Computability and Complexity of Julia sets

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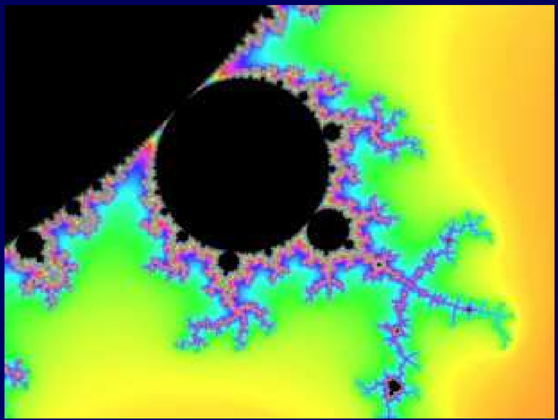
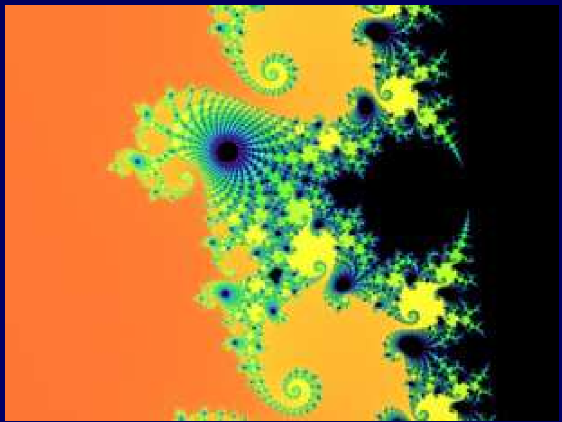
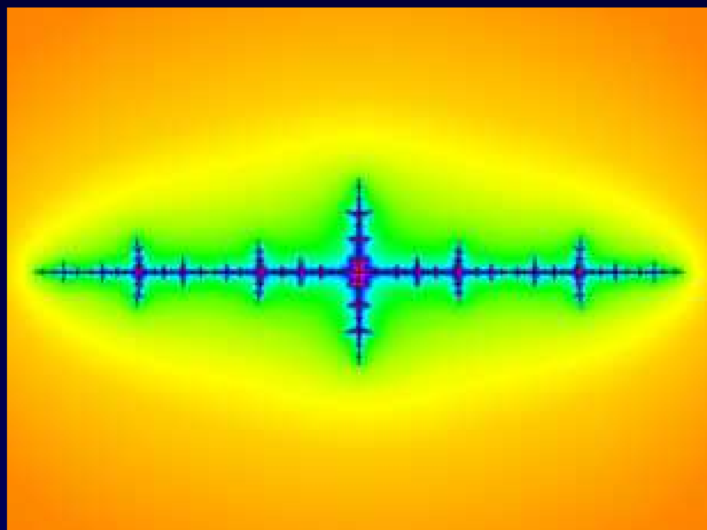
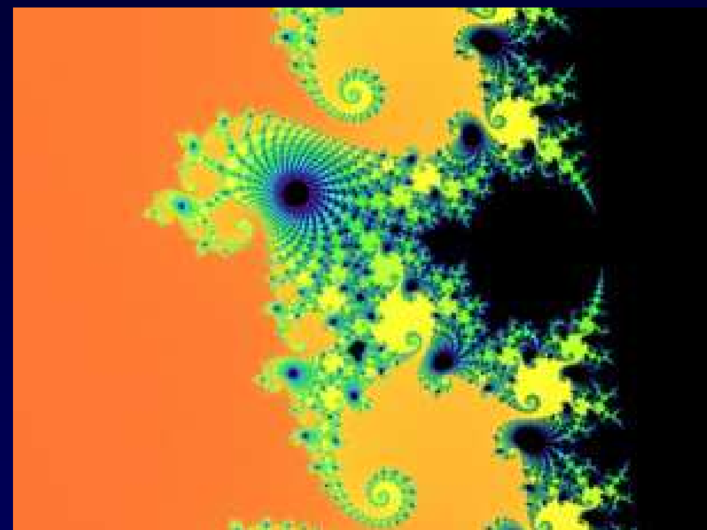
Based on joint works with Ilia Binder and Michael Yampolsky

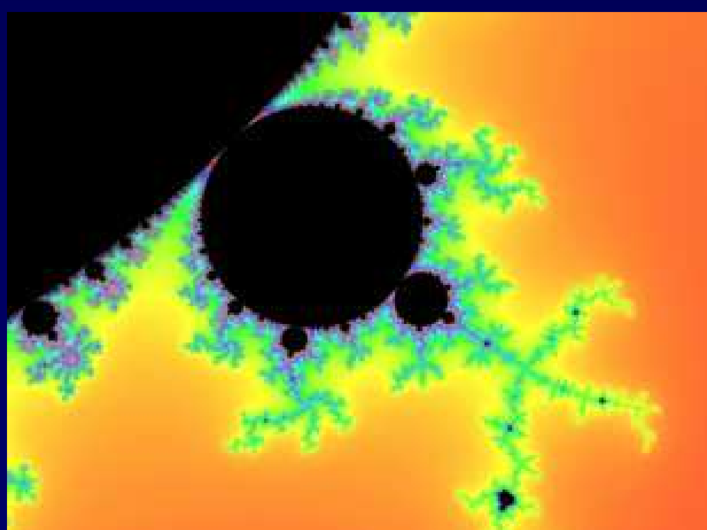


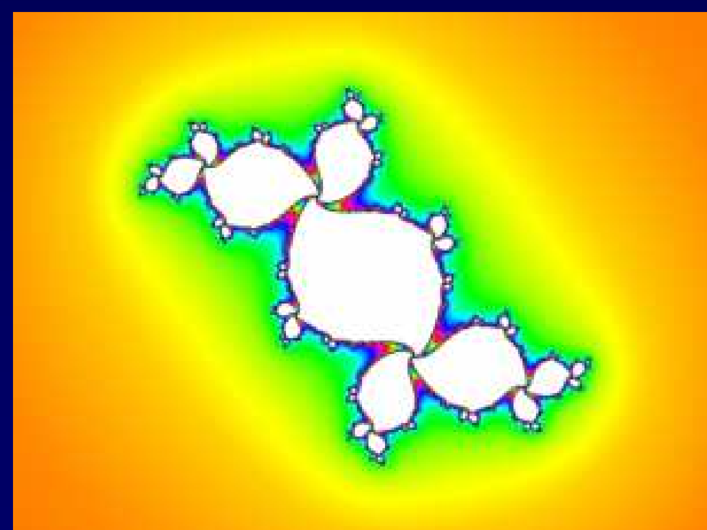


Julia Sets









Julia Sets

- Let c be a complex parameter.
- Consider the following discrete-time dynamical system on the complex plane:

$$z \longrightarrow z^2 + c$$

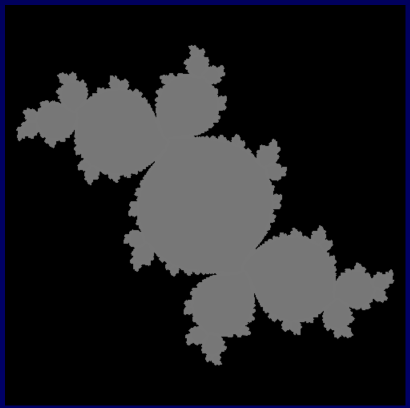
- The simplest non-trivial polynomial dynamical system on the complex plane.

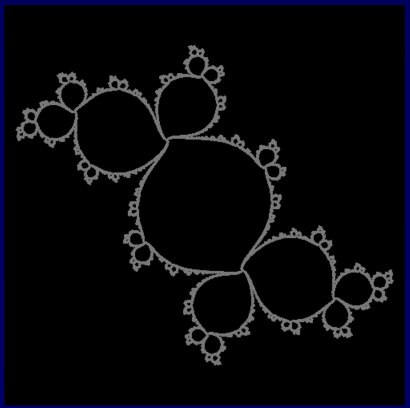
The Julia set

- The filled Julia set K_c is the set of initial z's for which the orbit does not escape to ∞.
- The Julia set J_c is the boundary of K_c:

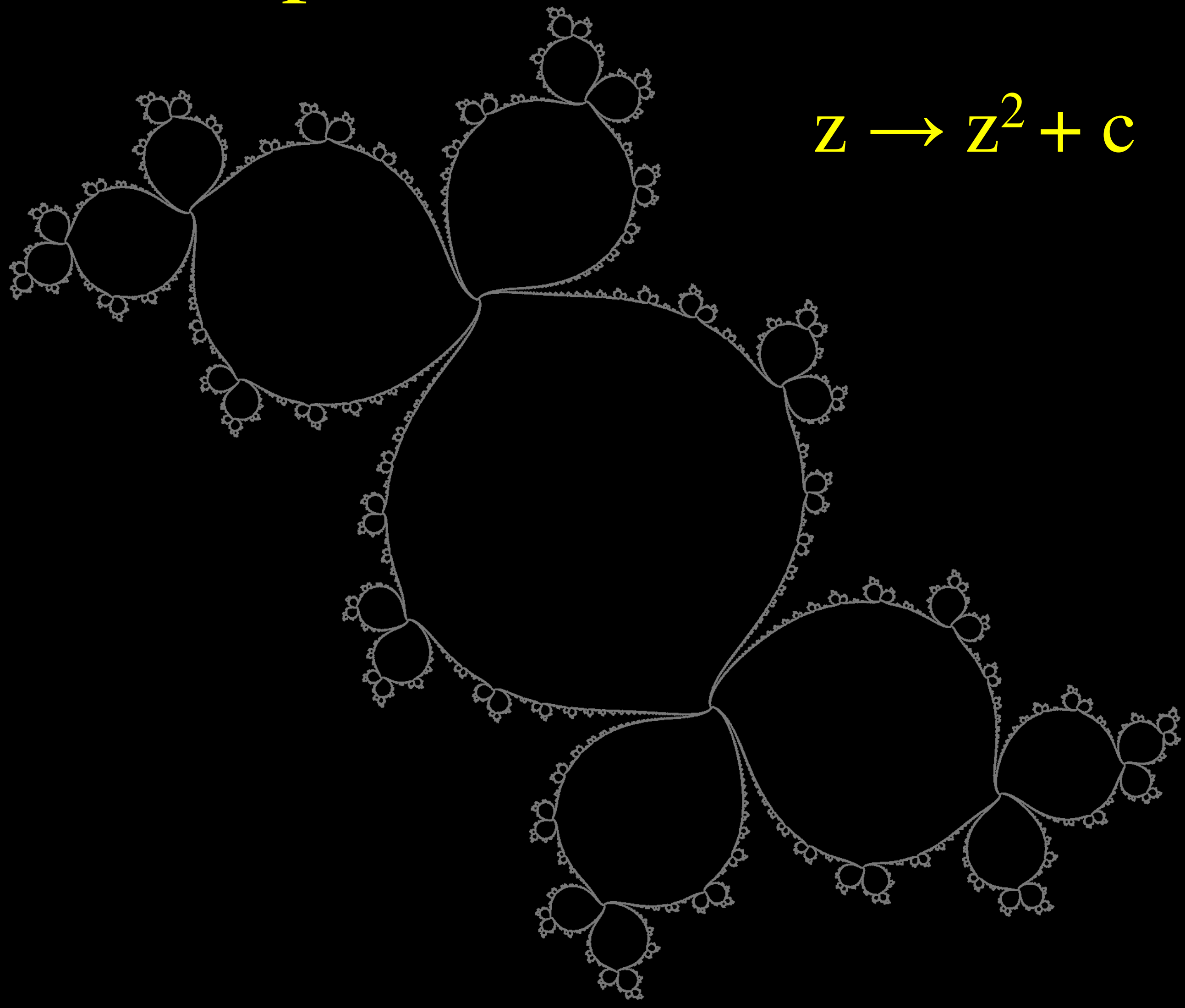
$$J_c = \partial K_c.$$

- J_c is also the set of points around which the dynamics is unstable.

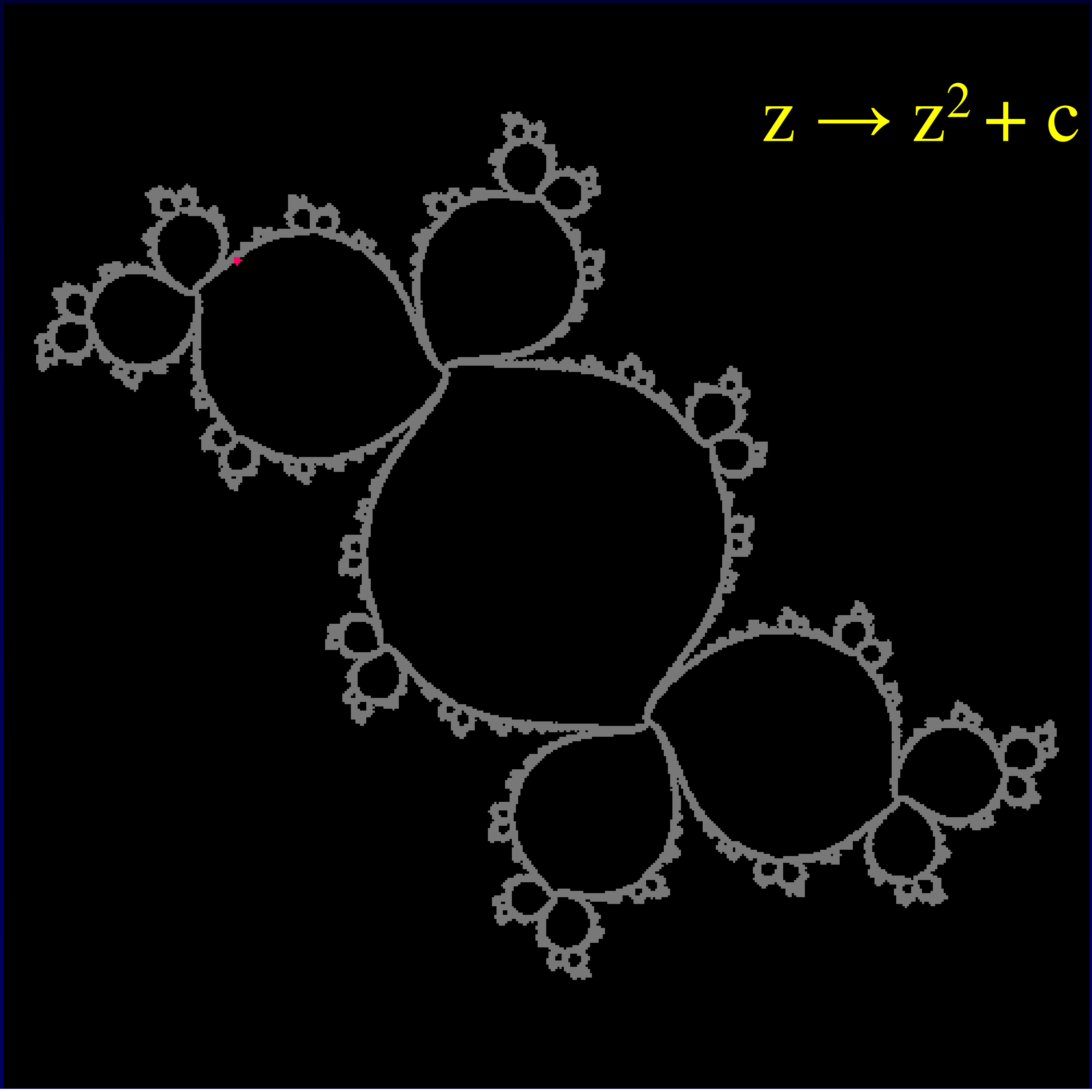




Example: c = -0.12 + 0.665 *i*



Iterating $z \rightarrow z^2 - 0.12 + 0.665\, i$



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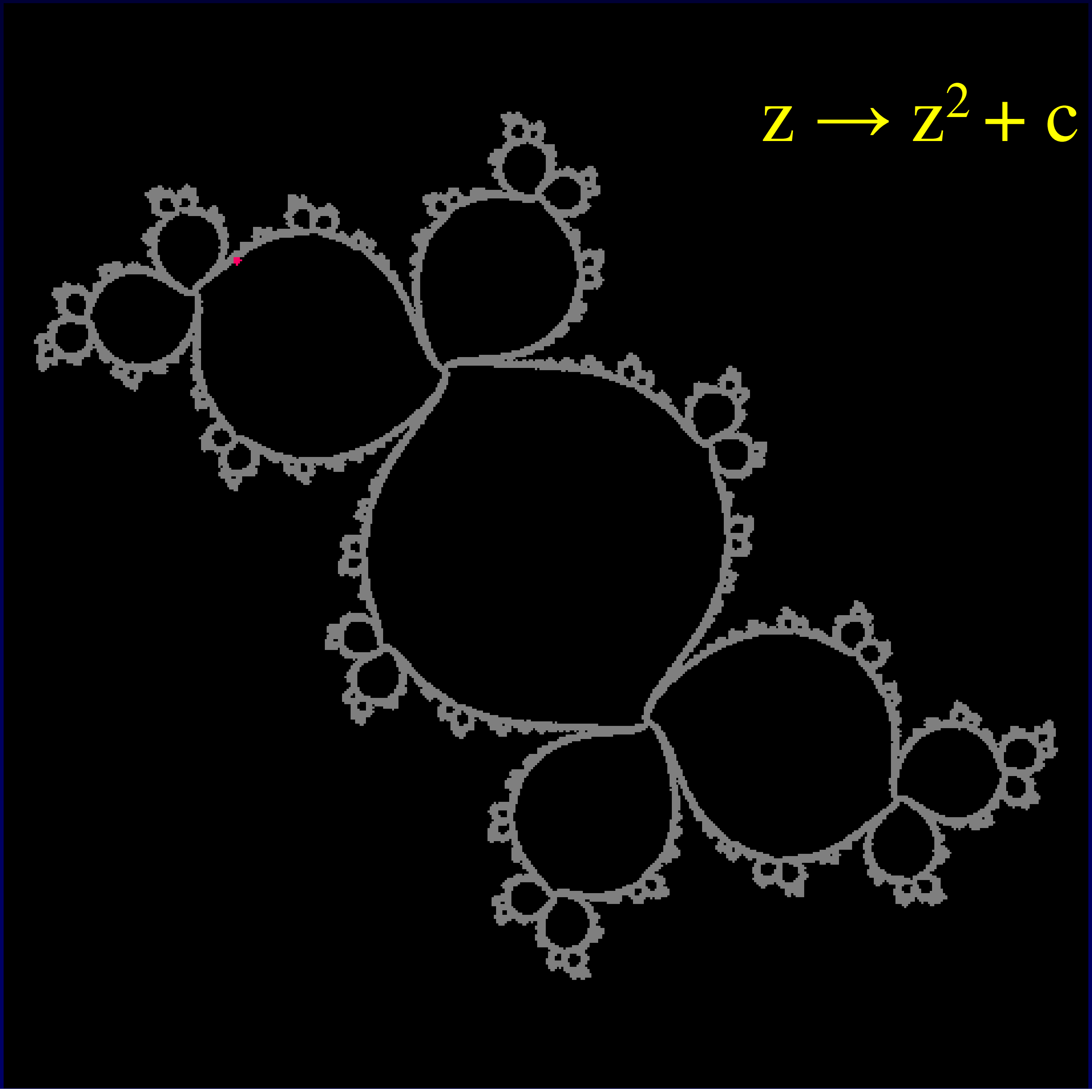

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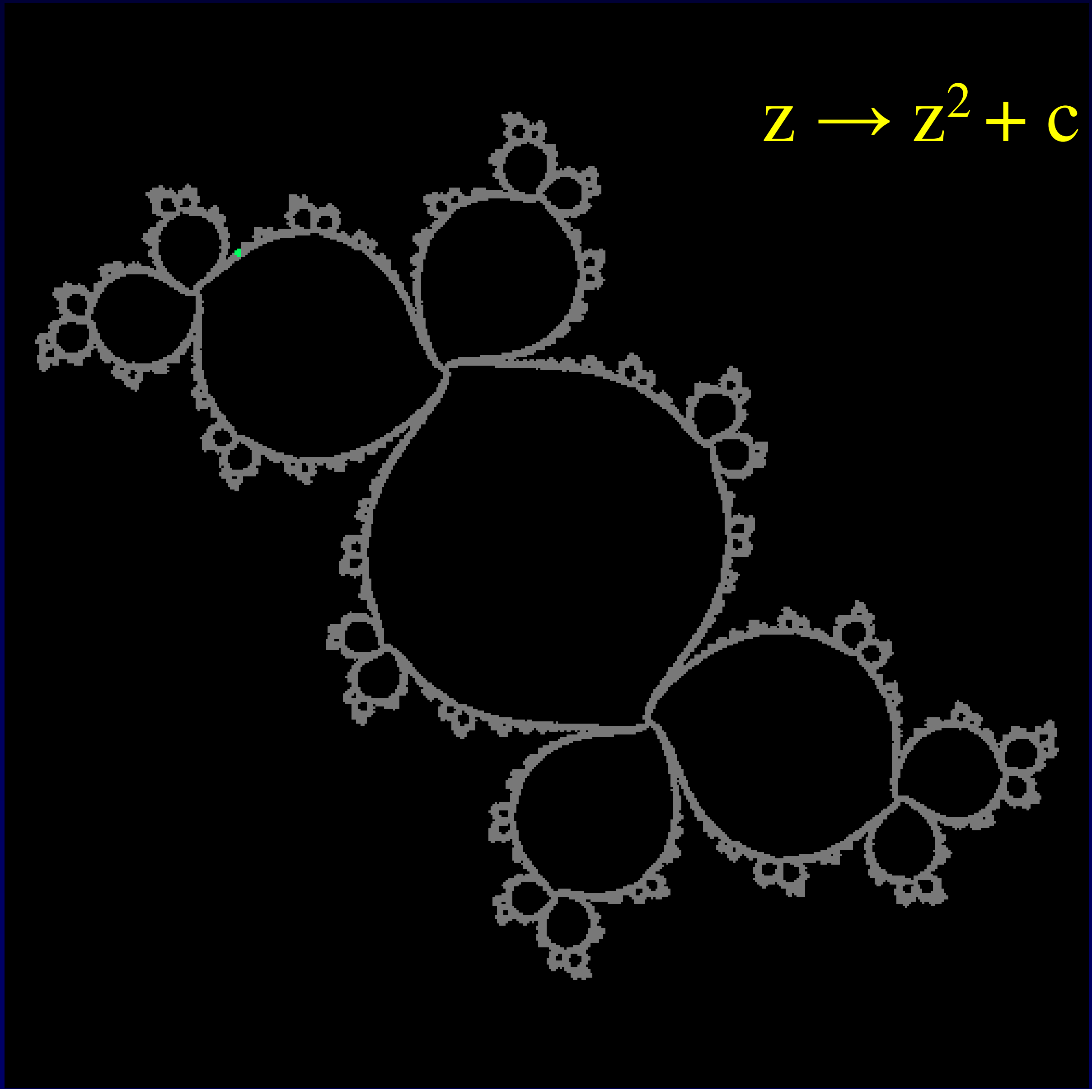

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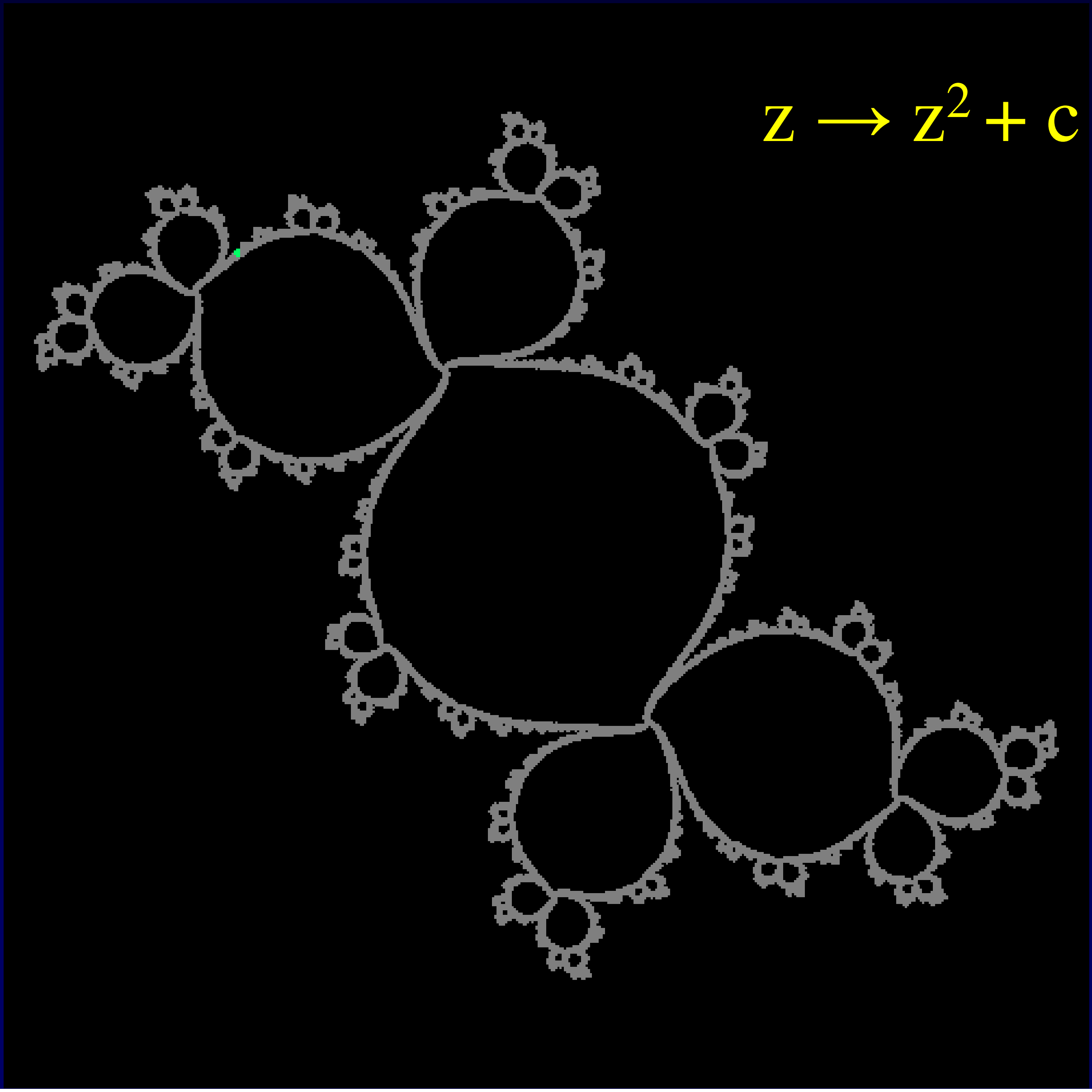

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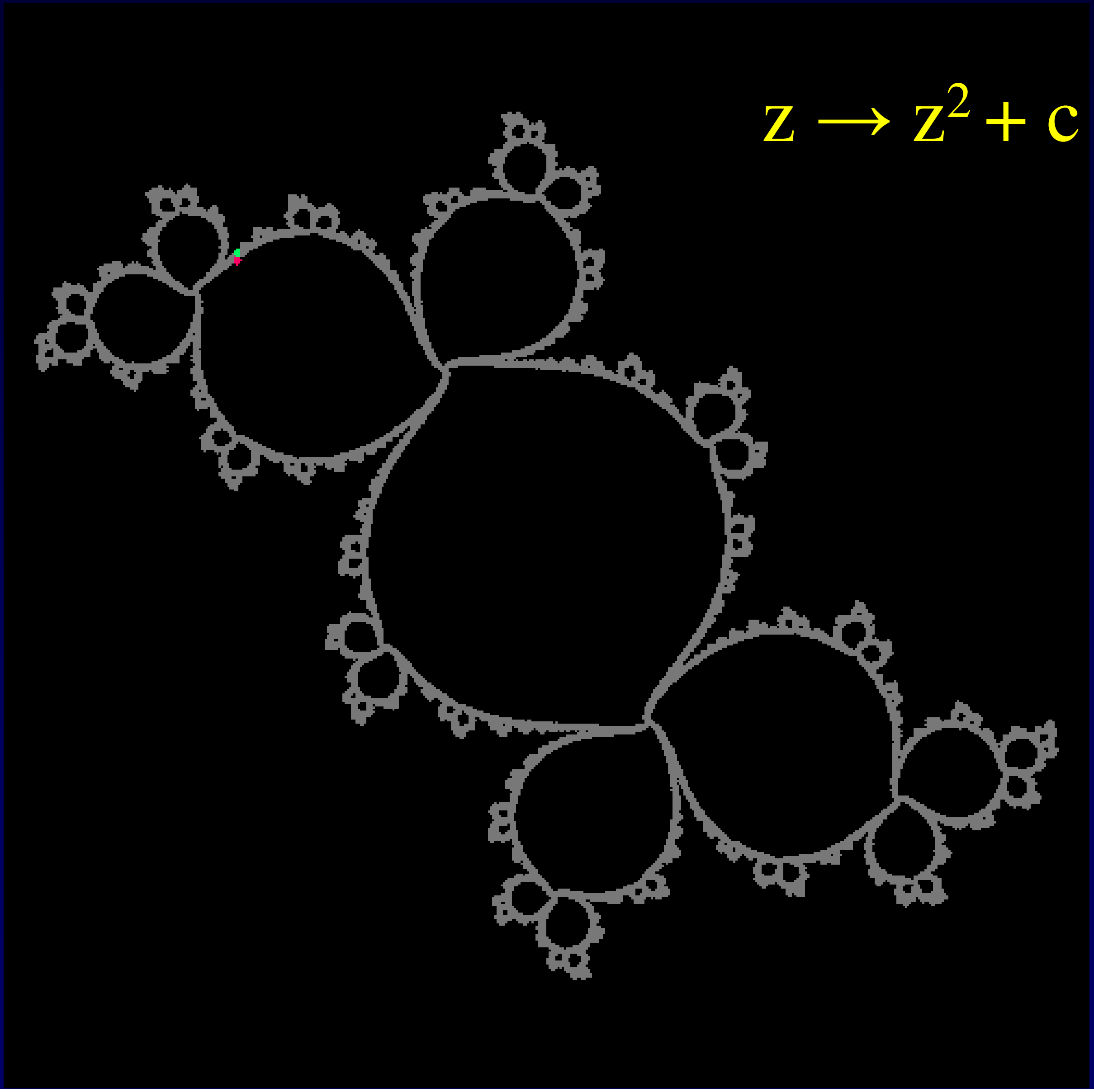
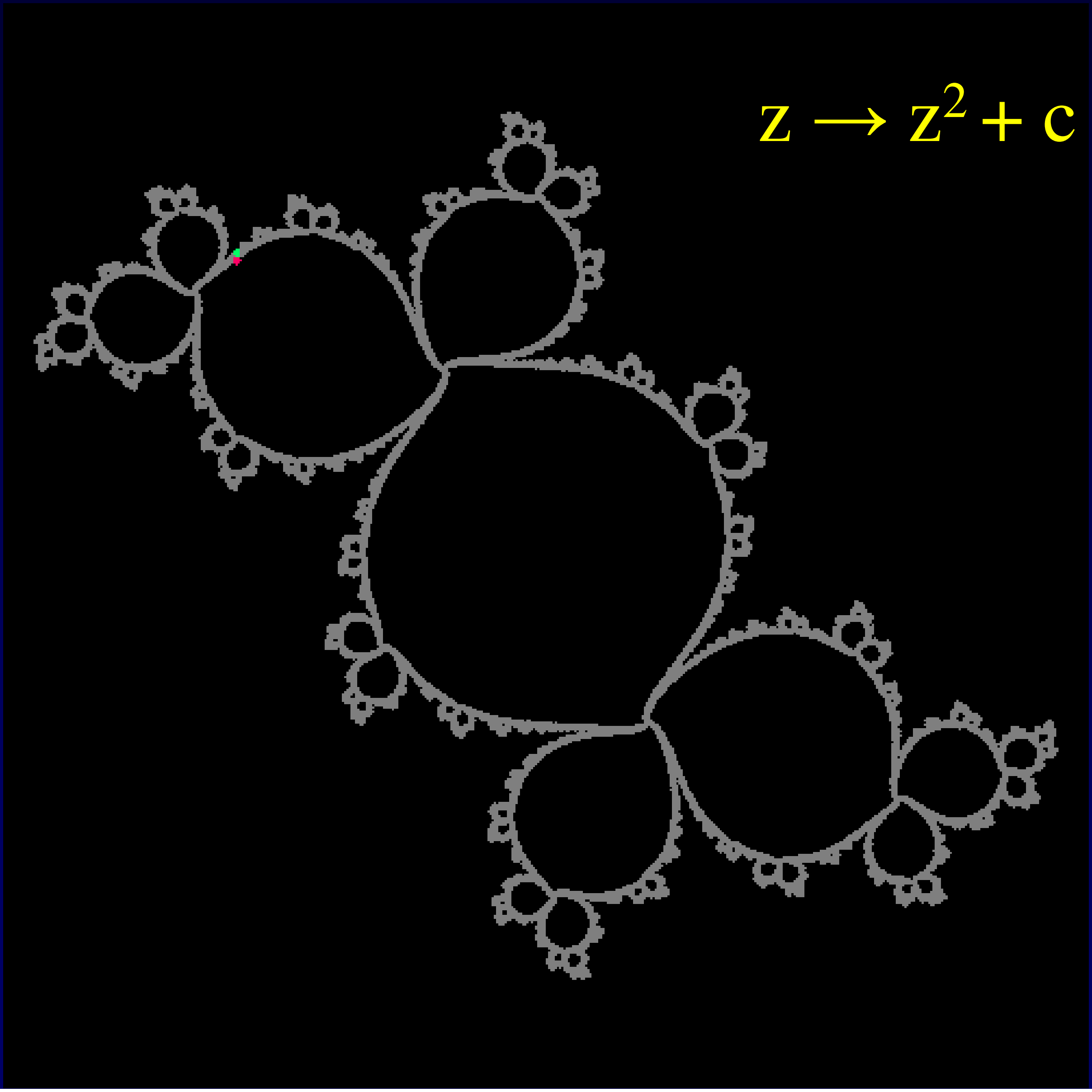


Iterating $z \rightarrow z^2 - 0.12 + 0.665\,i$



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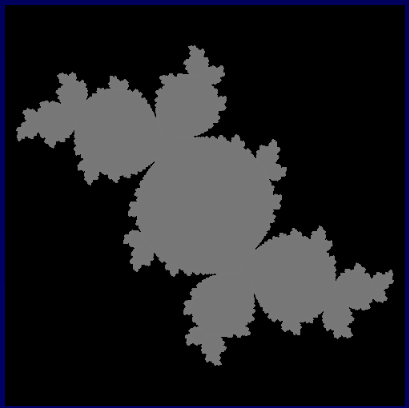
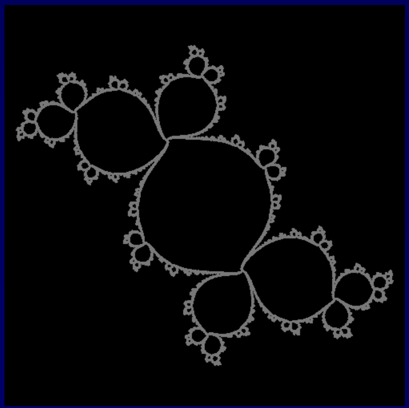
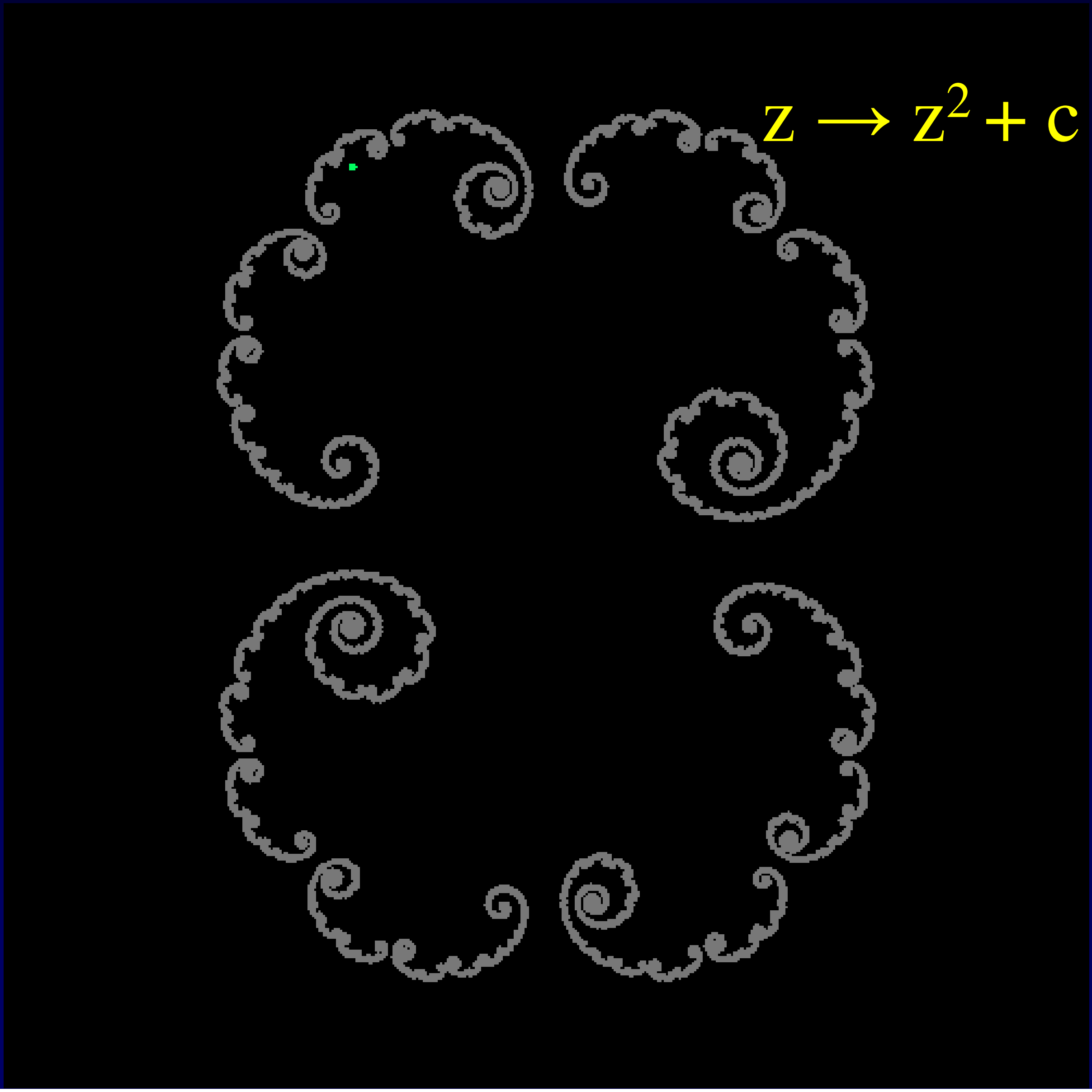


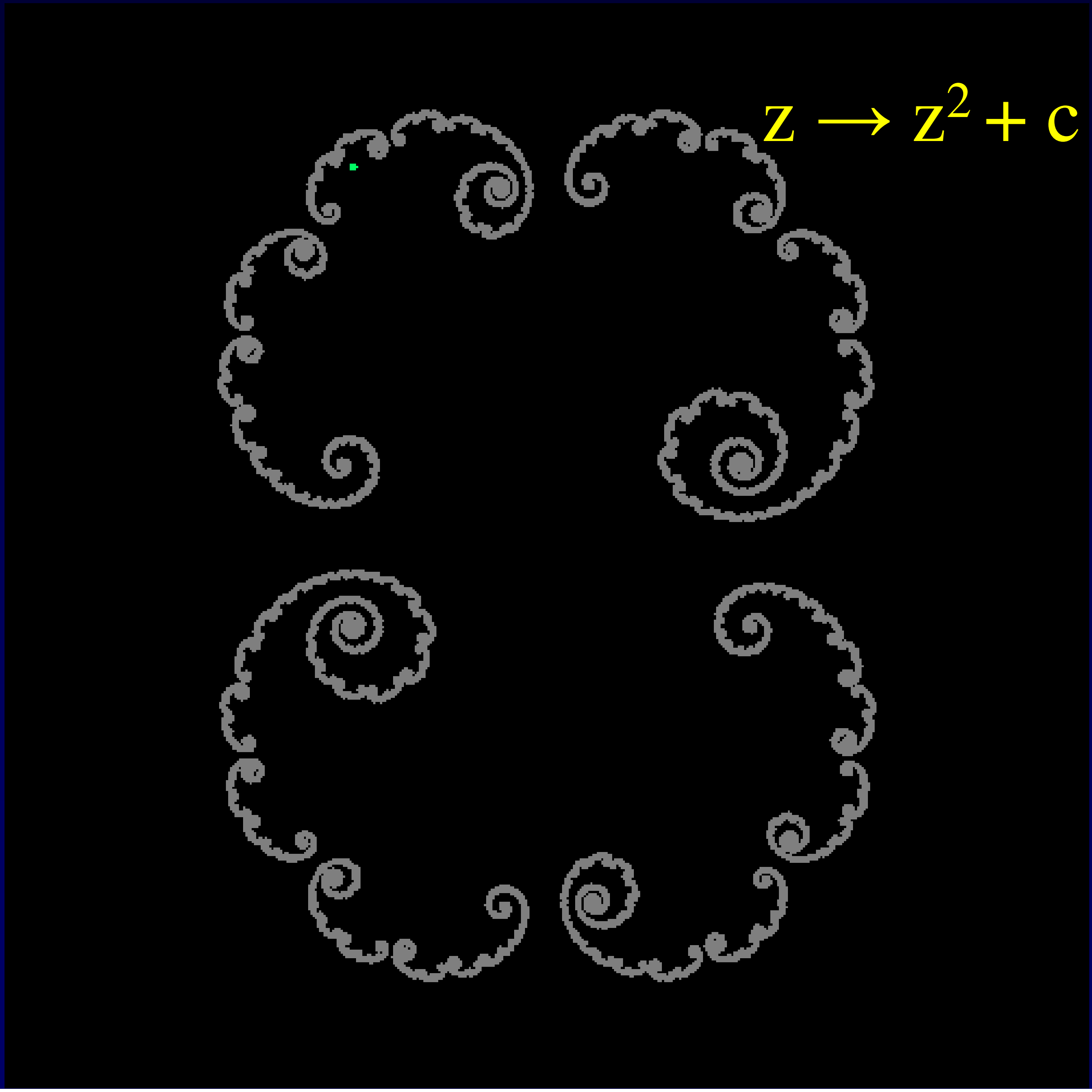


Example: $c = 0.29 + 0.005\,i$

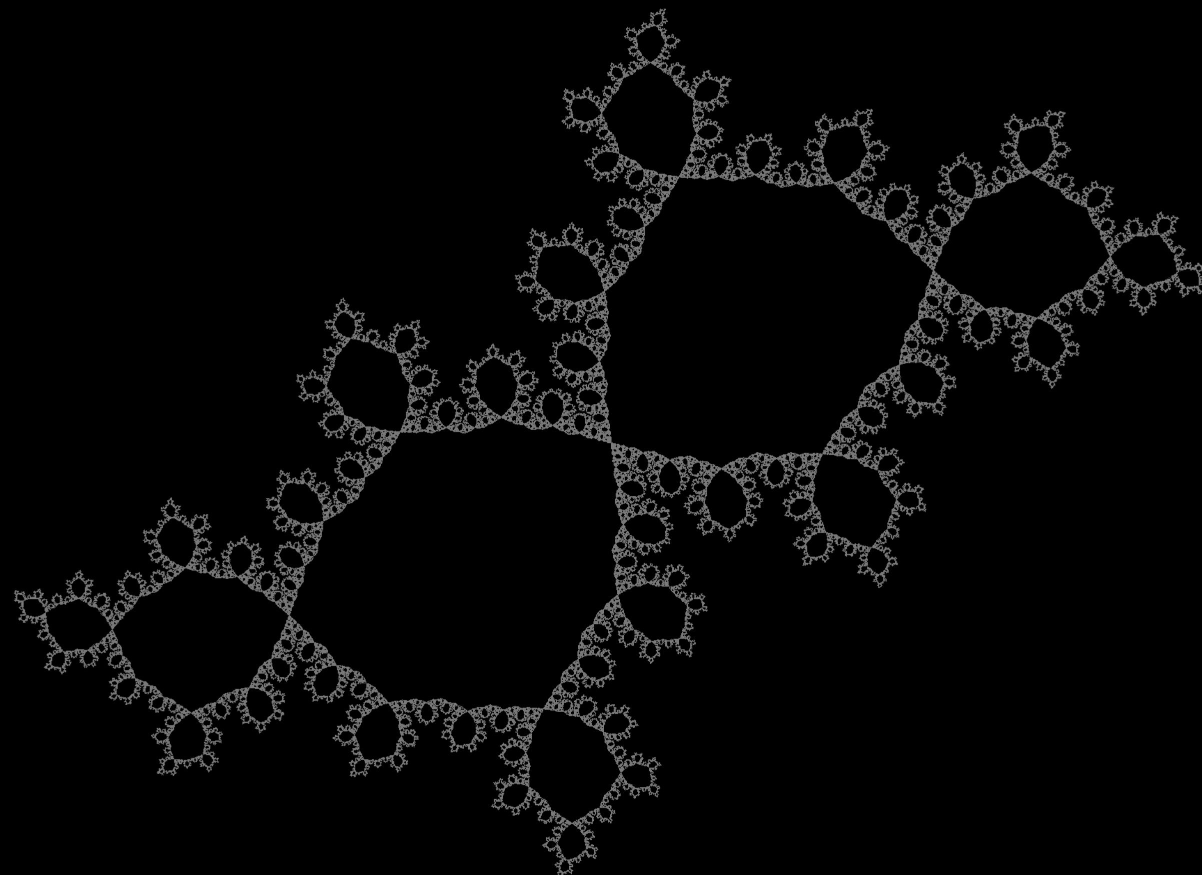
Iterating $z \rightarrow z^2 + 0.29 + 0.005\, i$



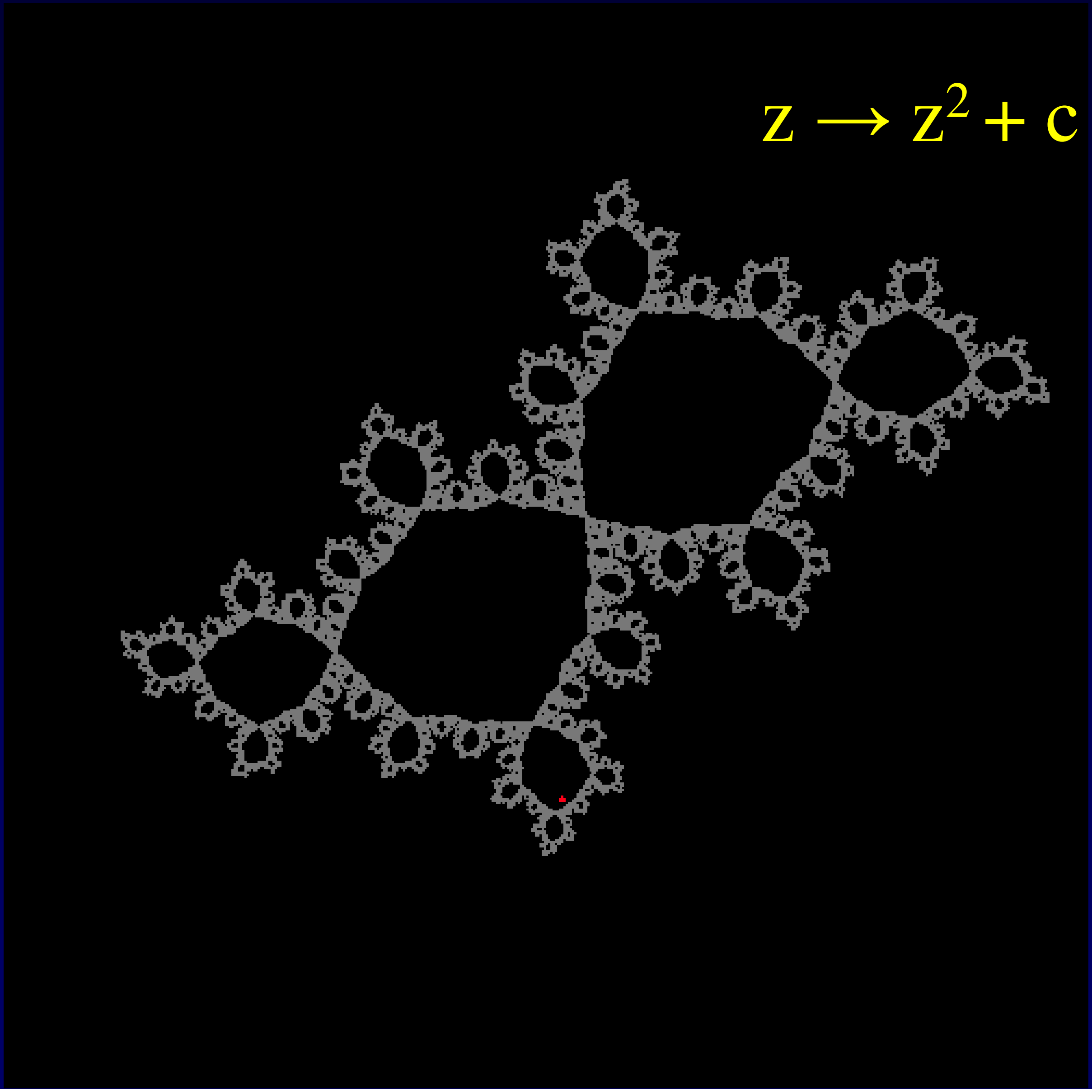
Iterating $z \rightarrow z^2 + 0.29 + 0.005\, i$



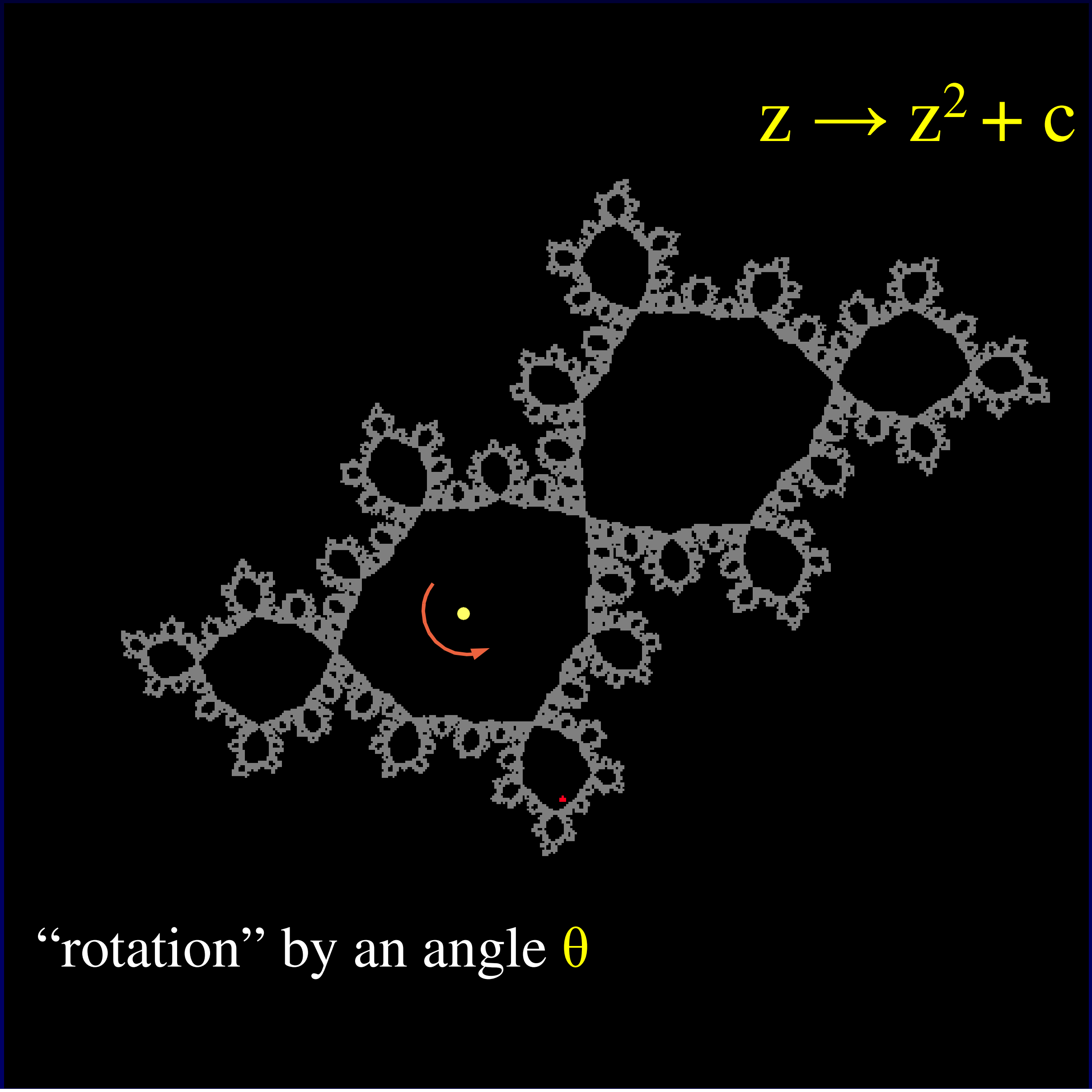
Example: c ≈ - 0.391 - 0.587 *i*



Iterating $z \to z^2 - 0.391 - 0.587\,i$



Iterating $z \rightarrow z^2 - 0.391 - 0.587\,i$

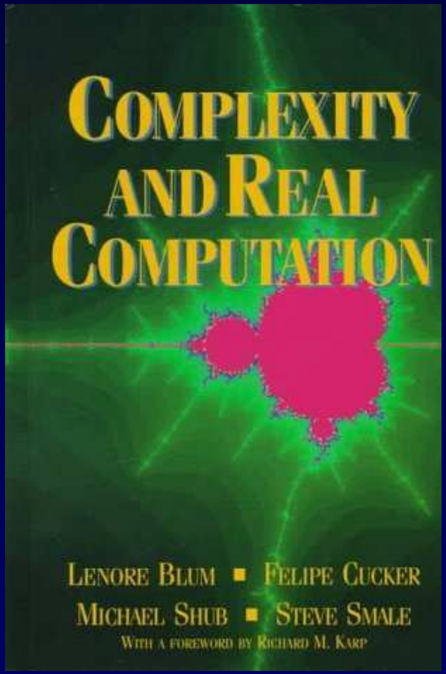


Computing K_c and J_c

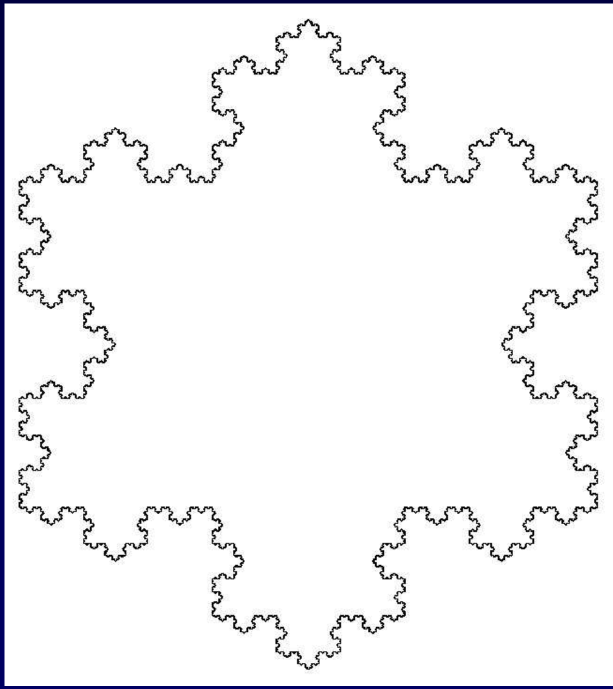
- Given the parameter c as an input, compute K_c and J_c.
- The parameter c describes the rule of the dynamics – “its world”.

The BSS model and Julia sets

- Model by [BlumShubSmale89].
- Use precise arithmetic machines with exact =, <, > and +,• to describe the set.
- Connects with algebraic geometry.
- Theorem [BCSS98]: The Mandelbrot set and almost all Julia sets are not BSS decidable.



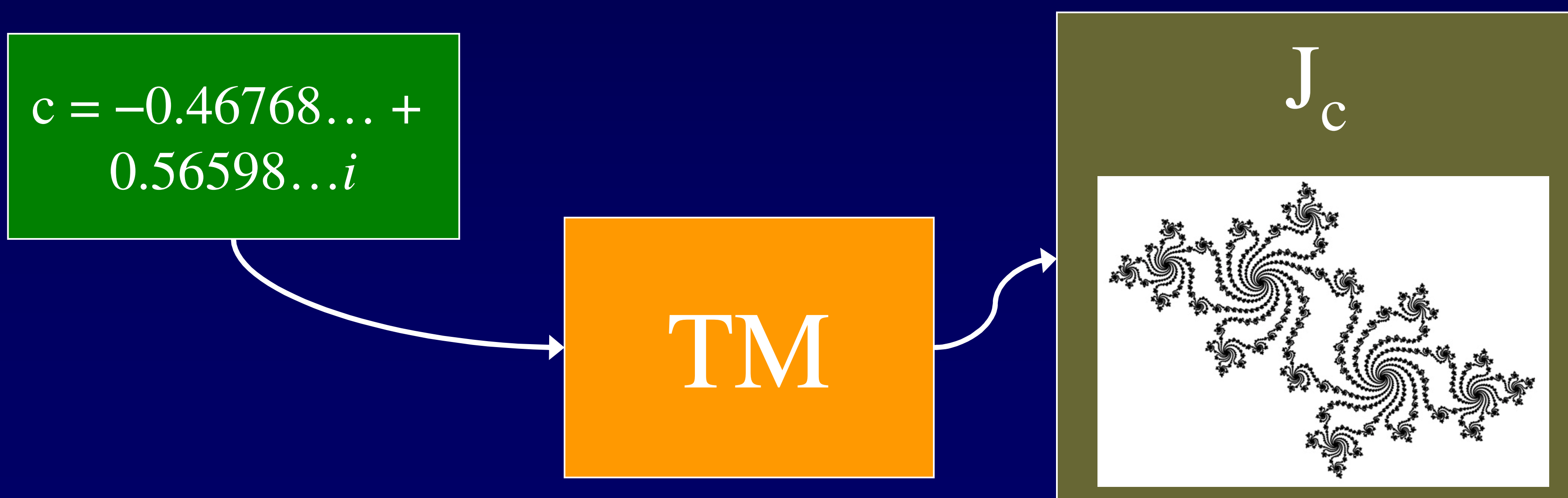
BSS model for sets?



- The graph of e^x on the [0,1] interval is not decidable in this model.
- Koch snowflake, having fractional Hausdorff dimension of $\log_3 4$, is not computable in this model.
- If we want to discuss computability of non-algebraically structured sets, need to make modifications.
- Once reasonable modifications are made, the BSS model becomes equivalent to Computable Analysis – the model that we use.

Computability model

- We use the Computable Analysis notion, which accounts for the cost of the operations on a Turing Machine.



Input – giving c to TM

- The input c is given by an oracle φ(m).
  - On query m the oracle outputs a rational approximation of c within an error of 2^{-m}.
- TM is allowed to query c with any *finite* precision.

$c = -0.46768\ldots + 0.56598\ldots i$

TM

Input – giving c to TM

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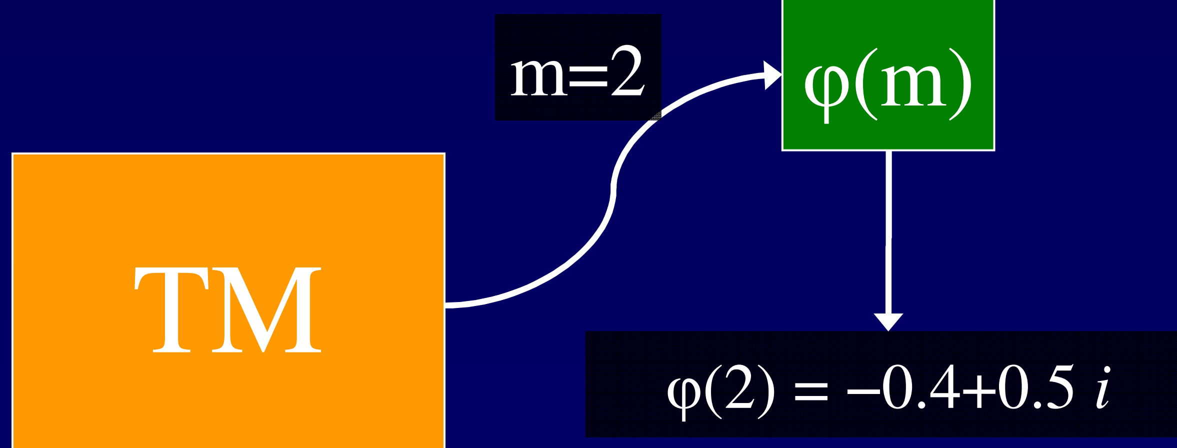

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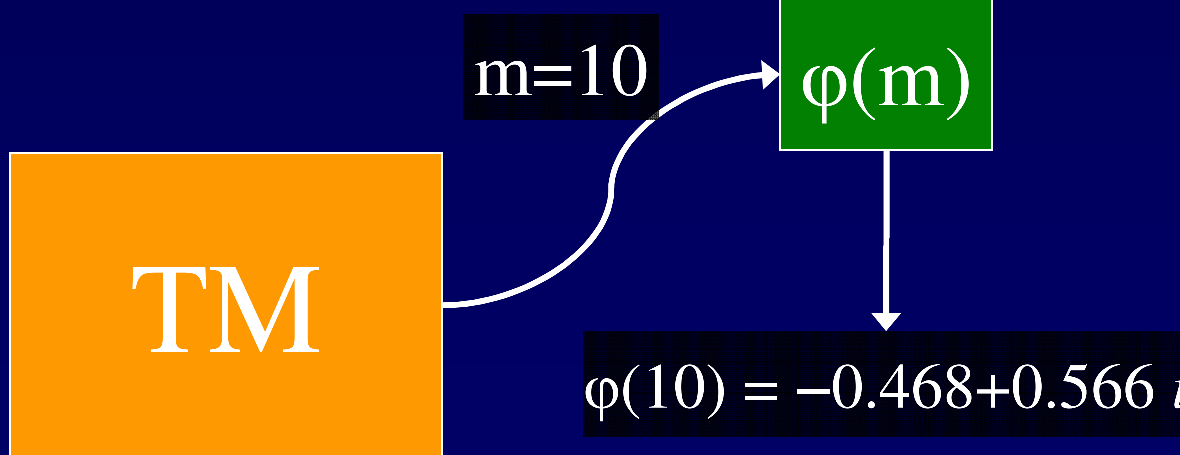

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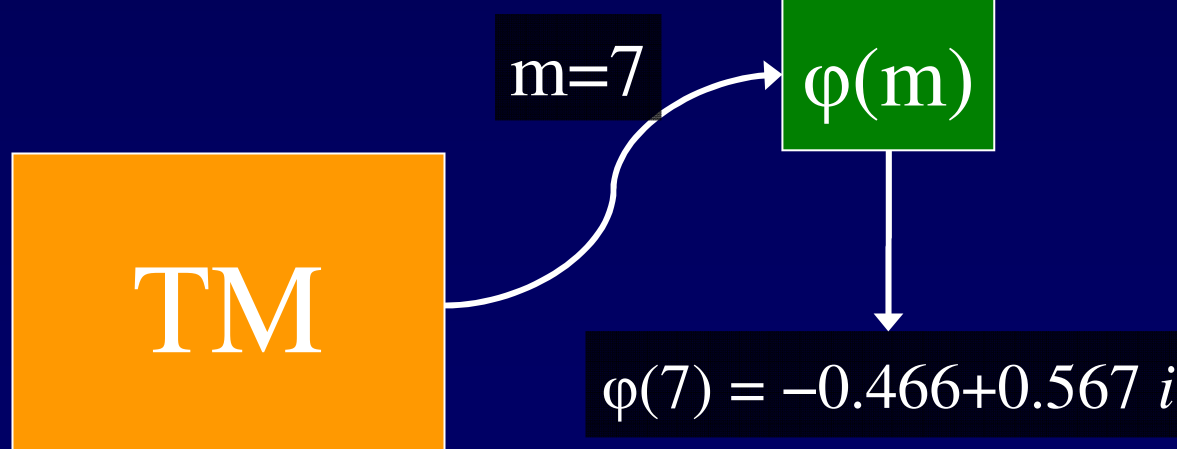
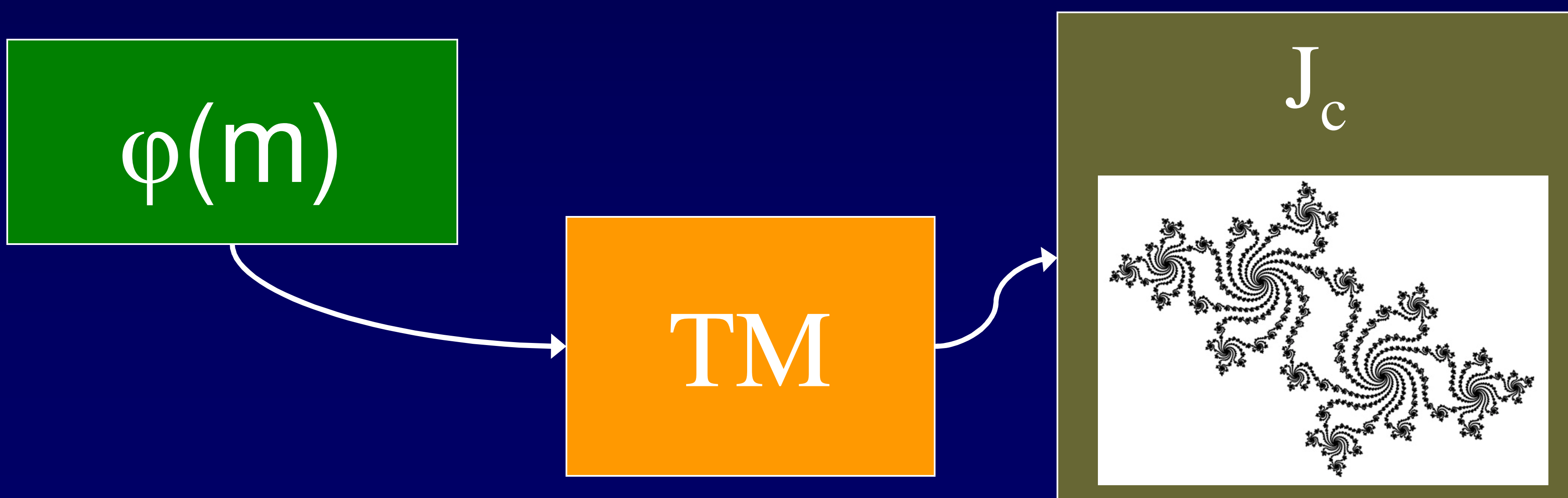


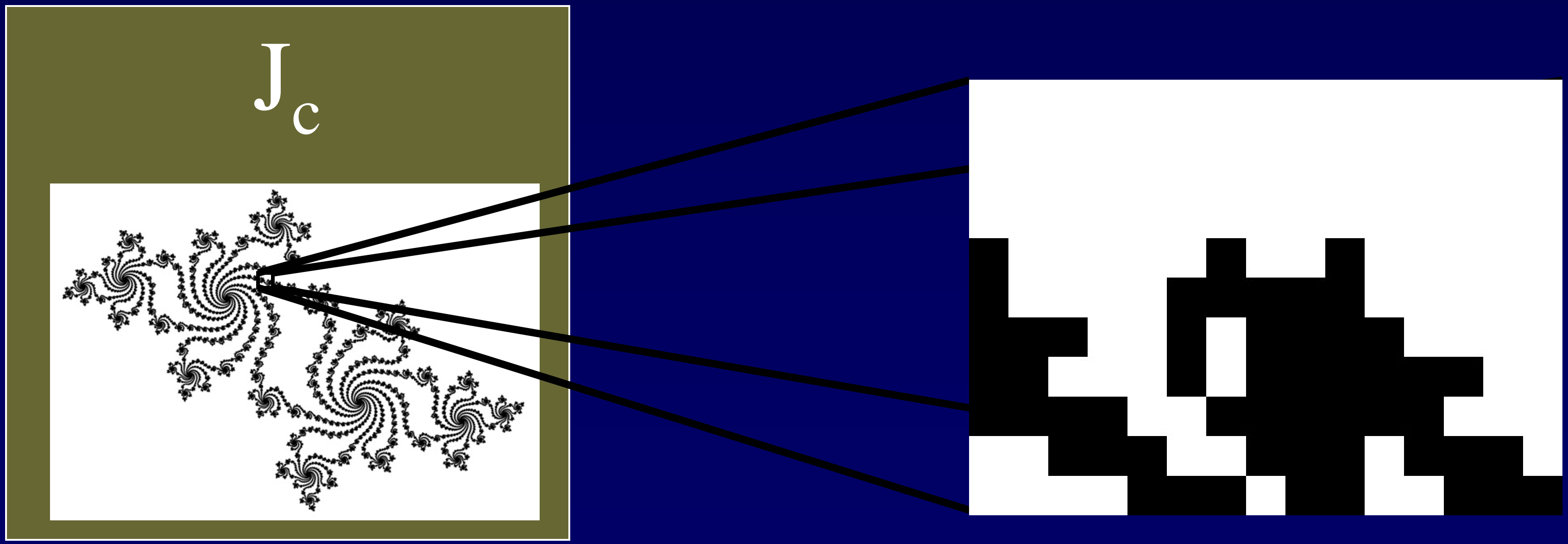
Output

- Given a precision parameter n, TM needs to output a 2^{-n}-approximation of J_c, which is a "picture" of the set.

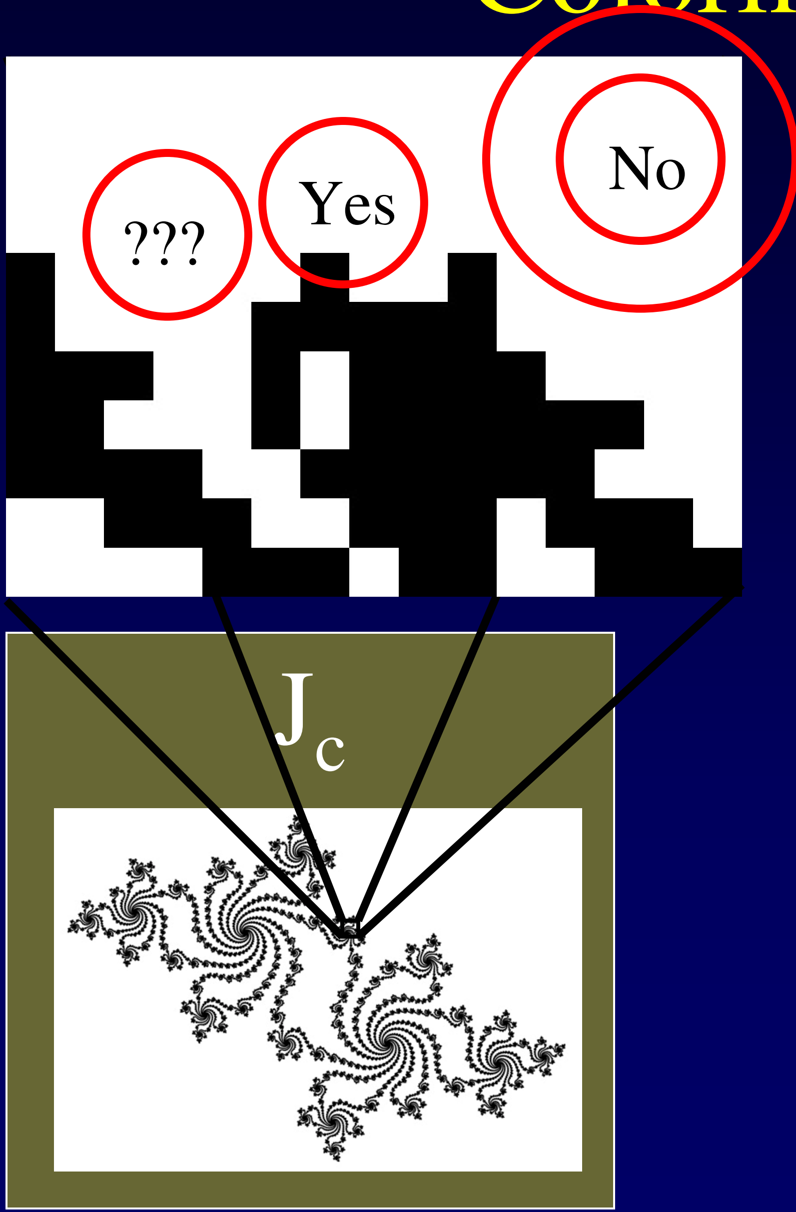


Output

- A 2^{-n}-approximation of J_c, is made of pixels of size $\approx 2^{-n}$.
- For each pixel, need to decide whether to paint it white or black.



Coloring a Pixel



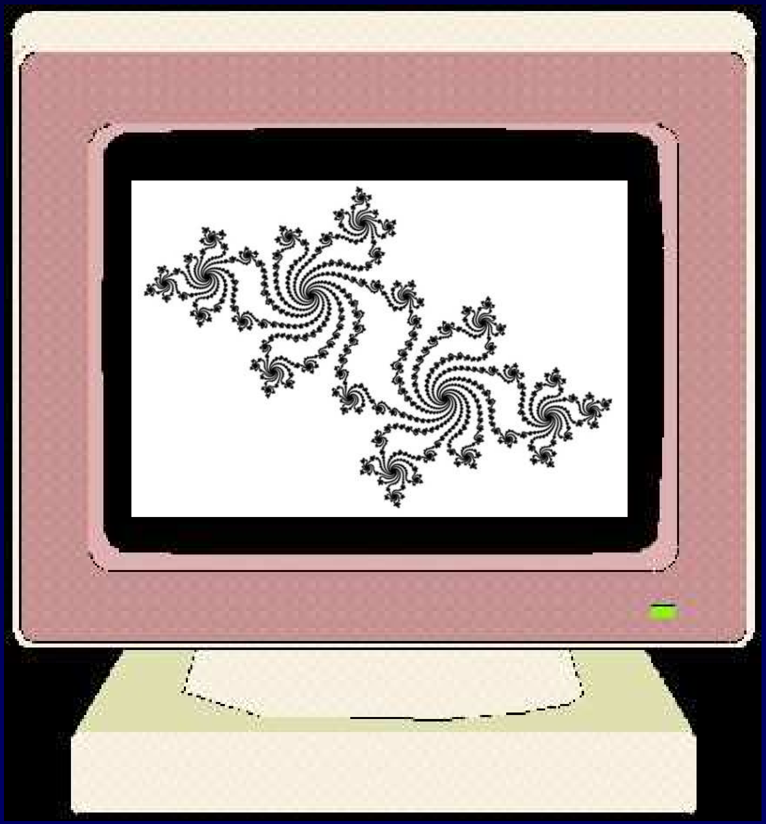
- We use round pixels – equivalent up to a constant.
- A pixel is a circle of radius 2^{-n} with a rational center.
- Put it in if it intersects J_c.
- If twice the pixel does not intersect J_c, leave it out.
- Otherwise, don't care.

$$f(q,n)=\begin{cases}1, & \text{if } B(q,2^{-n})\cap J_c \neq \varnothing \\ 0, & \text{if } B(q,2\cdot 2^{-n})\cap J_c = \varnothing \\ 0 \text{ or } 1 & \text{otherwise}\end{cases}$$

Complexity of real sets

- The time complexity $T_c(n)$ of computing J_c is defined as the *worst-case* time required to evaluate $f(q,n)$.
- Queries $\varphi(m)$ to the oracle are charged m time units.
- $T_c(n)$ measures the computational cost of zooming into J_c.

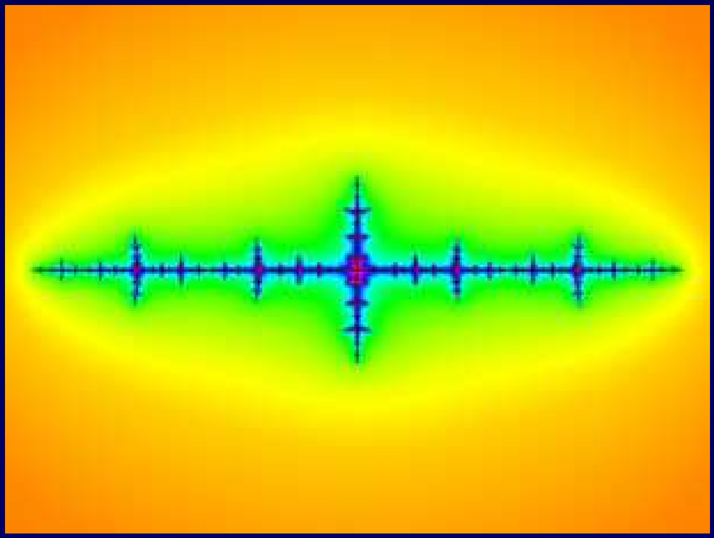
Cost of zooming in

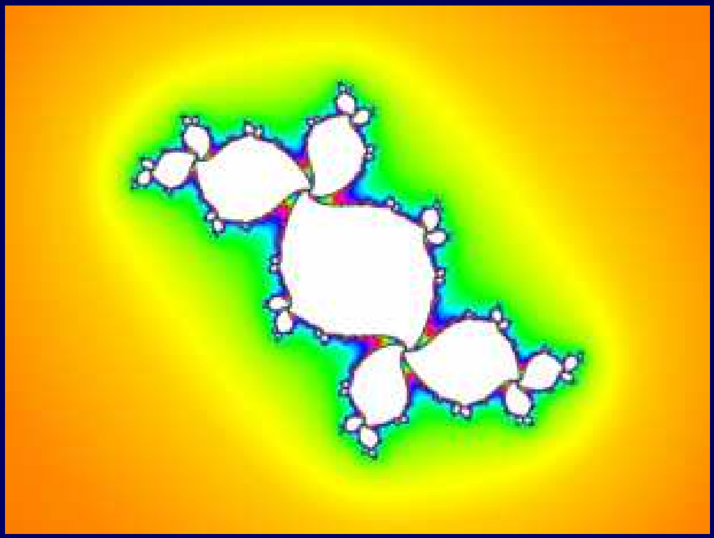


- Drawing a portion of J_c with 2^n-zoom-in on a 1000x1000 pixel display, requires $O(10^6 \cdot T_c(n))$ time, for any n.

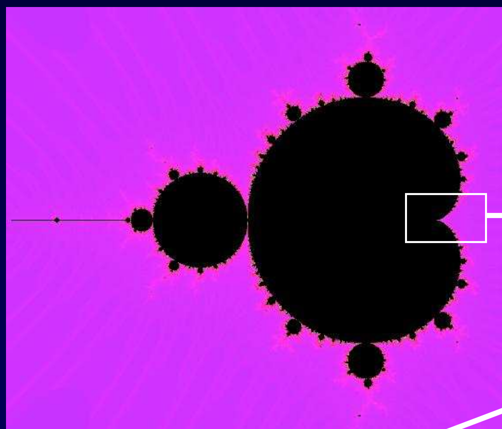
Computability and Complexity of Julia Sets

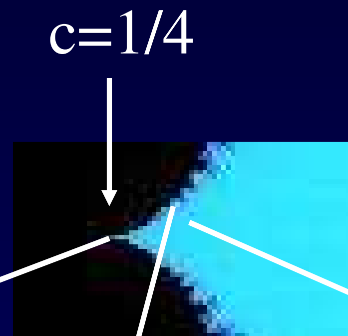
- Now that we have the model, we would like to address computational questions about Julia sets.
- Which Julia sets can be computed and how efficiently?



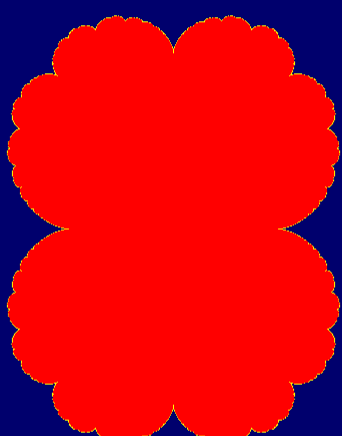


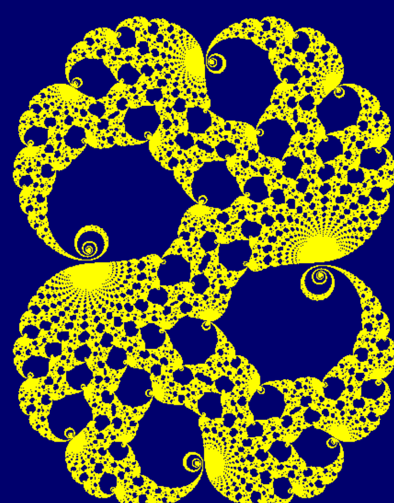
Discontinuity of J near c=1/4.

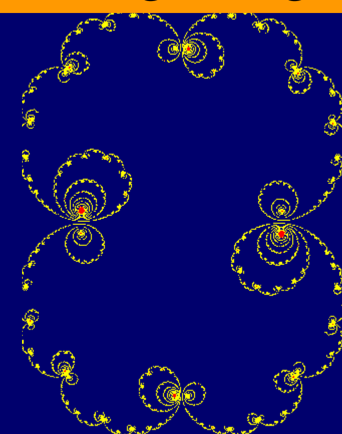




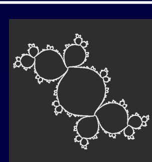
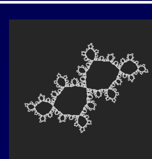
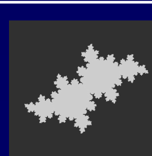
No hope of uniform computability!
(=computability by a single algorithm)







Summary

Type	Empirical and prior work	New
Hyperbolic 	empirically easy; some shown in poly-time	poly-time computable
Parabolic 	empirically computable (exp-time)	poly-time computable
Siegel 	empirically computable in many cases	some are computable some are not
Cremer 	no useful pictures to date	computable
Filled Julia set K_c 	thought to be tightly linked to J_c	always computable

Types of Julia sets

J_c

0 escapes to ∞?

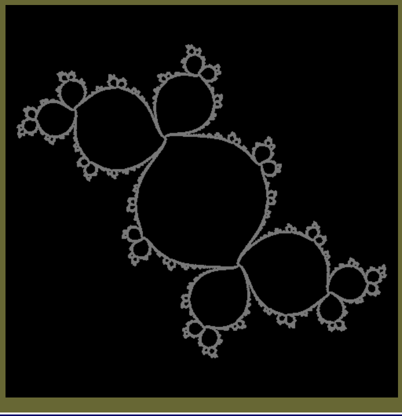
no

yes

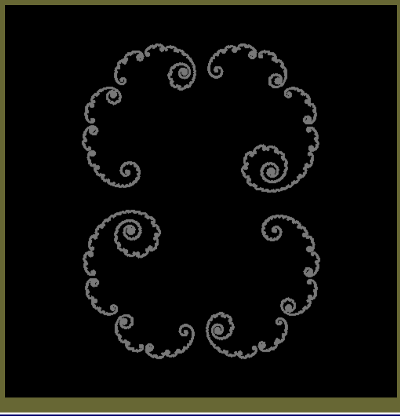
$\equiv$ c not in M?

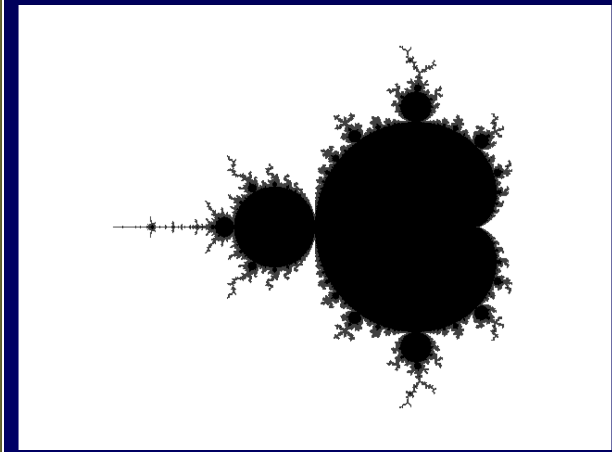
M = the Mandelbrot set

connected

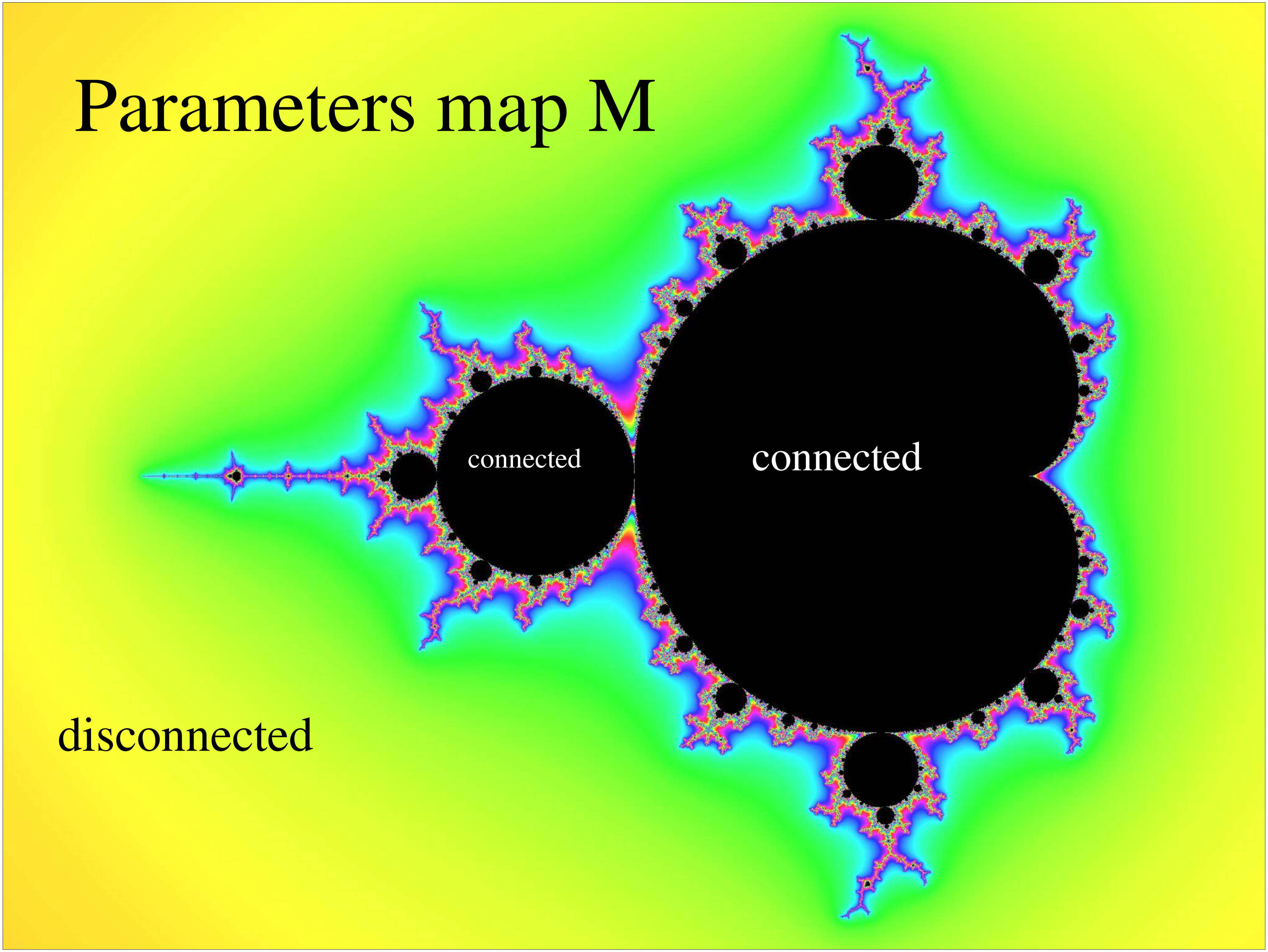


disconnected

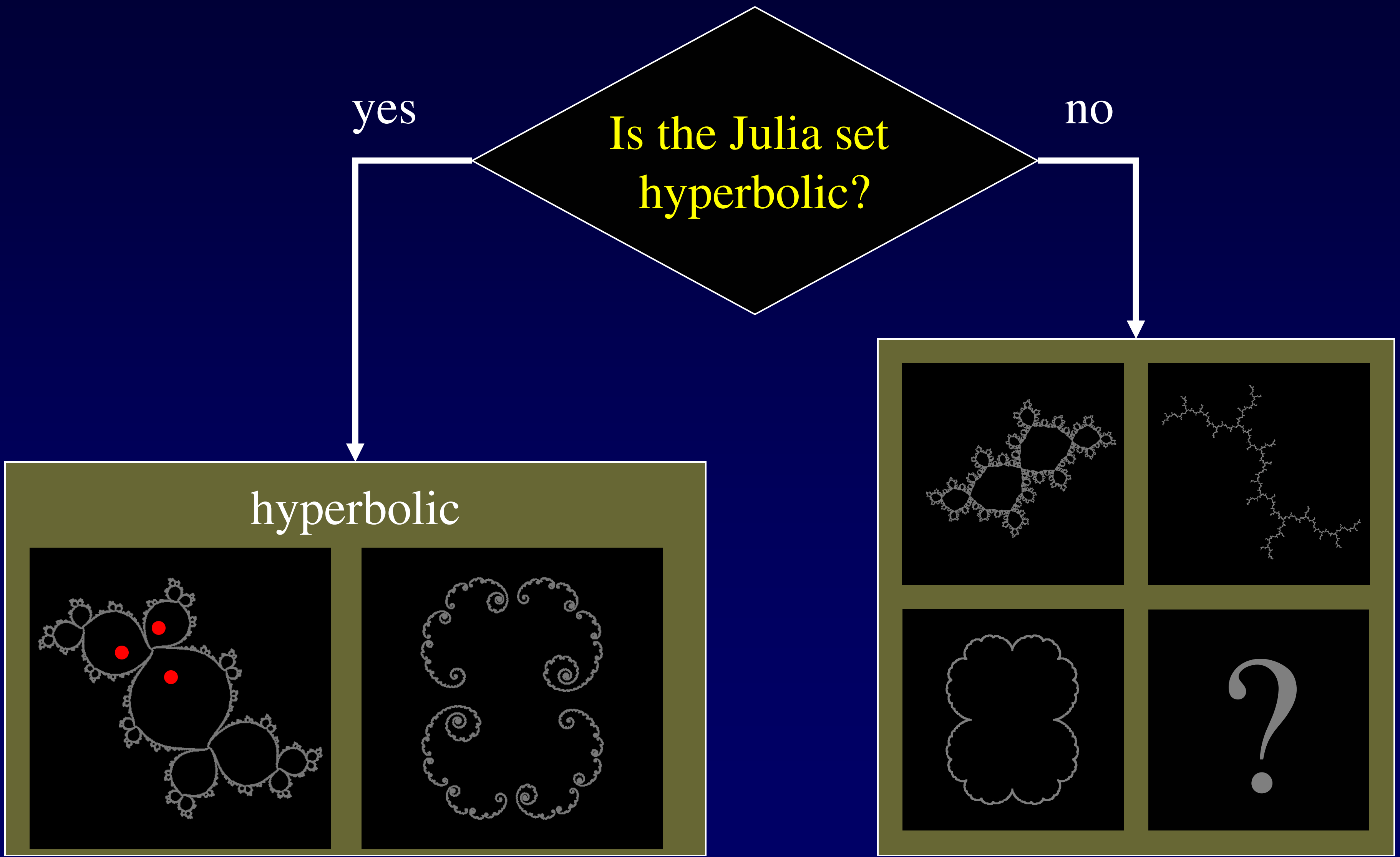


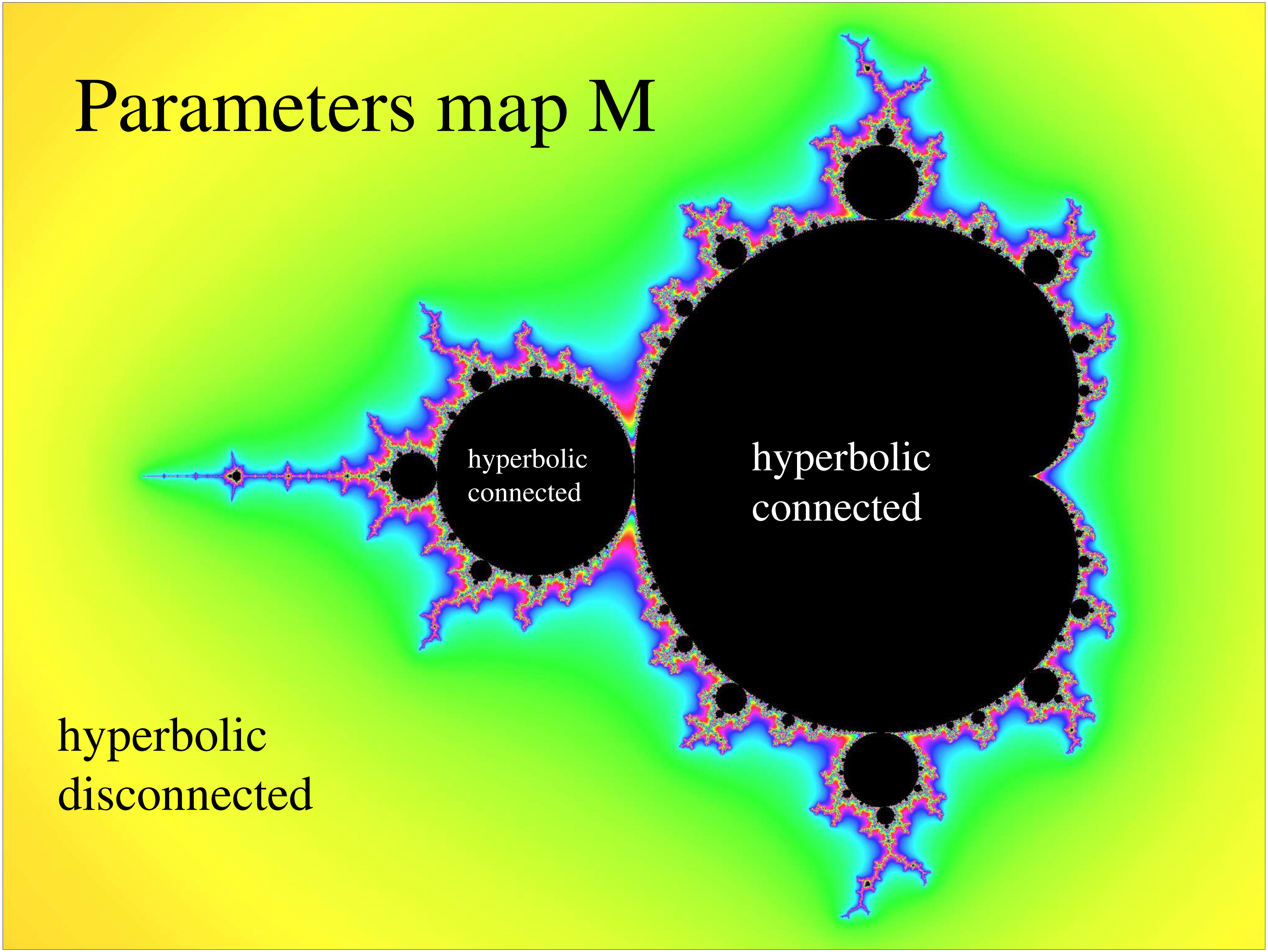


Parameters map M



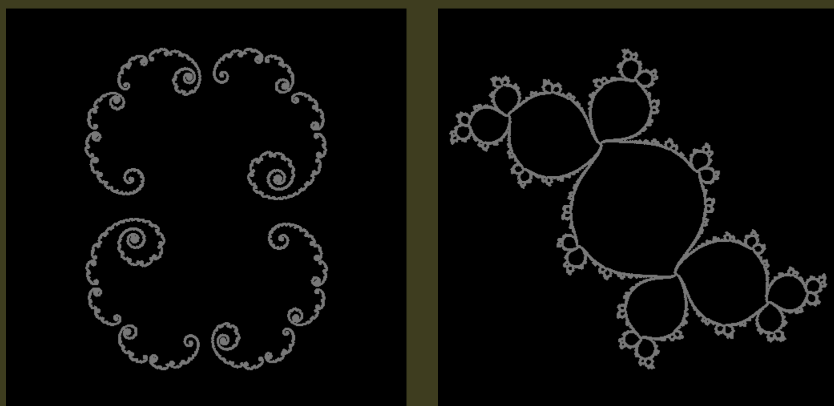
Types of Julia sets




Parameters map M
hyperbolic
connected
hyperbolic
connected
hyperbolic
disconnected

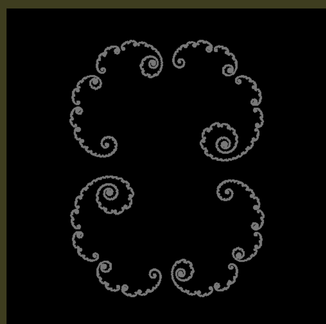
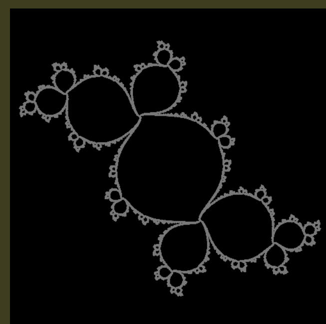
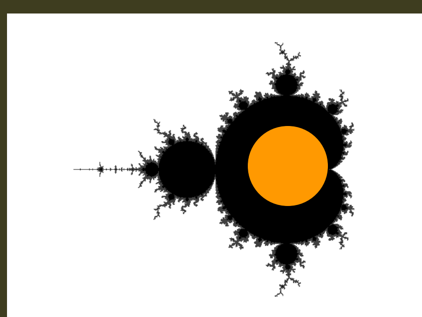
Prior work – empirical results

Hyperbolic Julia sets

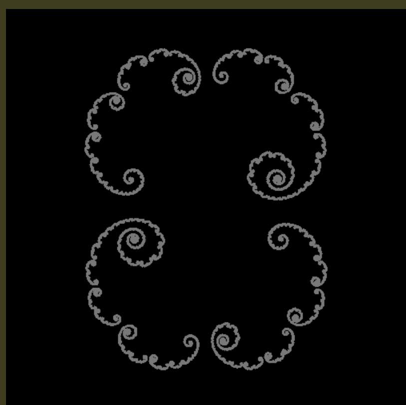
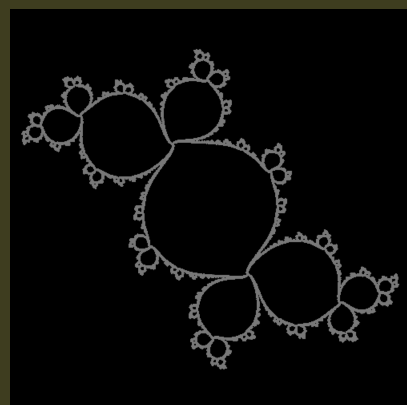


Very efficiently computable; many algorithms including Milnor's *Distance Estimator* [Fisher'88, Milnor'89, Peitgen'88]; many programs.

Prior work – formal results

Hyperbolic *polynomial* Julia sets  	Computable [Zhong*'98*]
Hyperbolic *quadratic* Julia sets with $\|c\|<1/4$ 	Poly-time computable [Rettinger,Weihrauch*'03*]

New Results – Positive

Hyperbolic Julia sets  	Poly-time computable. [B.'04]; [Rettinger'04].

Types of Julia sets

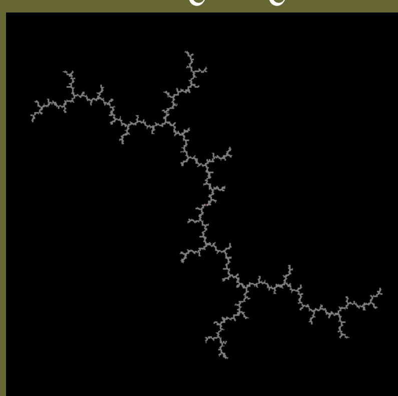
J_c is connected but not hyperbolic

empty interior? $K_c=J_c$?

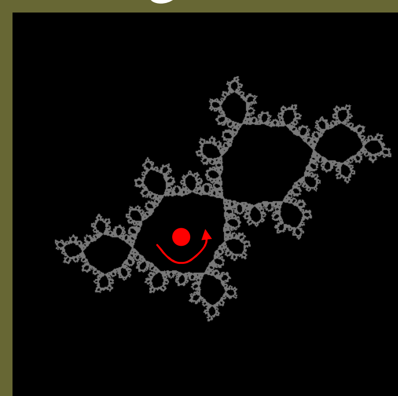
yes

no

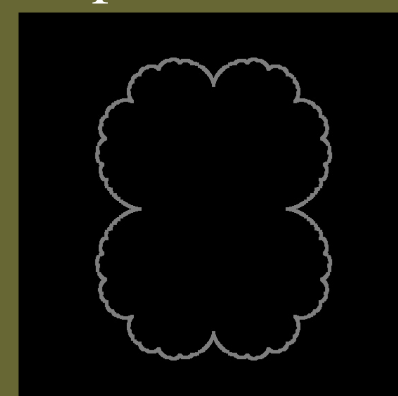
$K_c=J_c$



Siegel disc

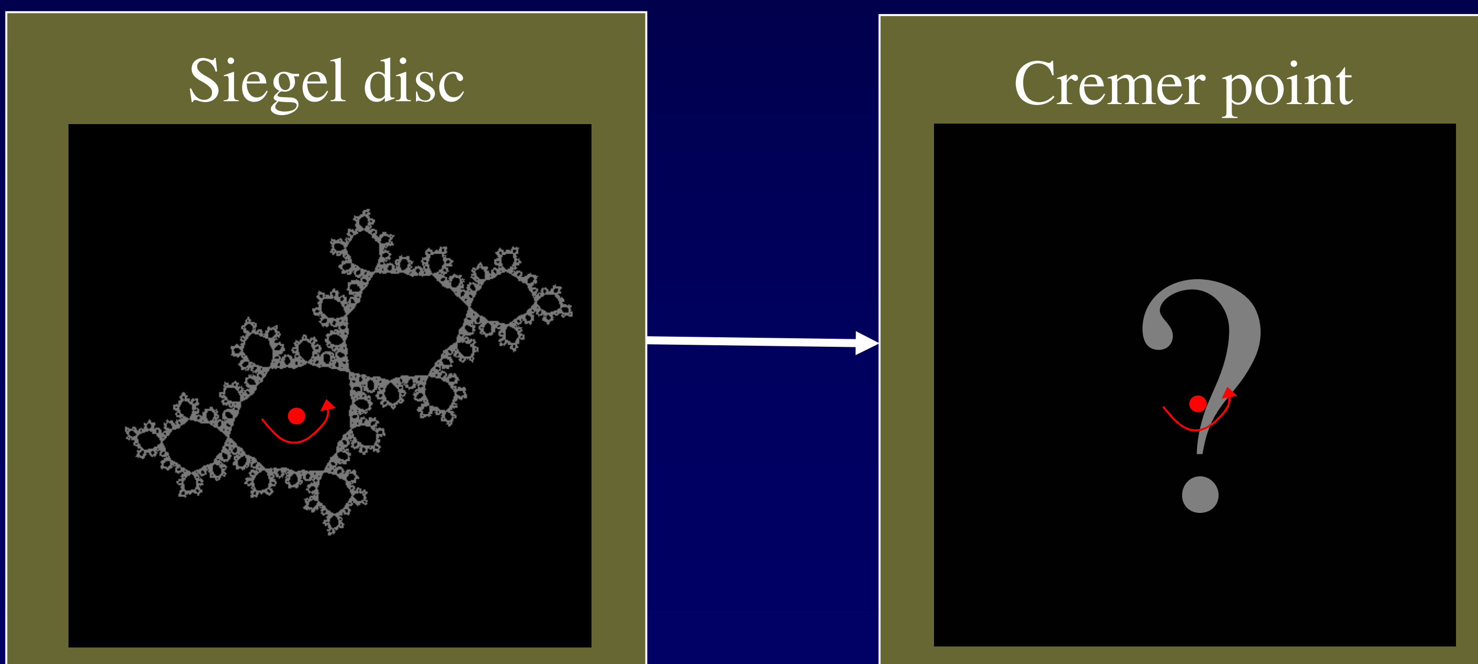


parabolic

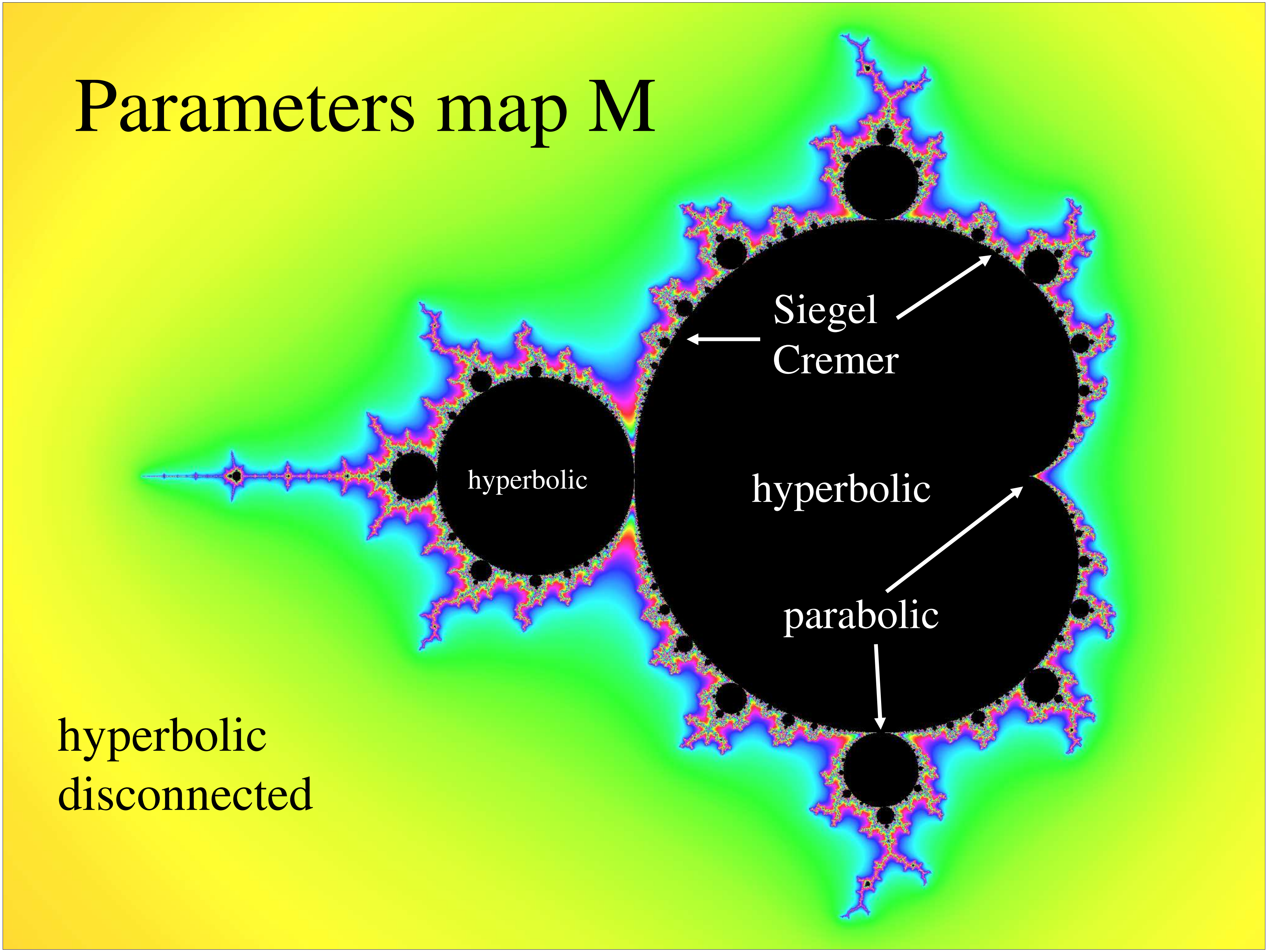


Cremer Julia sets

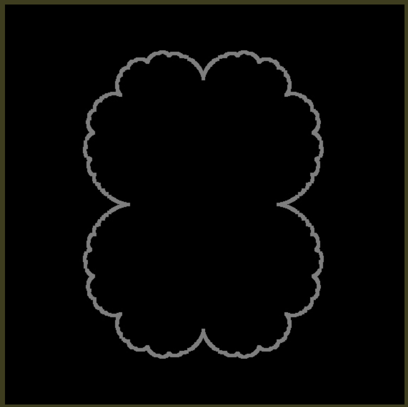
- A special case of $K_c=J_c$.
- A Siegel disc does not exist for all rotation angles θ.
- For some rotation angles the disc “disappears”.



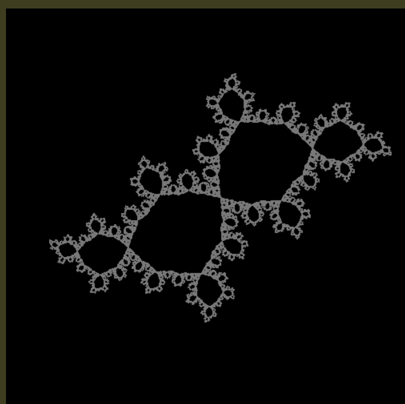
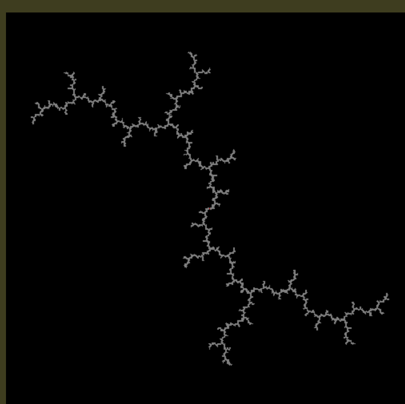
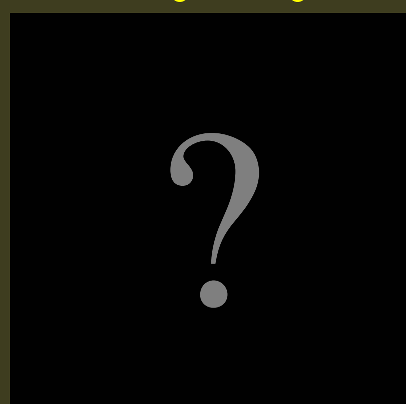
Parameters map M



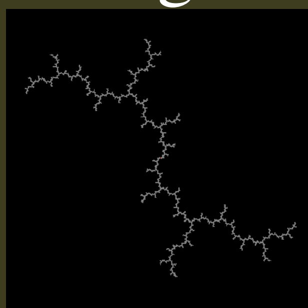
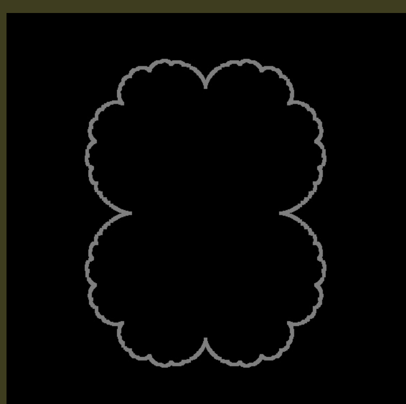
Prior work – empirical results

Parabolic Julia sets 	The *Distance Estimator* and other algorithms still work, but require exponential time. Still may be viable if we don't try to zoom into the set.

Prior work – empirical results

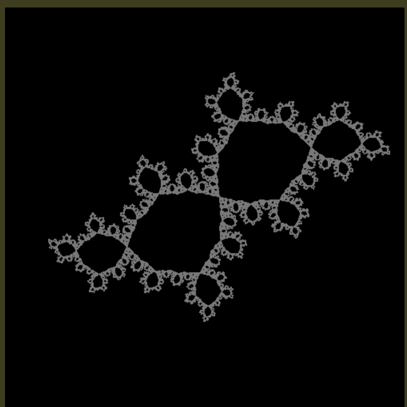
Julia sets with a Siegel disc 	For "good" parameters, pictures can be produced for practical purposes.
Connected J's with $J_c = K_c$  	Reasonable pictures in some cases. No useful pictures to date for Julia sets with Cremer points.

New Results – Positive

Connected J's with $J_c=K_c$, including Cremer Julia sets.  	Always computable. No running time guarantees. [Binder B. Yampolsky '*07*].
Parabolic Julia sets 	Poly-time computable. [B. '*06*] A possible building block for producing pictures of Cremer Julia sets.

New Results – Negative

Julia sets with a Siegel disc

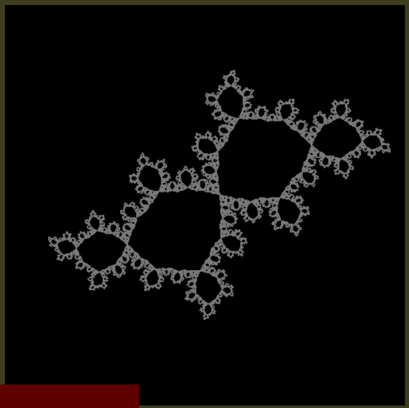


There *exist* non-computable Julia sets with a Siegel disc [B.Yampolsky *'06*]

Can construct computable Julia sets with a Siegel disc of an arbitrarily high computational complexity [Binder B.Yampolsky *'06*]

New Results – Negative

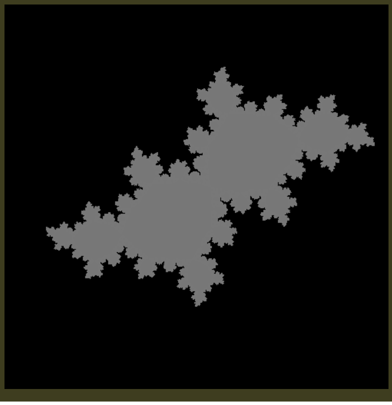
Julia sets with a Siegel disc



Can construct an explicit *computable* parameter c such that computing J_c is as hard as solving the Halting Problem. [B.Yampolsky *'07*]

In contrast:

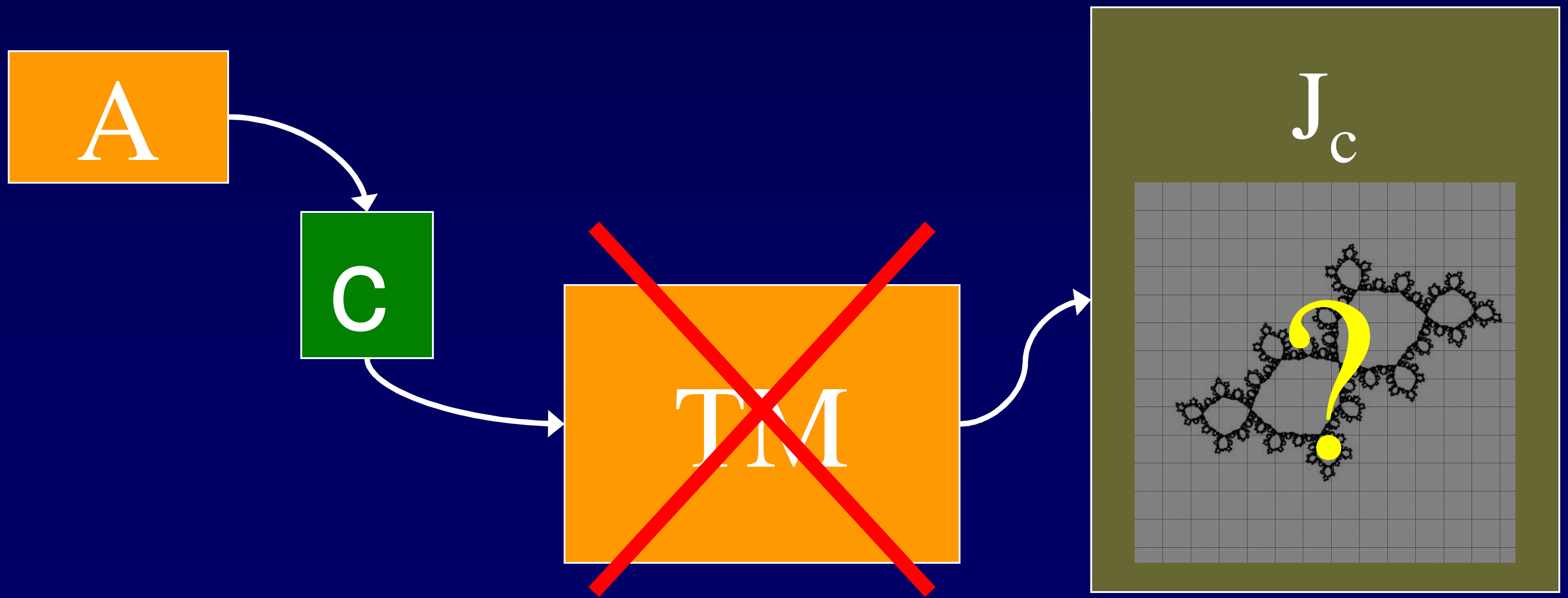
Filled Julia sets



Theorem [B. Yampolsky*'07*]
The *filled* Julia set K_c is always (non-uniformly) computable.

Theorem [BY07]: There is an algorithm A that computes a number c such that no machine with access to c can compute J_c.

- Under a reasonable conjecture from Complex Dynamics, c can be made *poly-time* computable.



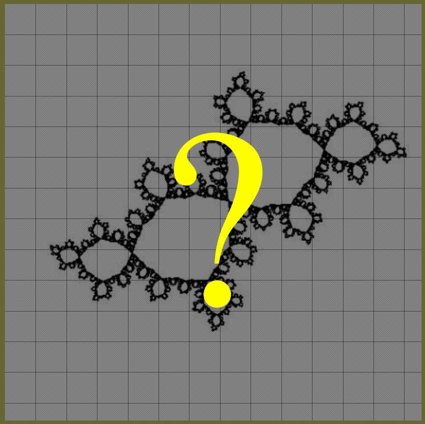
A

the rotation angle $2\pi\theta$ of the Siegel disc

c such that J_c has a Siegel disc

~~TM~~

J_c



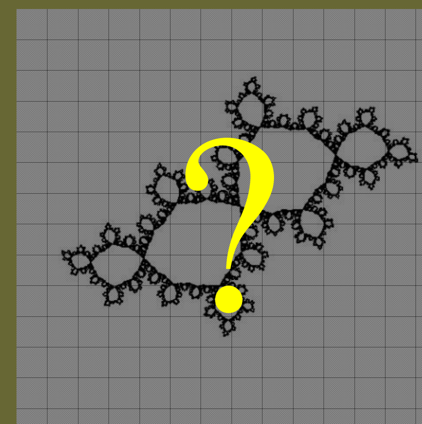
A

the rotation angle $2\pi\theta$ of the Siegel disc

c such that J_c has a Siegel disc

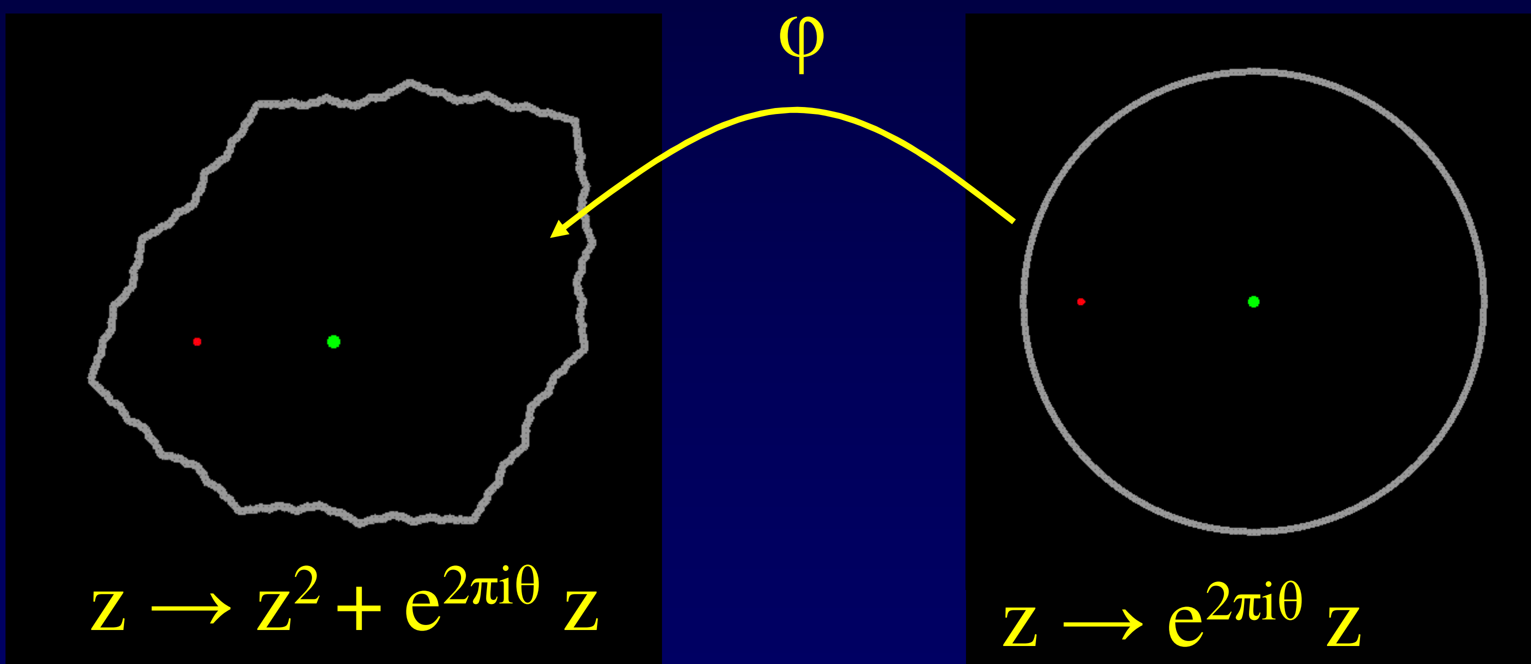
~~TM~~

J_c



Conjugating to rotation

- The conformal Riemann map φ from the unit disc to the Siegel disc Δ_θ conjugates f_θ to an actual rotation.



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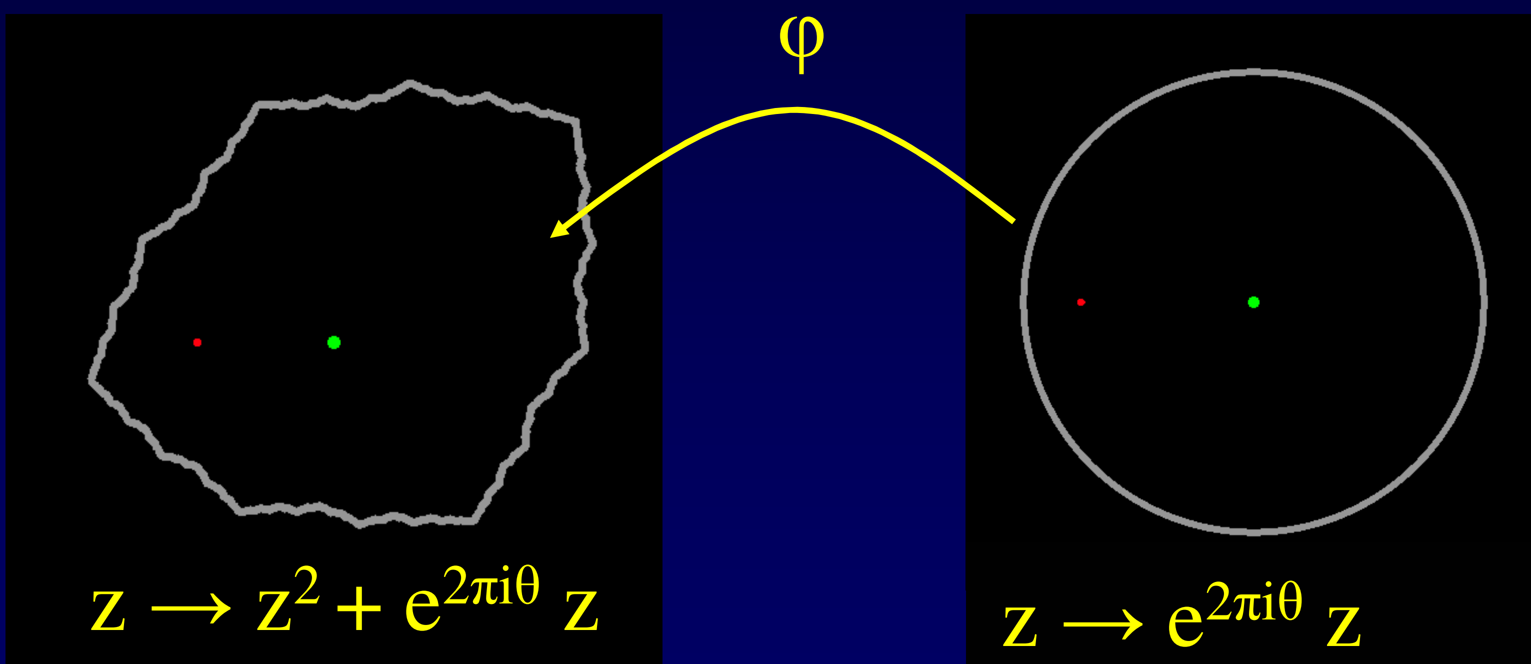
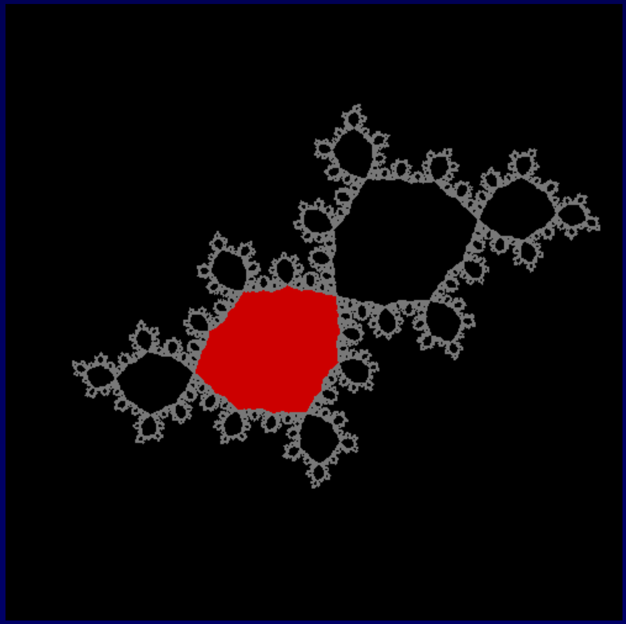


- $r(\theta) := |\varphi'(0)|$ is the *conformal radius* of Δ_θ.

Conformal radius

- The conformal radius $r(\theta)$ measures the size of the Siegel disc Δ_θ.
- Theorem [BBY'05]: A Julia set J_c with a Siegel disc Δ_θ is computable iff $r(\theta)$ is computable.



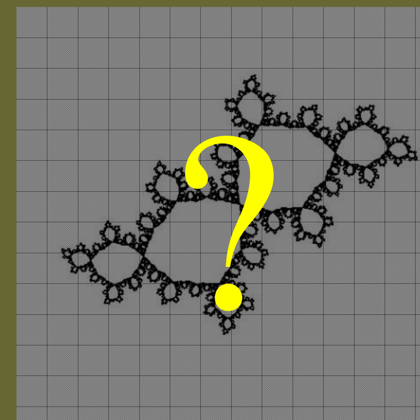
A

the rotation angle $2\pi\theta$ of the Siegel disc

~~TM~~

$r(\theta)$

J_c



A

the rotation angle $2\pi\theta$ of the Siegel disc

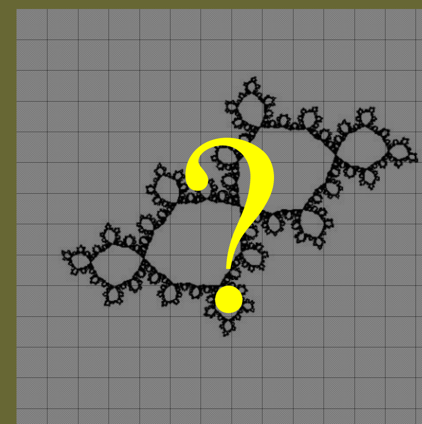
algebraic

~~TM~~

$r(\theta)$

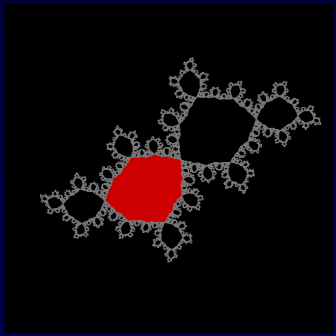
geometric

J_c



?

Proving the non-computability theorem



- Consider the family $z \to z^2 + e^{2\pi i\theta} z$.
- When is there a Siegel disc?
- <u>Theorem</u> [Brjuno'65]: When the function

$$\Phi(\theta) = \sum_{n=1}^{\infty} \theta_1\theta_2 \ldots \theta_{n-1} \log \frac{1}{\theta_n}; \;\; \theta_1 = \theta, \;\; \theta_{i+1} = \left\{ \frac{1}{\theta_i} \right\}$$

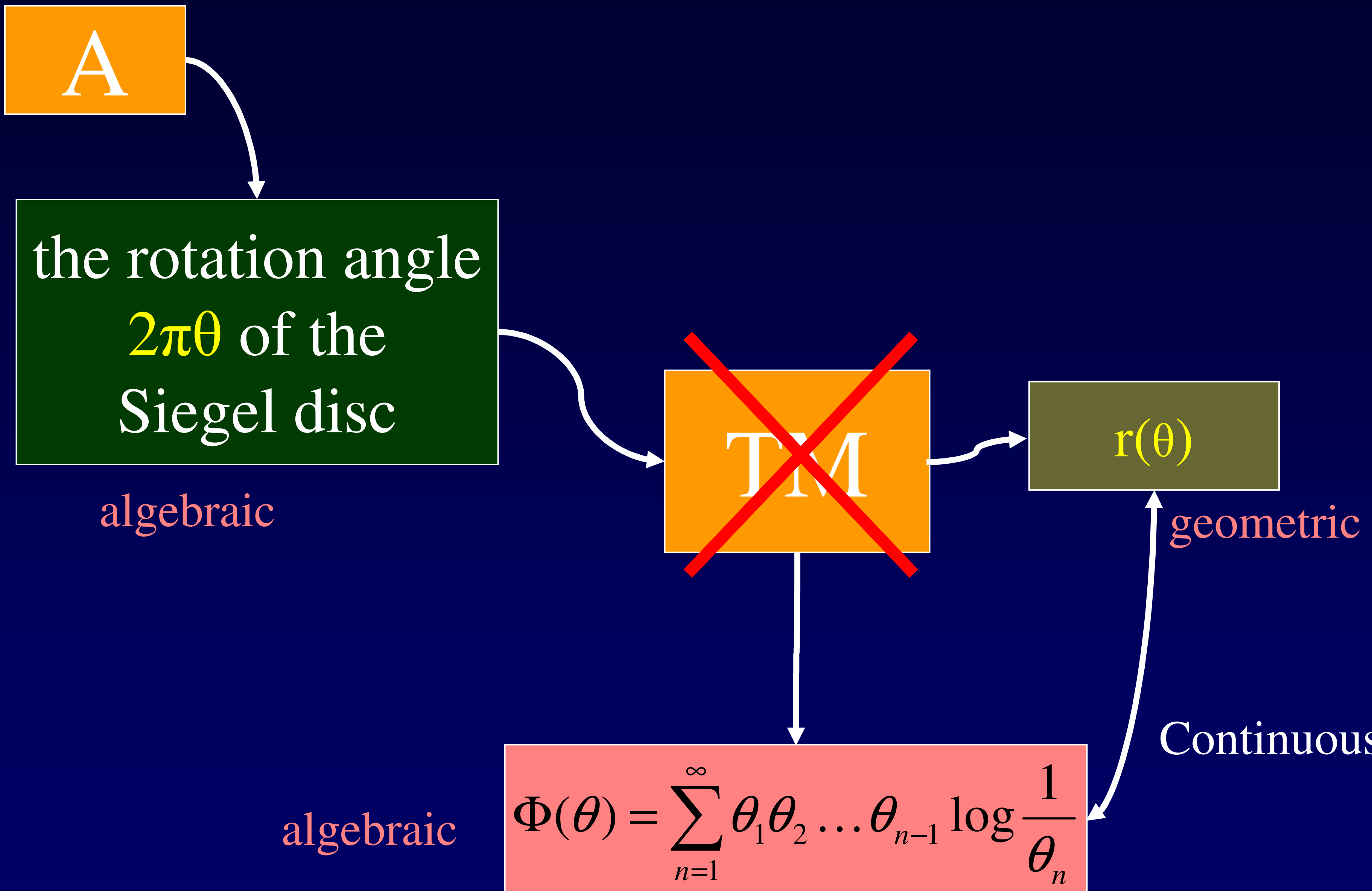
converges.

Geometric Meaning of $\Phi(\theta)$

$$\Phi(\theta) = \sum_{n=1}^{\infty} \theta_1 \theta_2 \ldots \theta_{n-1} \log \frac{1}{\theta_n}; \;\; \theta_1 = \theta, \;\; \theta_{i+1} = \left\{ \frac{1}{\theta_i} \right\}$$

- Theorem [Yoccoz'88], [Buff,Cheritat'03]: The function $\Phi(\theta) + \log r(\theta)$ is continuous.
- In particular, when $\Phi(\theta)=\infty$, $r(\theta)=0$.

- Theorem [BY'07]: There is an explicit poly-time algorithm that generates a θ such that $\Phi(\theta)$ is as hard to compute as the Halting Problem.


A
the rotation angle $2\pi\theta$ of the Siegel disc
algebraic
TM
$r(\theta)$
geometric
Continuous
algebraic
$\Phi(\theta) = \sum_{n=1}^{\infty} \theta_1 \theta_2 \ldots \theta_{n-1} \log \frac{1}{\theta_n}$

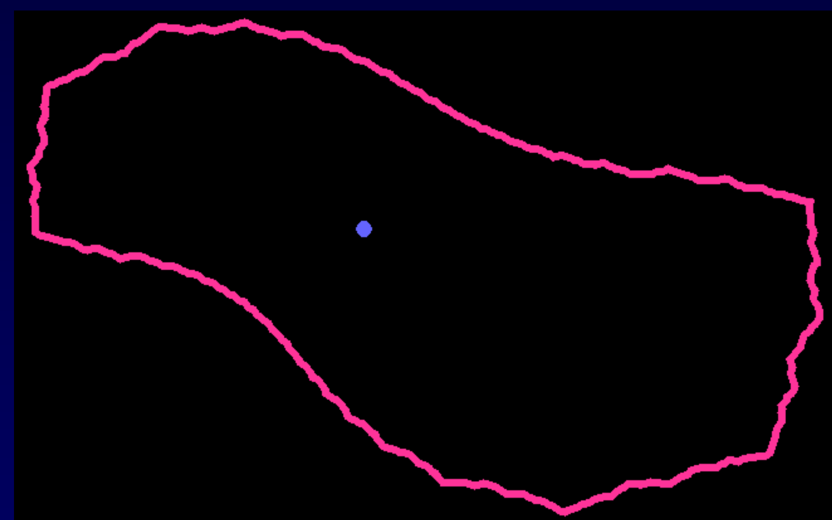
Controlling r(q) through F (q)

- The key idea in the non-computability proof is that we can drop the value of r(q) by a prescribed amount a < r(q) while changing q by no more than a given ¶ > 0.
- When q tends to *any* rational number, r(q) tends to 0.
- Can *carefully* approach a rational with an arbitrarily small change.
- F (q) is used to show that the argument works.

Controlling r(θ) in pictures

- $\theta_1 = [1,1,20,1,1,1,1,\ldots] =$

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{20+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}}}}}$$



$\approx$ 0.511838

Controlling $r(\theta)$ in pictures

- $\theta_2(N) = [1,1,20,1,1,N,1,\ldots] =$

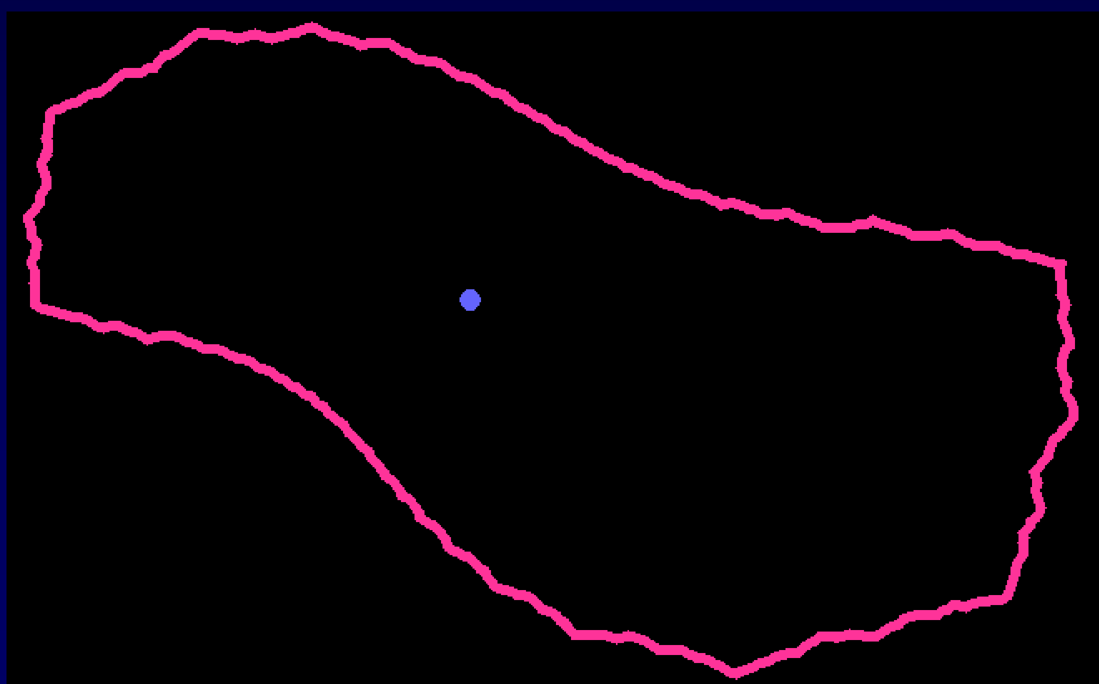
$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{20+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{N+\cfrac{1}{1+\cfrac{1}{\ddots}}}}}}}}$$

Change in θ small, but can implement any drop in $r(\theta)$.

$\theta_1 \approx 0.511\underline{838} < \theta_2(N) < 0.511\underline{905}$

Controlling r(q) in pictures

N=1

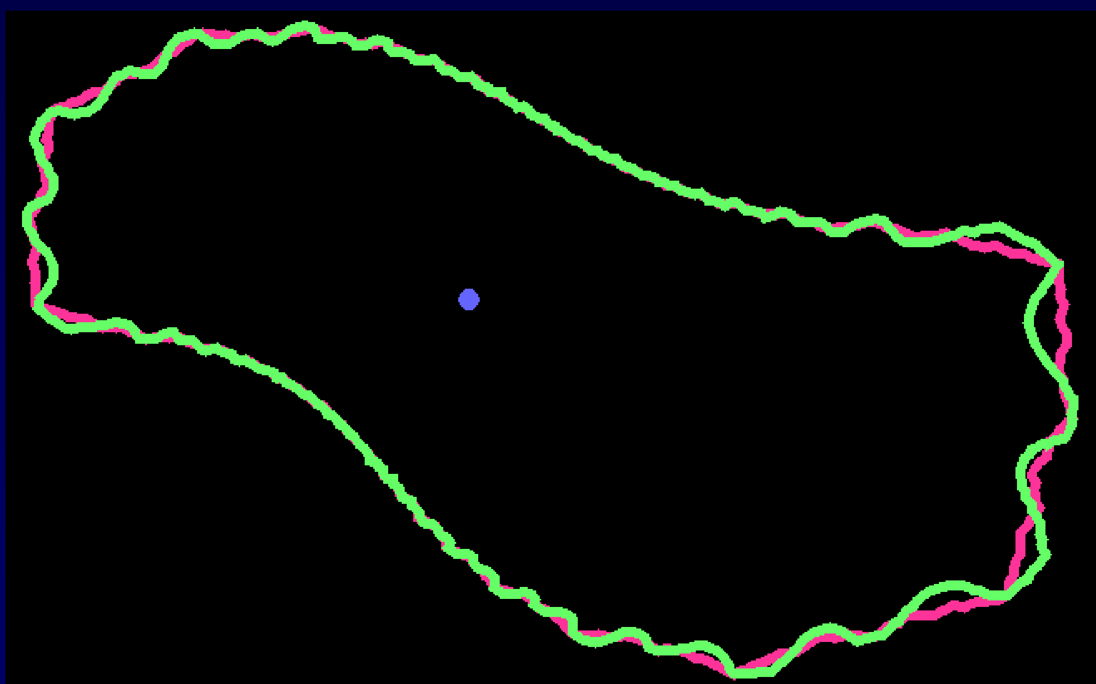


$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{20+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{N+\cfrac{1}{1+\ddots}}}}}}}$$

Controlling $r(\theta)$ in pictures

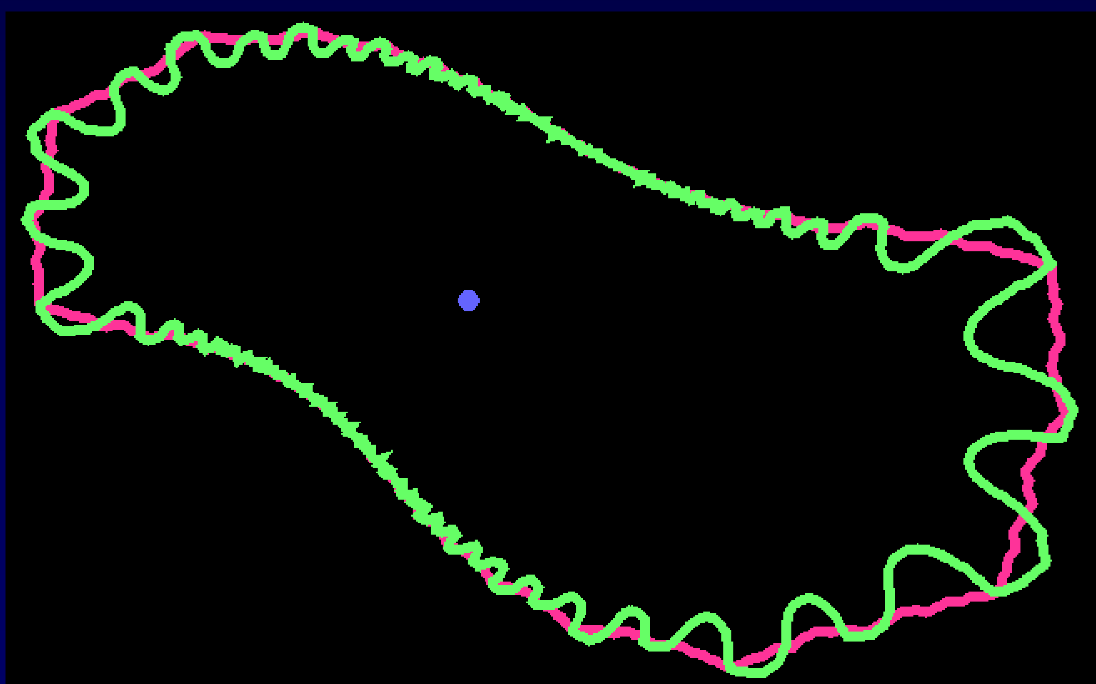
N=10

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{20+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{N+\cfrac{1}{1+\ddots}}}}}}}$$



Controlling $r(\theta)$ in pictures

N=100

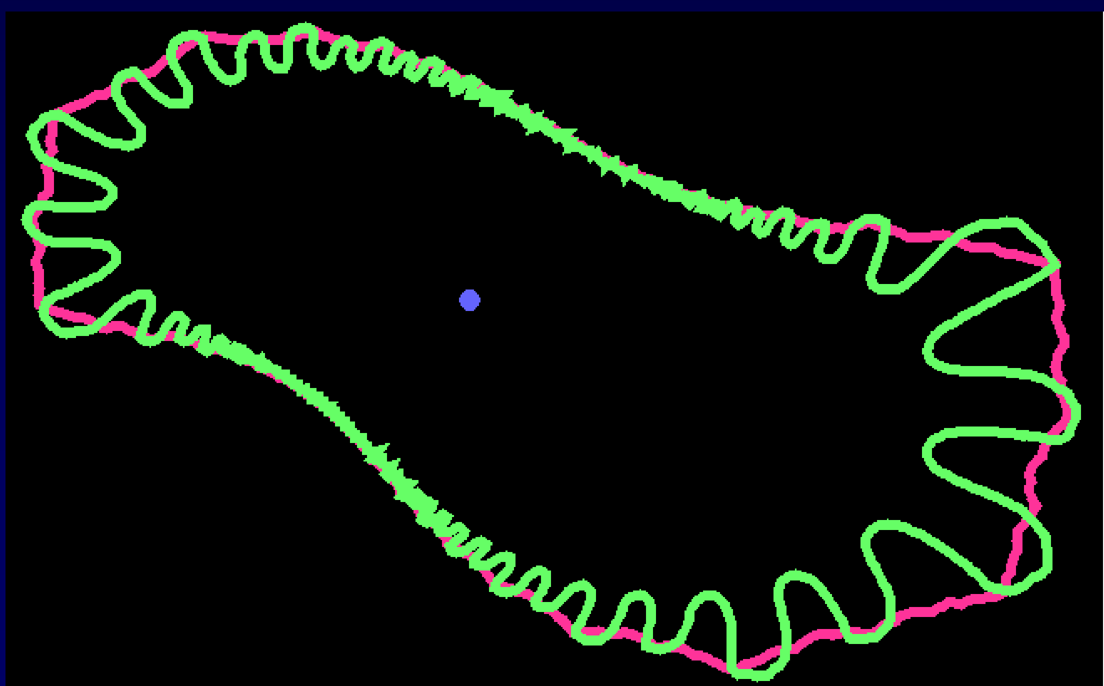


$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{20+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{N+\cfrac{1}{1+\ddots}}}}}}}$$

Controlling r(q) in pictures

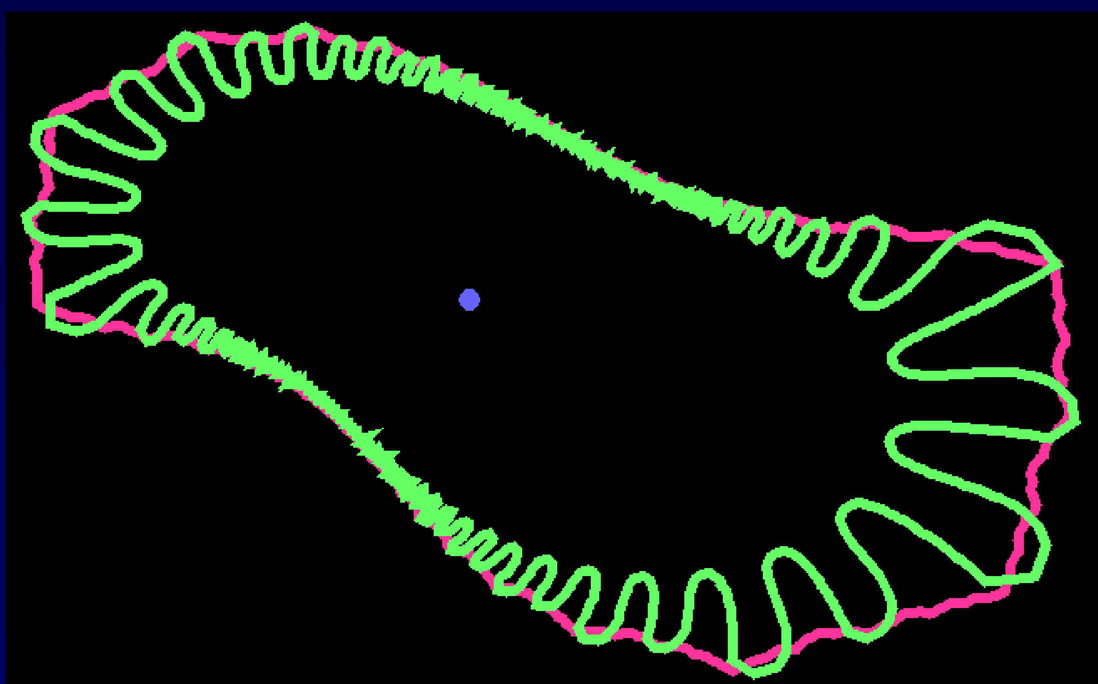
N=1000

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{20+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{N+\cfrac{1}{1+\ddots}}}}}}}$$



Controlling r(q) in pictures

N=10000



$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{20+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{N+\cfrac{1}{1+\ddots}}}}}}}$$

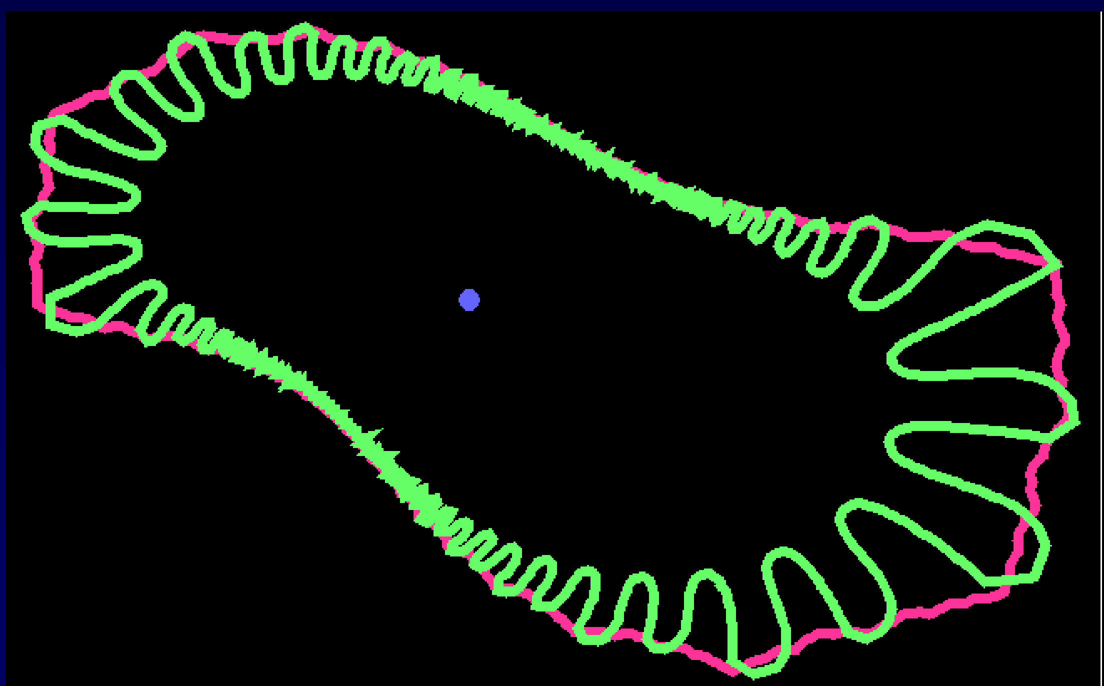
r(q $_2$(N)) fi 0
as N fi ¥

Any drop possible!

Controlling r(q) in pictures

N=10000

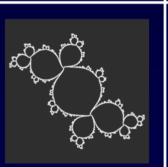
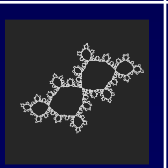
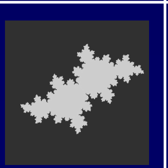
$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{20+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{N+\cfrac{1}{1+\ddots}}}}}}}$$



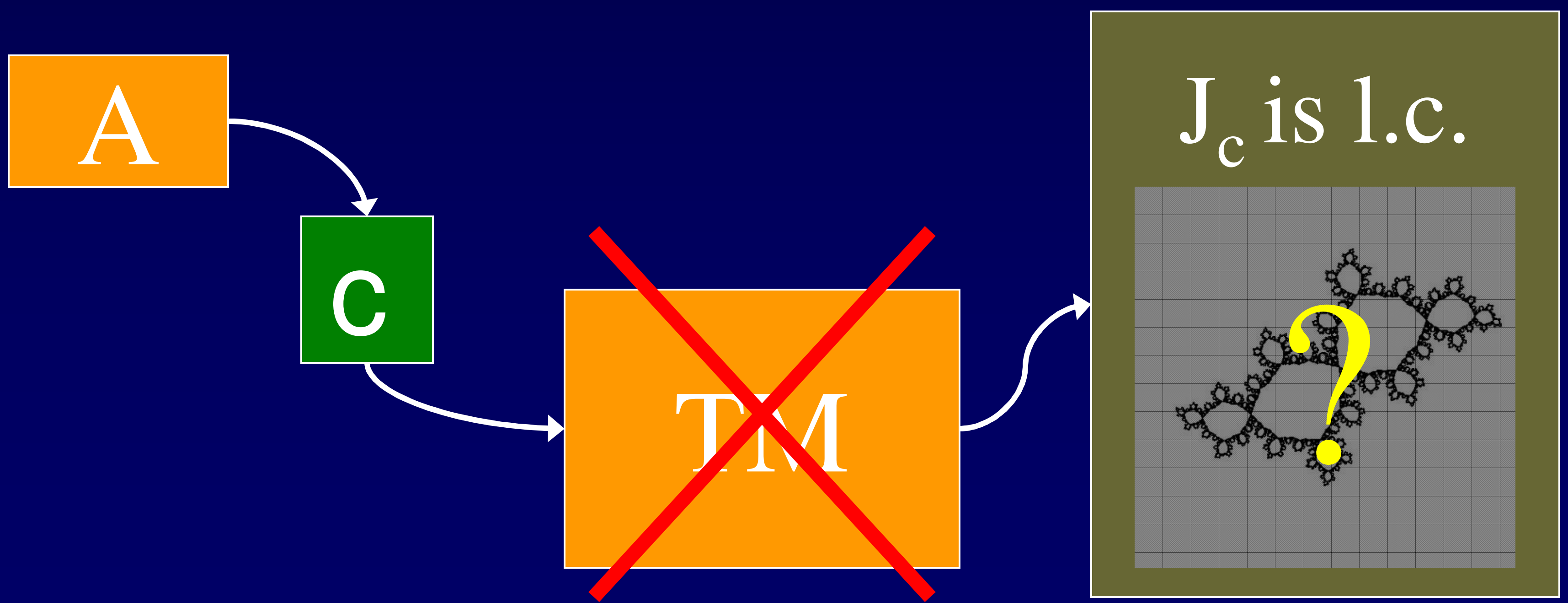
r(q $_2$(N)) fi 0
as N fi ¥

Any drop possible!

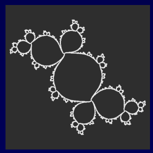
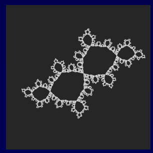
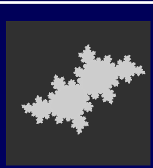
Summary

Type	Empirical and prior work	New
Hyperbolic 	empirically easy; some shown in poly-time	poly-time computable
Parabolic 	empirically computable (exp-time)	poly-time computable
Siegel 	empirically computable in many cases	some are computable some are not
Cremer 	no useful pictures to date	computable
Filled Julia set K_c 	thought to be tightly linked to J_c	always computable

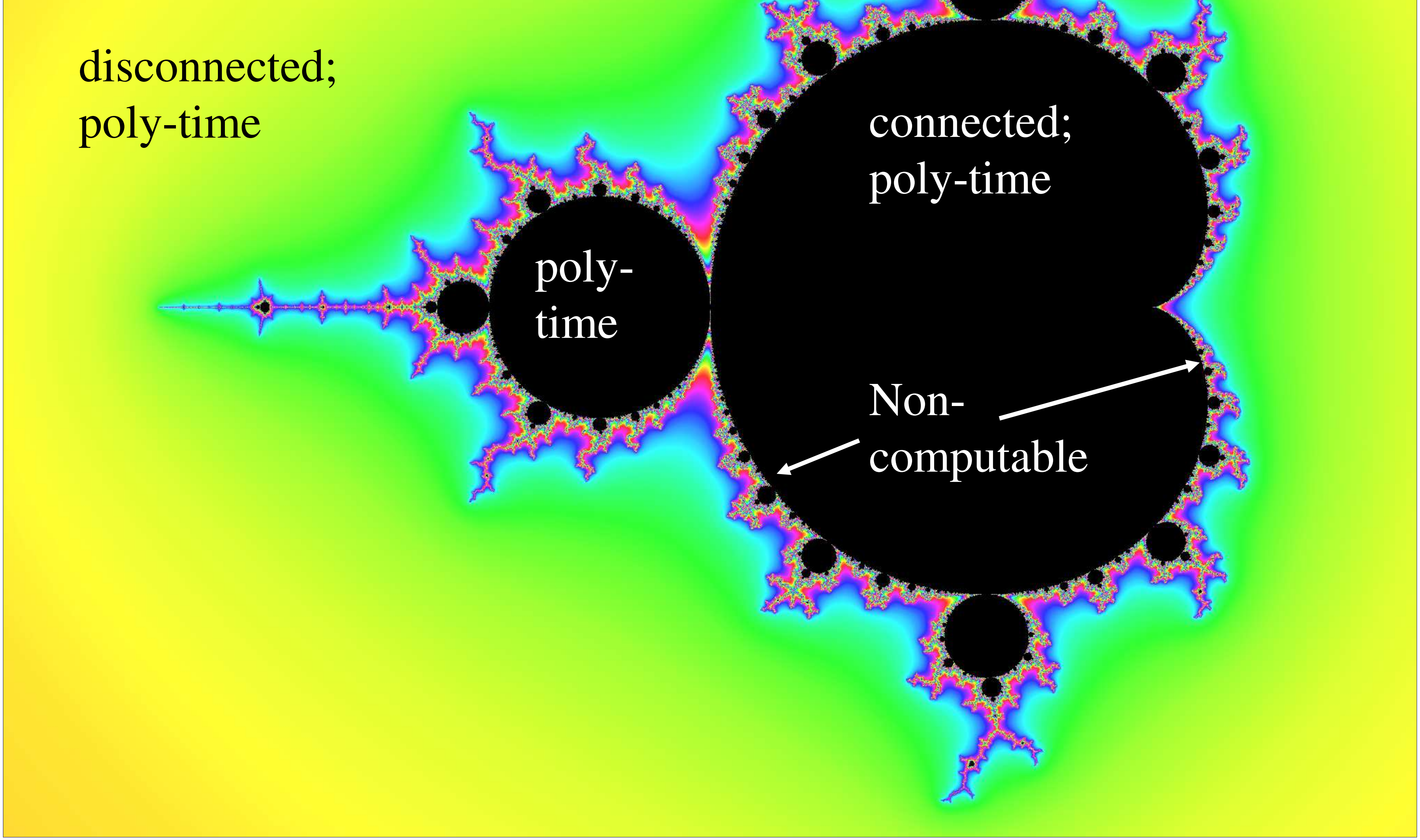
Theorem [BY09]: There is an algorithm A that computes a number c such that J_c *is locally connected* and no machine with access to c can compute J_c.



"Simplicity": topological vs. computational

	Computable	Non-computable
Locally connected	✓ e.g. hyperbolic 	✓ Siegel 
Not locally connected	✓ e.g. Cramer ?	✓ also Siegel 

Prevalence of non-computability



Thank You[1]

Accelerating parabolic computation

- <u>Example:</u> The simplest parabolic example is given by the map $f: z \rightarrow z + z^2$ (same as $z \rightarrow z^2+1/4$ via a change of coordinates).
- Want to iterate a point to see if its trajectory escapes.
- Suppose we are given $z_0 = 2^{-n}$.
- Need to see that its orbit escapes to ∞ in poly(n) steps.

Computing z_0's orbit

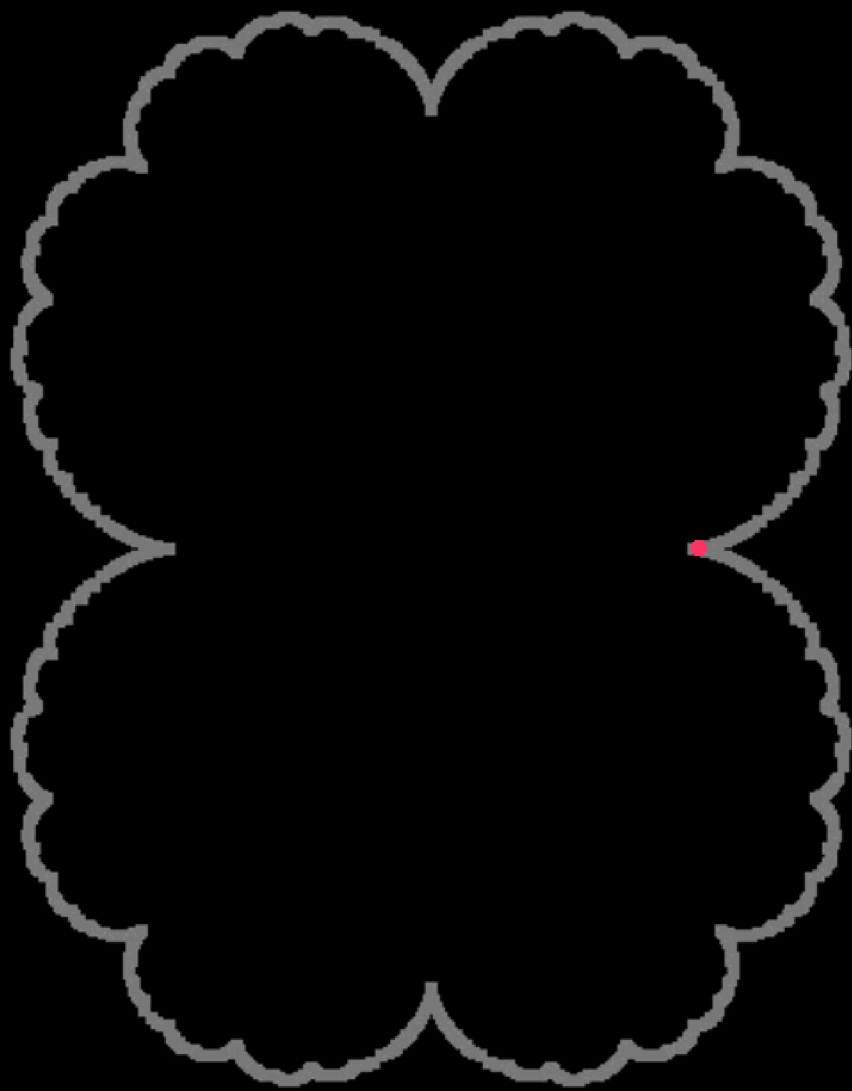
- $z_0 = 2^{-n}$;
- $z_1 = f(z_0) = z_0 + z_0^{\ 2} = 2^{-n} + 2^{-2n}$;
- $z_2 = f^2(z_0) = f(z_1) = z_1 + z_1^{\ 2} \approx 2^{-n} + 2{\cdot}2^{-2n}$;
- $z_3 = f^3(z_0) = f(z_2) = z_2 + z_2^{\ 2} \approx 2^{-n} + 3{\cdot}2^{-2n}$;

…

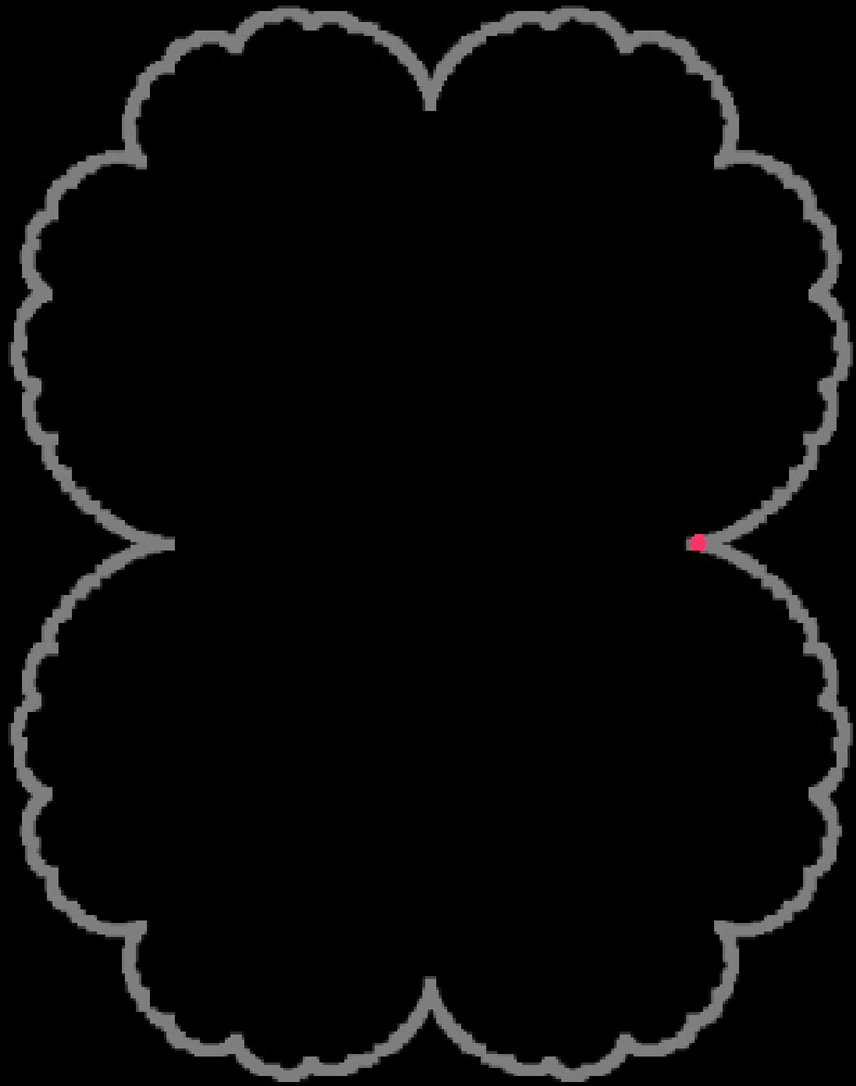
…

- Too slow! Will take 2^n steps to get anywhere!

Before:



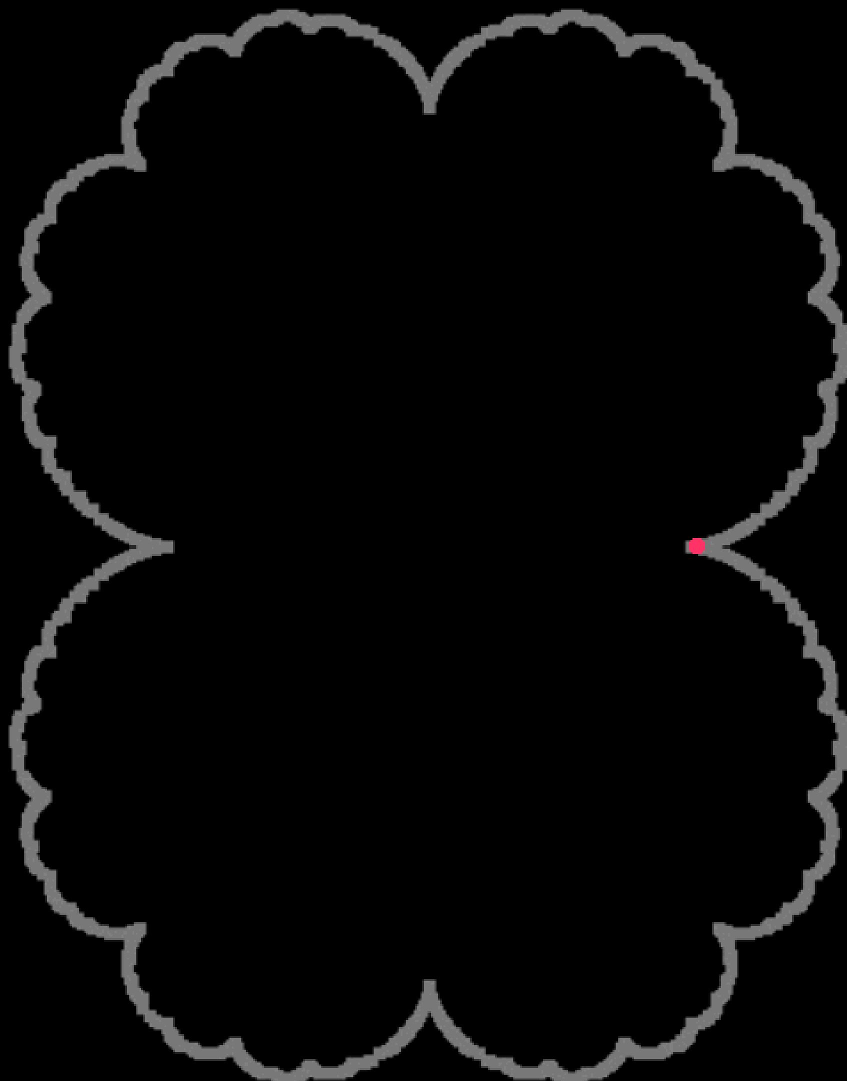
Before:



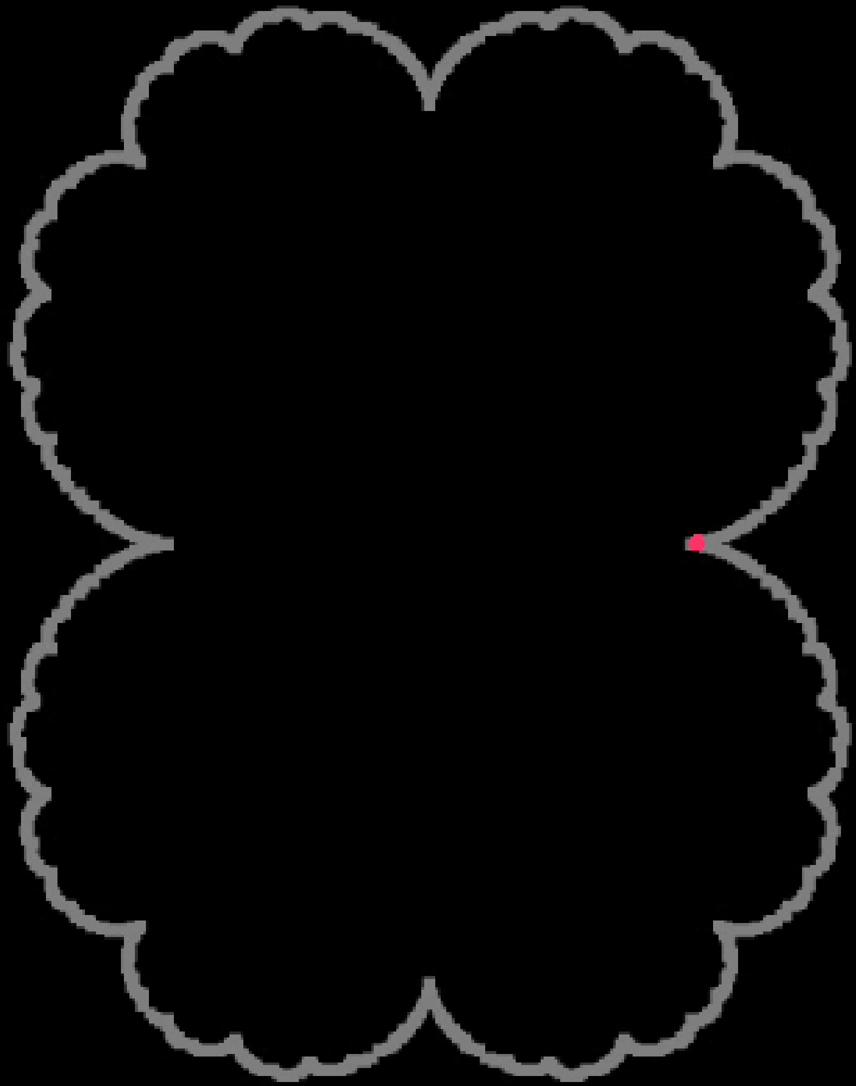
Computing z_0's orbit

- Instead, compute the orbit symbolically:
  - $f^1(z) = f(z) \quad = z + z^2$
  - $f^2(z) = f(f^1(z)) = z + 2\,z^2 + 2\,z^3 + z^4$
  - $f^3(z) = f(f^2(z)) = z + 3\,z^2 + 6\,z^3 + 9\,z^4 + \ldots$
  - $f^4(z) = f(f^3(z)) = z + 4\,z^2 + 12\,z^3 + 30\,z^4 + \ldots$
- In general,
  - $f^k(z) = z + k\,z^2 + (k^2\text{-}k)\,z^3 + (k^3\text{-}2.5\,k^2 + 1.5k)\,z^4 + \ldots$
- Coefficients can be computed symbolically.
- To get a good approximation of $f^{2^n}(z_0)$ enough to take $O(n)$ terms in the expansion of $f^k(z_0)$ and plug in $k=2^n$.

After:



After:



Thank You_2