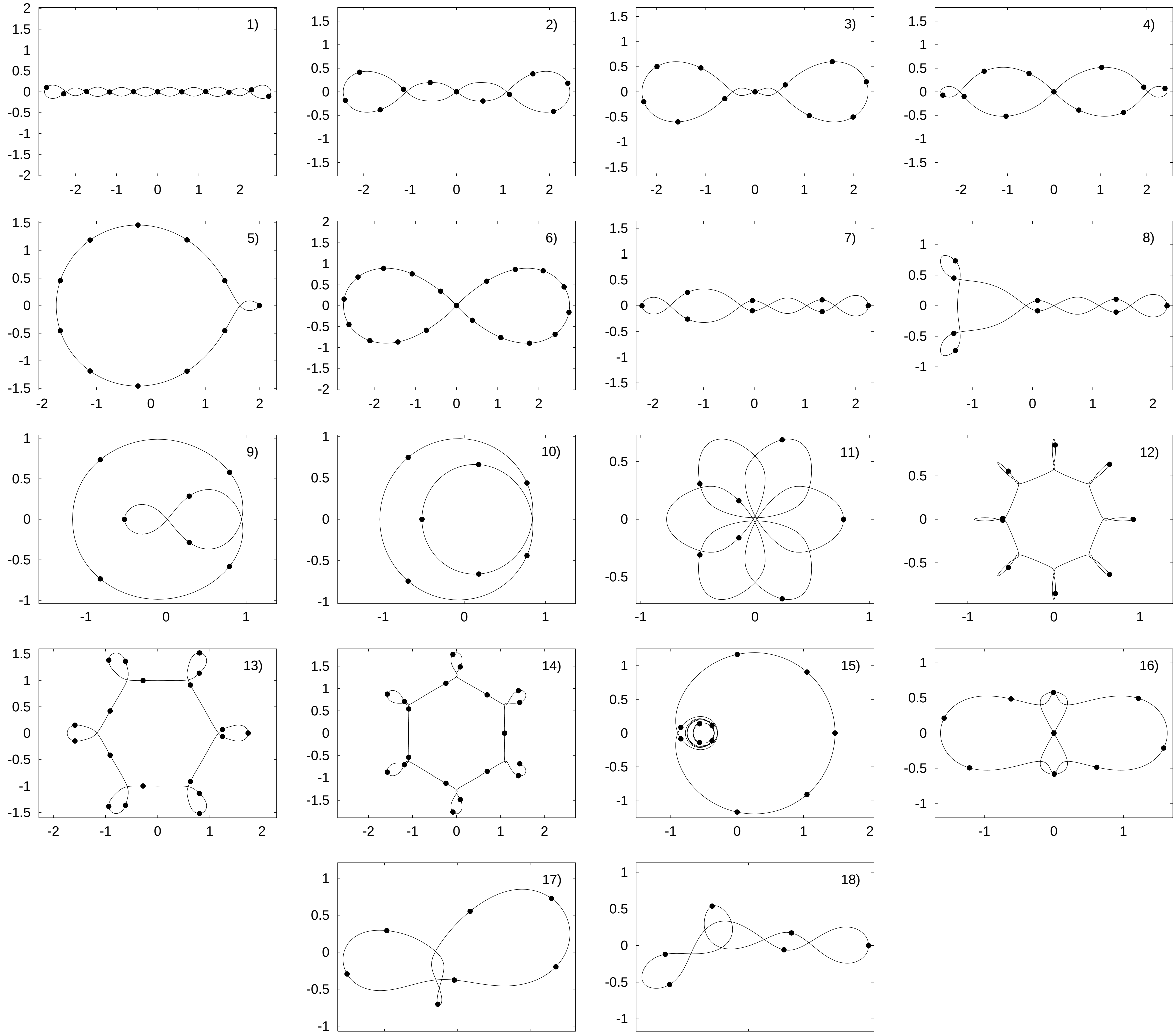
Some choreographies

Planar simple choreographies: Periodic solutions of planar N-body problem with equal masses with all the bodies moving in the same path.

A sample of choreographies for $N = 4$ is presented. Newtonian case.



And some additional examples come.



1)
2)
3)
4)
5)
6)
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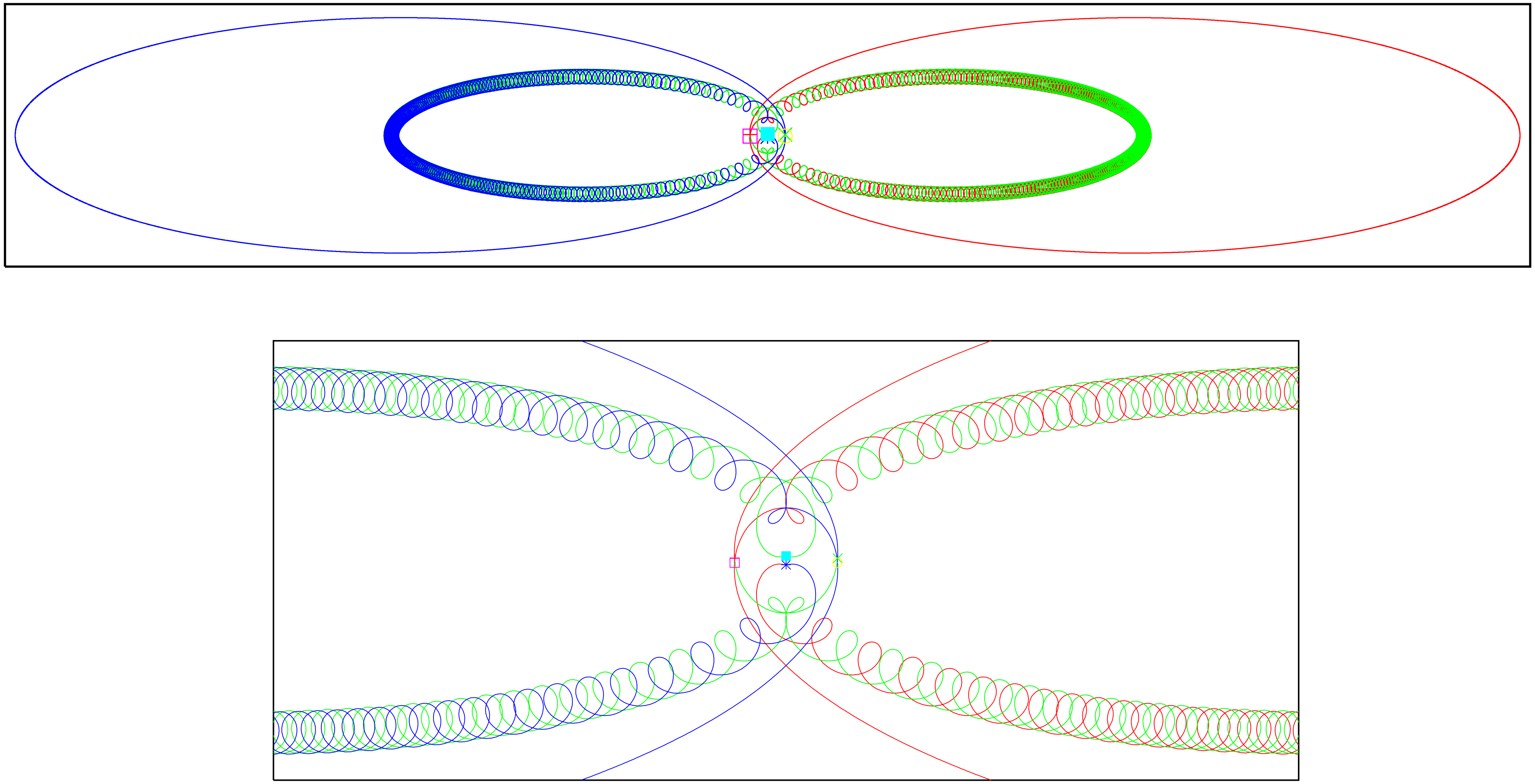
18)

How many?

A relevant question is whether the number even for $N = 3$ is **finite or not**. The answer is **NOT** (a routine Computer Assisted Proof can prove it).



Top: A choreography of the **3-body problem**. Bottom: A magnification of the central part. In each one of the binary portions, the bodies in the binary make **200 revolutions around their centre of masses**.

Jet transport and application to NEO

Consider an **IVP for an ODE** like $\dot{x} = f(t,x)$, $x(t_0) = x_0$, Assuming f **analytic** in a neighbourhood of $(t_0, x_0) \in \Omega \subset \mathbb{R} \times \mathbb{R}^n$ or $\Omega \subset \mathbb{C} \times \mathbb{C}^n$

Goal: to easily obtain the **Taylor expansion** $x(t_0 + h)$ for suitable values of h and use it as a one-step method.

For a very large class of functions the evaluation of f can be **split in simple expressions**

$$
\begin{aligned}
e_1 &= g_1(t,x), \\
e_2 &= g_2(t,x,e_1), \\
&\vdots \\
e_j &= g_j(t,x,e_1,\ldots,e_{j-1}), \\
&\vdots \\
e_m &= g_m(t,x,e_1,\ldots,e_{m-1}), \\
f_1(t,x) &= e_{k_1}, \\
&\vdots \\
f_n(t,x) &= e_{k_n}.
\end{aligned}
$$

Each expressions e_j contains a sum of arguments, a product or quotient of two arguments or an **elementary function** (like $\sin, \cos, \log, \exp, \sqrt{}, \ldots$) of a **single argument**.

Basic idea: to compute in a **recurrent way** the **power series expansion** of all the e_j. The g_j have to be seen as **operations with (truncated) power series**.

Input: t and the coefficients of order 0 of the components of x_0.

Step s: from arguments of g_j at order s we obtain order s of e_j. In particular for $f_j(t,x)$, which gives **order** $s+1$ for x_j (dividing by $s+1$).

The **representation** of $x(t_0+h)=(x_i)$ is $x_i=\sum_{s=0}^{N} a_i^{(s)} h^s$ for suitable N, h, such that the **truncation error** $\sum_{s>N} a_i^{(s)} h^s$ can be considered as **negligible** in front of the (unavoidable) **round off error**.

Example: $a(t)=\sum_{k\geq 0} a_k t^k$, $a_0 \neq 0, \alpha \in \mathbb{R}$ and $b(t)=a(t)^\alpha=\sum_{k\geq 0} b_k t^k$:

$$b_0=a_0^\alpha, \quad b_n=-\frac{1}{n a_0}\sum_{k=0}^{n-1} b_k a_{n-k}[k-\alpha(n-k)],\ n>0,$$

the determination being fixed by the one used for b_0. To compute to order N has a **cost** $\mathcal{O}(N^2)$. This is true for the **most expensive elementary operations and functions**.

Similar recurrences can be obtained for **any elementary function**.

For (near) integrable Hamiltonian systems and $\Lambda_{\max} \approx 0$ (zero maximal Lyapunov exponent) the **errors in actions** are $\mathcal{O}(t^{1/2})$ and **in angles** are $\mathcal{O}(t^{3/2})$ due to **random walk-like** behaviour of round off.

Jet transport: Assume the i.c. are $x_0 + \xi$, where ξ are some variations of i.c. It is enough **to replace all operations** with numbers by operations with **polynomials in ξ** up to the desired order. It is elementary to include as components of ξ all **relevant parameters**.

Can be **implemented in efficient way**, to produce **rigorous estimates of the tails** at every step and to obtain **intervals** which contain the **correct values of all the coefficients**.

If the initial data are in some **uncertainty set**

a) The coordinates can be **adapted** according to the **shape** of the initial uncertainty set.

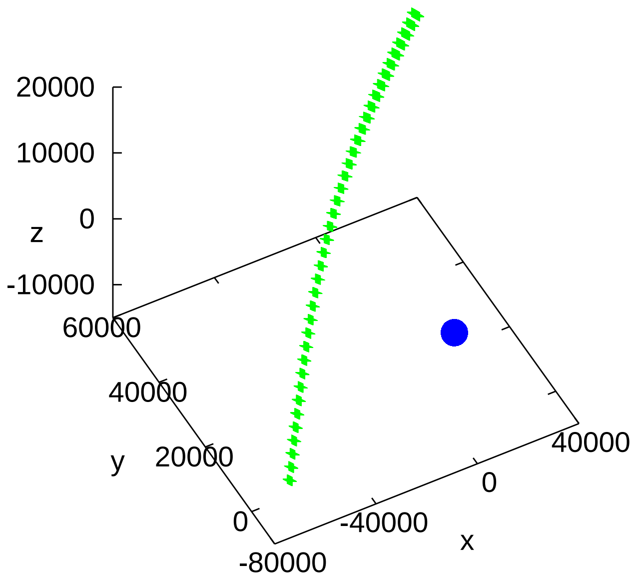
b) The effect of the **uncertainty in parameters** can be easily included.

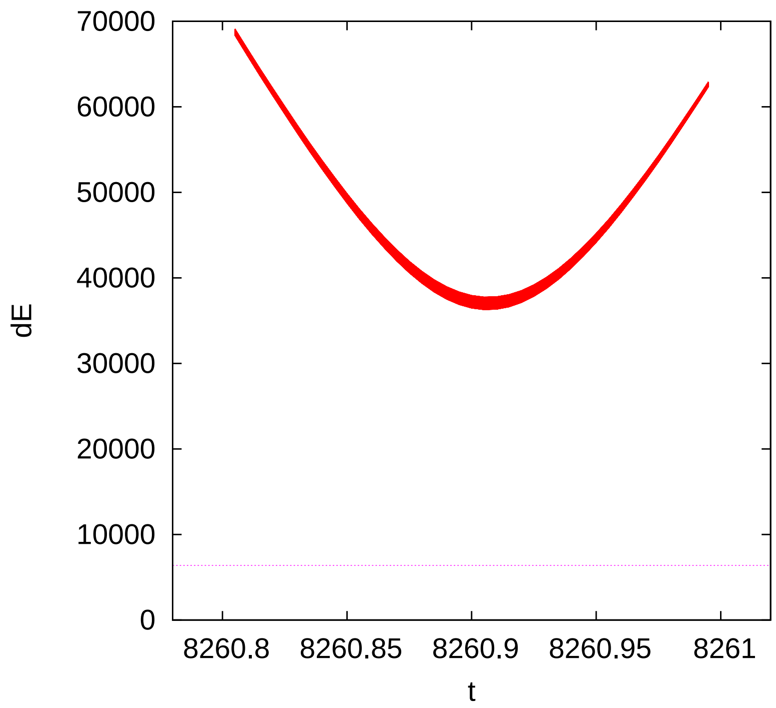
c) It can be be modified to take into account **different distributions** for the uncertainty.

The case of the **asteroid (99942) Apophis**, which will experience **close approaches with the Earth in 2029** and, perhaps, between 2036 and 2037. We want to understand the **effect of initial uncertainties**. Standard deviations are ≈ 1.5 km and ≈ 3 km **across and along the orbit**.

In 2029 Apophis will get at about **36000 km w.r.t. the centre of the Earth** (on Friday, April 13, near 9 pm UT). An accurate description of that passage can be obtained by means of a **3rd order integration** in ξ.

Most significant changes of Apophis orbit at the close encounter: **inclination, semi-major axis and thus period**, slowing down ≈ 3 km/s.





Escape/Capture boundaries

One of the outstanding problems in Celestial Mechanics is the **detection and computation** of **capture and escape boundaries**.

They are related to the **existence of some invariant objects at infinity** which have **invariant manifolds**.

But these invariant objects **are not hyperbolic**. They are only **parabolic** in the sense of Dynamical Systems.

It is well known that this fact was **partially analysed by Moser and McGehee**. The manifolds **exists** and they are **analytic except, perhaps, at infinity**. Related results are due to **C.Robinson**.

Standing question: Which is the **regularity class** of these manifolds? How can we **compute them** with rigorous error control, so that they can be used to obtain **capture and escape boundaries**?

The simplest problem to analyse is the **Sitnikov problem**:

$$q' = \Psi q^3 p, \quad p' = \Psi q^4 \left(1 + \Psi^2 q^4\right)^{-3/2}, \quad \Psi = (1 - e\cos(E))/4, \quad ' = d/dE.$$

We look for a **parametric representation** of the manifolds of the p.o. as

$$p(E,e,q)=\sum_{k\geq 1} b_k(e,E)q^k=\sum_{k\geq 1}\sum_{j\geq 0}\sum_{i\geq 0} c_{i,j,k}e^i\,\mathrm{sc}(jE)\,q^k,$$

where $b_k(e,E)$ are trigonometric polynomials in E with polynomial coefficients in e, $c_{i,j,k}$ are rational coefficients, sc denotes sin or cos functions.

Theorem: The manifolds $W_{\pm}^{u,s}$ are **exactly Gevrey-1/3** in q uniformly for $E\in\mathbb{S}^1$, $e\in(0,1]$. Concretely, let a_n denote the norm of b_n. Then there exist constants c_1,c_2, $0<c_1<c_2$ such that, for $n\geq 5$ except for $n=6,7,10$ one has

$$\mathbf{c_1}\rho^{\mathbf{n}}<\mathbf{a_n}/\boldsymbol{\Gamma}((\mathbf{n}+\mathbf{1})/\mathbf{3})<\mathbf{c_2}\rho^{\mathbf{n}},\quad \rho=(3/4)^{1/3}.$$

Recall: a **formal power series** $\sum_{n\geq 0}a_n\xi^n$ **is of Gevrey class** s **if** $\sum_{n\geq 0}a_n(n!)^{-s}\xi^n$ **is analytic around the origin**.

Theorem: The formal expansion gives an **asymptotic representation** of the invariant manifolds of $\mathrm{p.o.}_\infty$. Concretely, the **truncation of the series at order** n has an error which is bounded by the sum of the norms of next three terms

$$C(a_{n+1}q^{n+1}+a_{n+2}q^{n+2}+a_{n+3}q^{n+3}),\quad C\approx 1.$$

Given q the **optimal order is $n_{\mathbf{opt}} \approx 4/q^3$**. Using optimal order **the error bound is $< N \exp(-4/(3q^3)), N < 1$**.

The method opens the way to other **more relevant problems, like 2DCR3BP, 3DCR3BP, 3DER3BP, general 3BP, etc**.

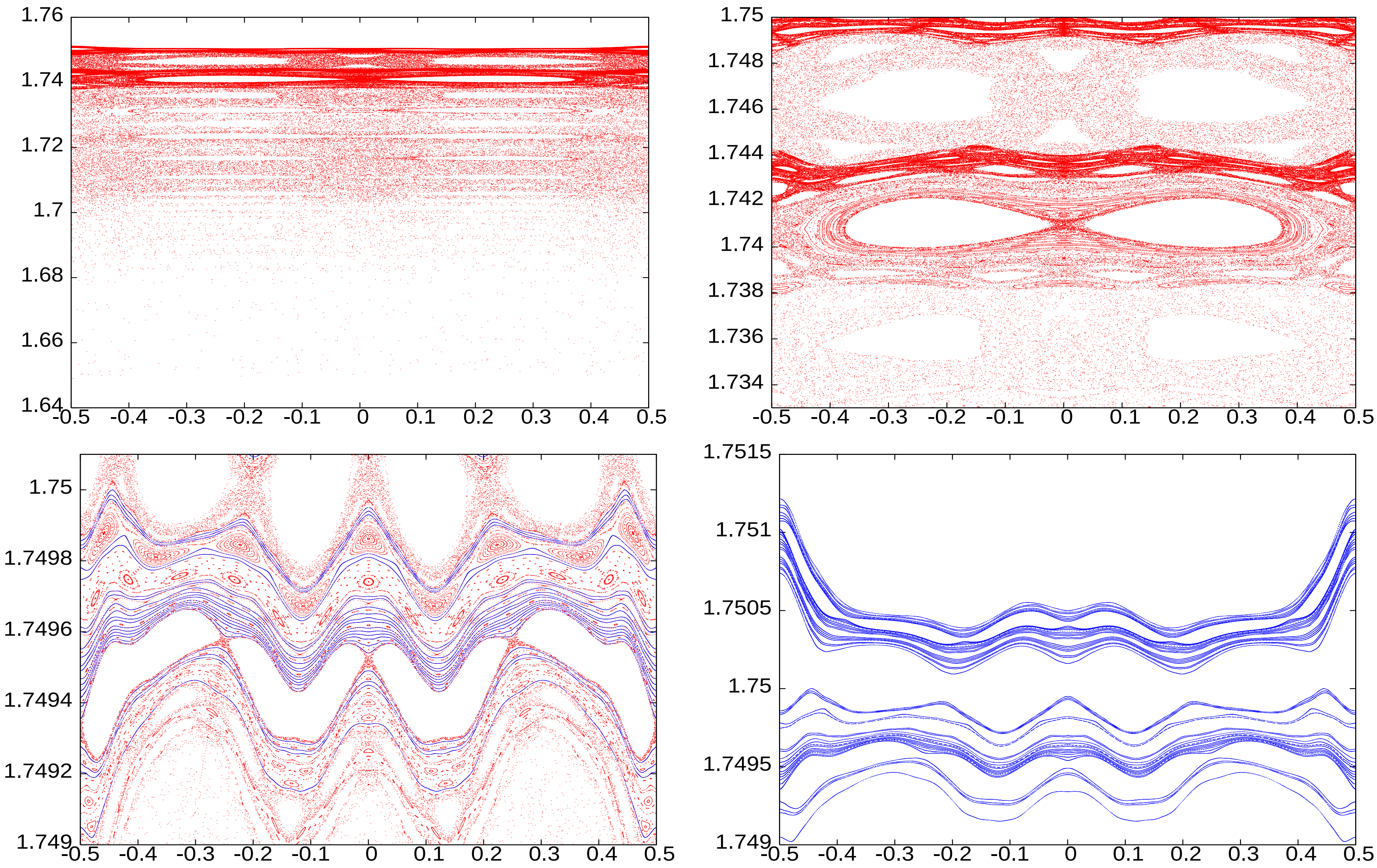
In these problems there are **parameters** playing **quite different roles**: small eccentricity and masses play an $\approx$ **linear role** while some **energy** plays and **exponentially small role**.

One recovers and enlarges previous results about **splitting of separatrices** obtained so far for **2DCR3BP, 2DER3BP, planar general 3BP**.

Combined with **return maps: separatrix maps and extensions** one obtains **barriers or "practical" barriers for escape**.

In next page these ideas are applied to a **comet** outside Jupiter orbit, in the planar RTBP, $\mu = 10^{-3}, C = 3.6$, and **escape is not possible if $e <$0.75**.

The **blue curves** identify approximate locations of **KAM tori**.



Poincaré map obtained by **pericentre passage** in (revolutions,radius) and details. For **small values of the pericentre distance** the effect of Jupiter is strong and some orbits **end in escape**.

Practical stability and codimension 1 manifolds

The **triangular libration points** of the RTBP (and many other examples) are **practically stable** in the sense that the rate of escape is **exponentially small** w.r.t. the distance ρ to the fixed point. Time to move to a finite distance behaves like $\exp(c/\rho^d), c > 0, d > 0$ **(Nekhorosev-like estimates)**.

This follows from **Normal Forms estimates** and/or **averaging theory**.

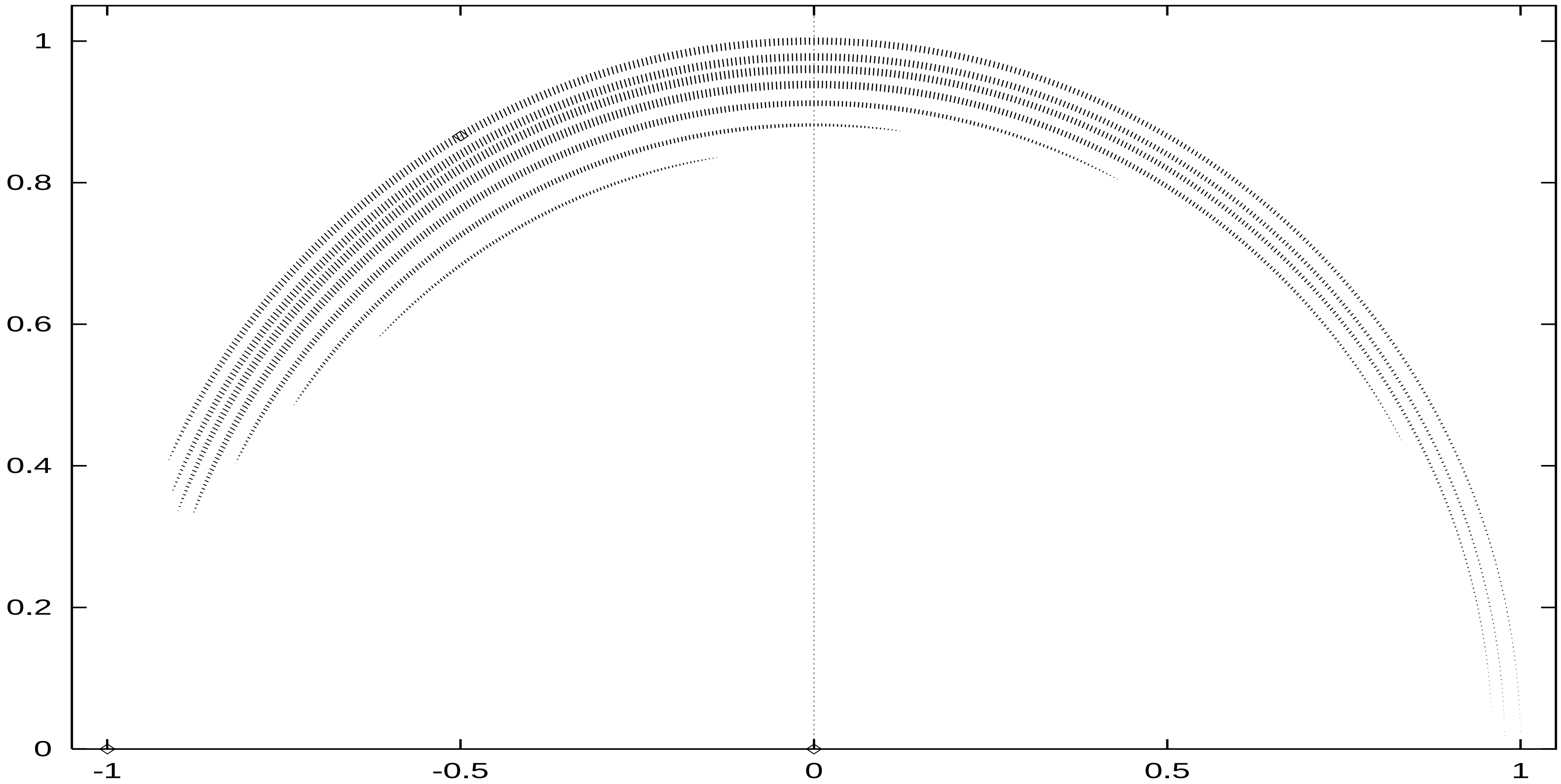
But plenty of numerical simulations (in many families of problems) show a **practically "stable domain"** much larger than the one following from these estimates.

We consider the case $\mu = 0.0002$ because:

a) it is **small enough** so that **perturbation theory** can be useful,

b) it is **large enough** so that some escapes do not require **too much computational time**,

c) it is also close to the **Saturn–Titan** mass-ratio.



Take points with **zero synodical velocity** and (changing C)

$$X = \mu + (1+\rho)\cos(2\pi\alpha), \quad Y = (1+\rho)\sin(2\pi\alpha), \quad Z$$

Escape criterion: $Y(t) < Y^*$ for some $Y^* < 0$ (e.g. $Y^* = -0.5$).

Time span: 10^4 revolutions of primaries (circa 400 years). No essential differences with $10^5, 10^6$.

Displayed results: Z =0.0, 0.3, 0.4, 0.5, 0.6, 0.7 and 0.8

Comments on results

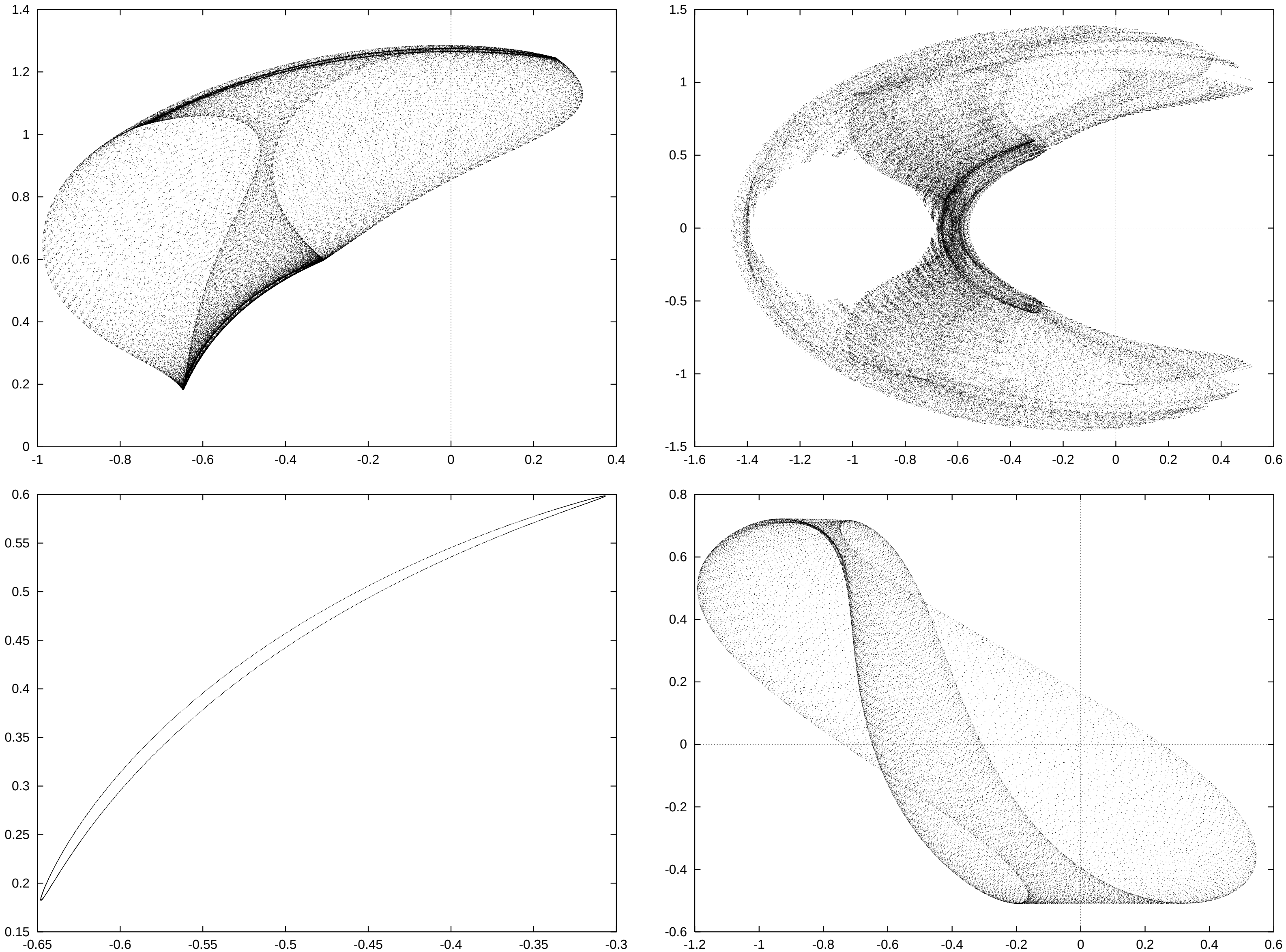
1) For the **planar case** the domain of non-escaping points is bounded by the intersection of $W^{u,s}(W^c_{L_3})$ (a **codimension 1 manifold** of the planar problem) and $v_{\text{syn}}=\mathbf{0}$. If the ZVC on the L_3 level is given by $\rho = K_\pm(\alpha)\mu^{1/2} + \mathcal{O}(\mu)$ then the domain boundary is $\rho \approx \frac{1}{2}K_\pm(\alpha)\mu^{1/2} + \mathcal{O}(\mu)$.

2) The full set of points in the "practical stability" zone with $v_{\text{syn}} = 0$, in the **spatial RTBP**, is on a **thin shell**. Its shape is **near circular** in (X, Y) and **parabolic** in the vertical direction. Cutting the "stable" zone by $\alpha = 1/3$, the central point, as a function of Z, is close to $\rho = -0.245Z^2$.

3) There are families of **unstable 2D tori** which play also a role in the **boundary of the "stability" region**, see next plots. If the initial Z is small these tori reach a **vicinity of** L_3 and their sections through $Z = 0$ have **positive** X **values**. For large initial Z, the sections through $Z = 0$ of these 2D tori have **negative** X **values**.

4) **Continuation** of these 2D shows that they belong to the W^c **of a family of p.o. which is partially hyperbolic**: $W^c_{\text{p.o.family}}$. The $W^{u,s}$ of that object is a **codimension 1 manifold**. Several of them play a role in different parts.

$\alpha = 1/3, Z = 0.8, \rho = -0.1506066340$ vs $\rho = -0.1506066339$ and $\mathbb{T}^2$.



Libration point orbits

Our interest in now to understand the structure of the set of **non-escaping orbits around $L_{1,2}$** in the RTBP. One has to **skip the unstable terms** of the Hamiltonian around an E×E×H point. This is done by **reduction to the 4D centre manifold W^c**.

After linear symplectic change the Hamiltonian is

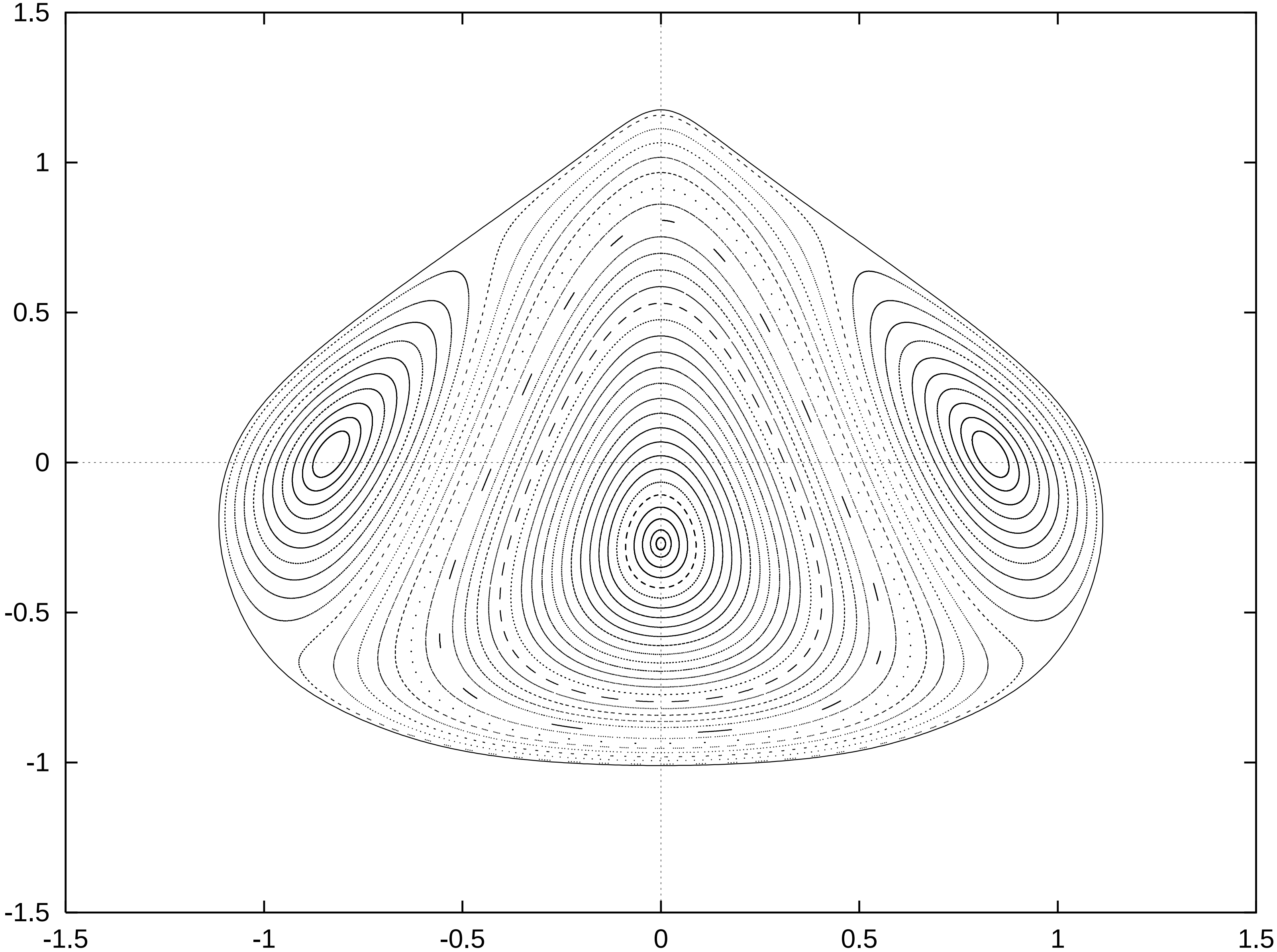
$$M = \lambda x_1 y_1 + \frac{1}{2}\omega_1(x_2^2 + y_2^2) + \frac{1}{2}\omega_2(x_3^2 + y_3^2) + \sum_{j\geq 3} M_j(x_1, x_2, x_3, y_1, y_2, y_3),$$

where M_j denotes a homogeneous polynomial of degree j.

One must **cancel all terms of total degree 1 in x_1, y_1**. No small divisors show up, there is **no convergence in general** (but the divergence is mild). We obtain

$$\widetilde{M} = M^0(x_2, x_3, y_2, y_3) + \sum_{j_1+j_2>1} x_1^{j_1} y_1^{j_2} M^{j_1,j_2}(x_2, x_3, y_2, y_3)$$

and $x_1 = y_1$**=0 gives** W^c.



Poincaré section of the flow of the Hamiltonian reduced to W^c of the L_2 point in the Earth-Moon case, for a given level of the energy, h. The Moon is located outside the figure, in the negative vertical axis of the plot.

One can **read off** the structure from this plot on a section Σ which is $\approx \{z=0\}$ in the initial variables.

The **boundary p.o.** corresponds to the **planar Lyapunov orbit**, unstable inside W^c for that value of h.

Near the **centre of the plot** there is a fixed point, corresponding to the **vertical Lyapunov orbit**.

The **two additional fixed points** correspond to **halo orbits**.

Invariant curves around fixed points correspond to **tori** (also named Lissajous orbits for that problem).

The zones **between the domains of curves** contain the **intersections of $W^{u,s}$** of the planar Lyapunov orbits with Σ. **Tiny chaos** appears there, hard to see.

The plot helps the preliminary **design of space missions around the libration point**. Improvement using perturbations of other bodies is required. In particular the **solar effects** are quite relevant.

Summary and Outlook

The systematic use of **invariant objects** and, when applicable its **centre, stable, unstable manifolds**, that is the **skeleton of the system**, provides useful tools to understand the **global dynamics**.

I consider the **most challenging problem** to study **statistical properties** like rates of diffusion, rates of escape, mass transport, "practical" ergodicity.

One has to face **diffusion problems** which are **highly heterogeneous and highly anisotropic**.