

Dessert: Hilbert's 13th Problem, in Full Colour

Dror Bar-Natan, Toronto November 2009. All sources at <http://www.math.toronto.edu/~drorbn/Talks/Fields-0911>

Abstract. To end a week of deep thinking with a nice colourful light dessert, we will present the Kolmogorov-Arnol'd solution of Hilbert's 13th problem with lots of computer-generated rainbow-painted 3D pictures.

In short, Hilbert asked if a certain specific function of three variables can be written as a multiple (yet finite) composition of continuous functions of just two variables. Kolmogorov and Arnol'd showed him silly (ok, it took about 60 years, so it was a bit tricky) by showing that **any** continuous function f of any finite number of variables is a finite composition of continuous functions of a single variable and several instances of the binary function “+” (addition). For $f(x, y) = xy$, this may be $xy = \exp(\log x + \log y)$. For $f(x, y, z) = x^y/z$, this may be $\exp(\exp(\log y + \log \log x) + (-\log z))$. What might it be for (say) the real part of the Riemann zeta function?

The only original material in this talk will be the pictures; the math was known since around 1957.



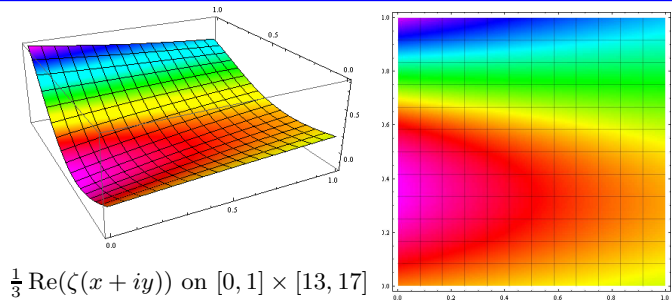
Hilbert



Kolmogorov



Arnol'd (by Moser)



Fix an irrational $\lambda > 0$, say $\lambda = (\sqrt{5} - 1)/2$. All functions are continuous.

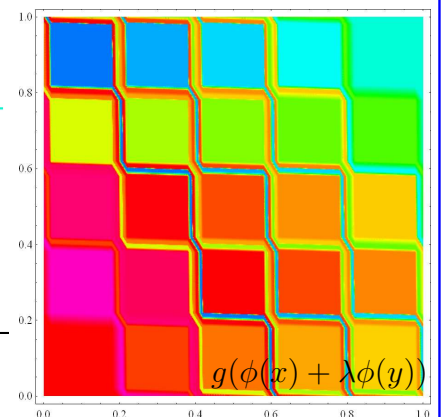
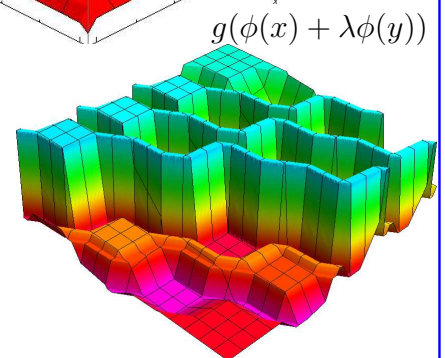
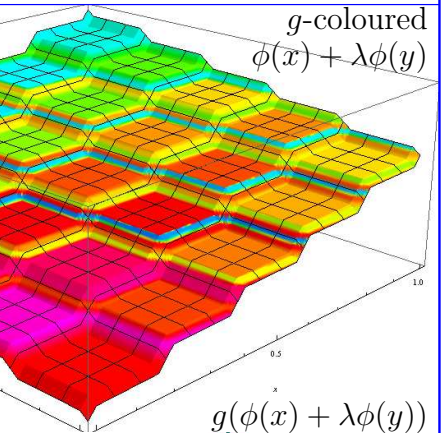
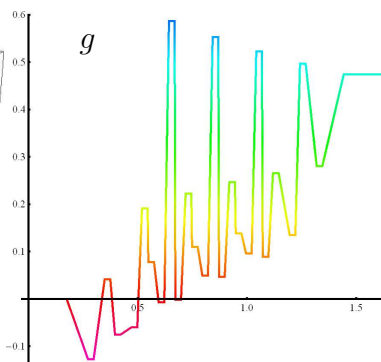
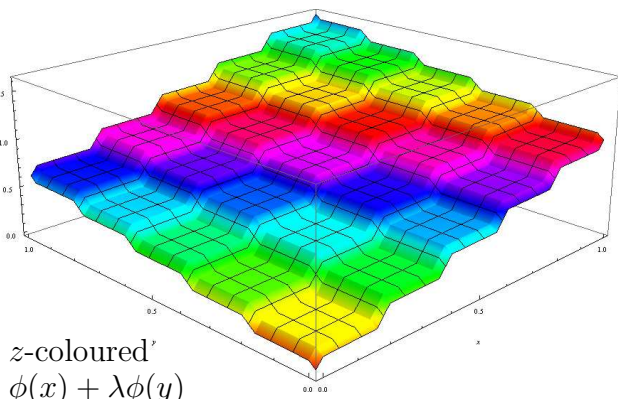
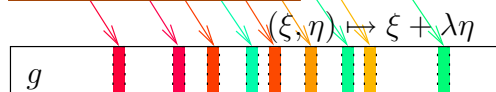
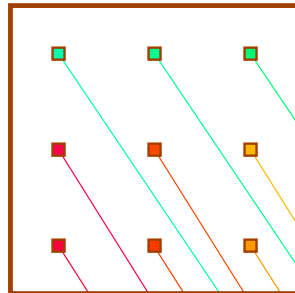
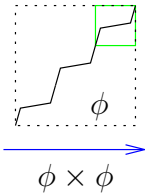
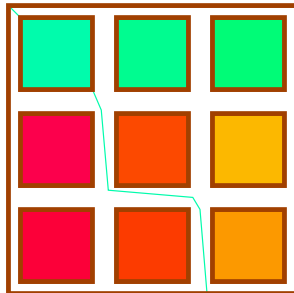
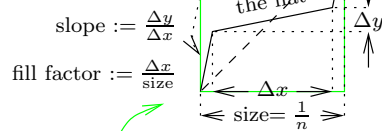
Theorem. There exist five $\phi_i : [0, 1] \rightarrow [0, 1]$ ($1 \leq i \leq 5$) so that for every $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ there exists a $g : [0, 1 + \lambda] \rightarrow \mathbb{R}$ so that

$$f(x, y) = \sum_{i=1}^5 g(\phi_i(x) + \lambda \phi_i(y))$$

for every $x, y \in [0, 1]$.

Step 1. If $\epsilon > 0$ and $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$, then there exists $\phi : [0, 1] \rightarrow [0, 1]$ and $g : [0, 1 + \lambda] \rightarrow \mathbb{R}$ so that $|f(x, y) - g(\phi(x) + \lambda \phi(y))| < \epsilon$ on at least 98% of the area of $[0, 1] \times [0, 1]$.

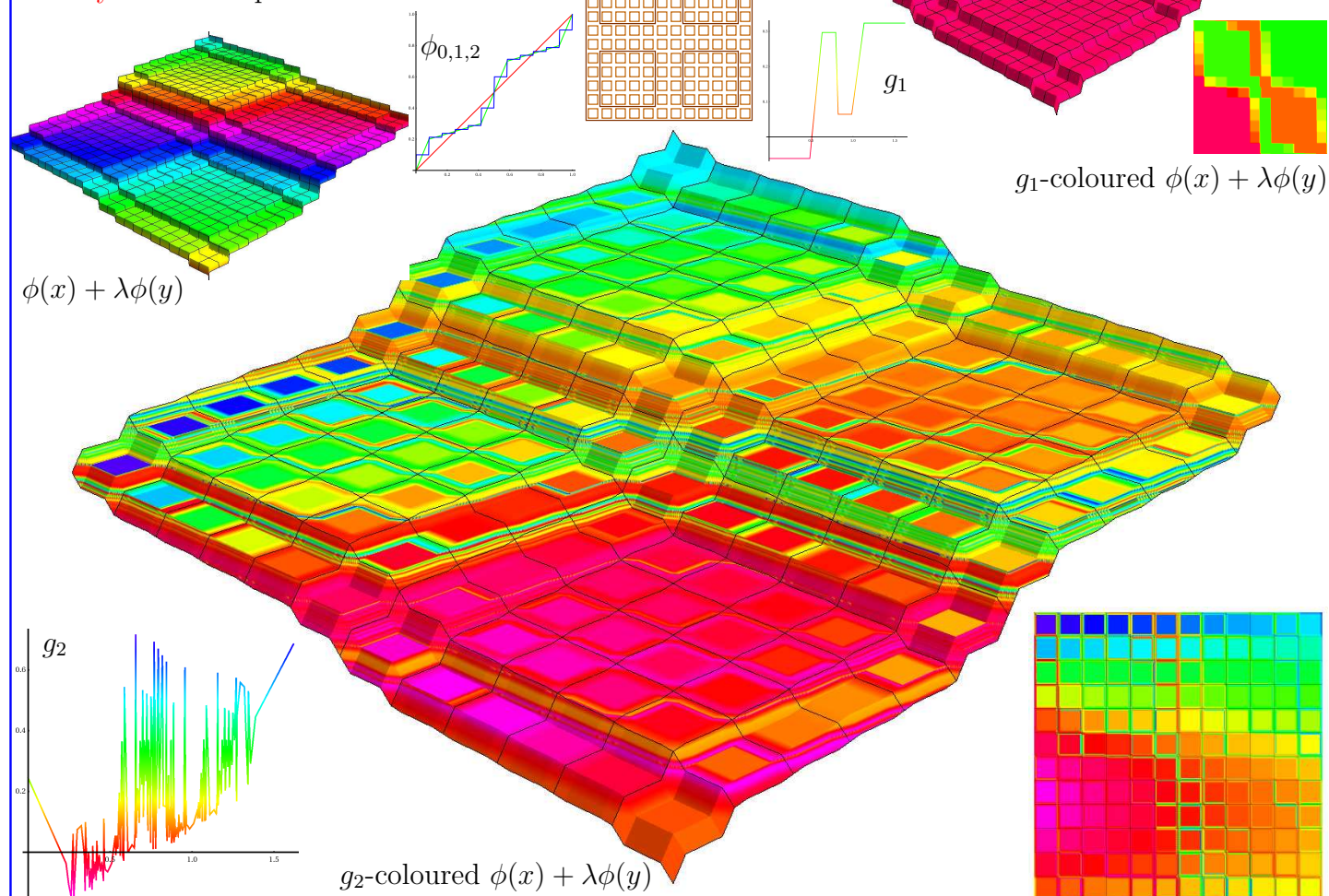
The key. “Poorify” chocolate bars.



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Step 2. There exists $\phi : [0, 1] \rightarrow [0, 1]$ so that for every $\epsilon > 0$ and every $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ there exists a $g : [0, 1 + \lambda] \rightarrow \mathbb{R}$ so that $|f(x, y) - g(\phi(x) + \lambda\phi(y))| < \epsilon$ on a set of area at least $1 - \epsilon$ in $[0, 1] \times [0, 1]$.

The key. "Iterated poorification".

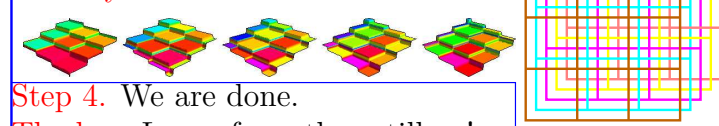


Step 3. There exist $\phi_i : [0, 1] \rightarrow [0, 1]$ ($1 \leq i \leq 5$) so that for every $\epsilon > 0$ and every $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ there exists a $g : [0, 1 + \lambda] \rightarrow \mathbb{R}$ so that

$$|f(x, y) - \sum_{i=1}^5 g(\phi_i(x) + \lambda\phi_i(y))| < \left(\frac{2}{3} + \epsilon\right) \|f\|_\infty$$

for every $x, y \in [0, 1]$.

The key. "Shift the chocolates"...



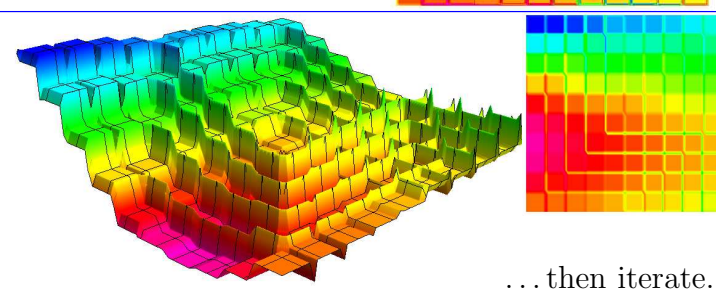
Step 4. We are done.

The key. Learn from the artillery!

Set $Tg := \sum_{i=1}^5 g(\phi_i(x) + \lambda\phi_i(y))$, $f_1 := f$, $M := \|f\|$, and iterate "shooting and adjusting". Find g_1 with $\|g_1\| \leq M$ and $\|f_2 := f_1 - Tg_1\| \leq \frac{3}{4}M$. Find g_2 with $\|g_2\| \leq \frac{3}{4}M$ and $\|f_3 := f_2 - Tg_2\| \leq (\frac{3}{4})^2 M$. Find g_3 with $\|g_3\| \leq (\frac{3}{4})^2 M$ and $\|f_4 := f_3 - Tg_3\| \leq (\frac{3}{4})^3 M$. Continue to eternity. When done, set $g = \sum g_k$ and note that $f = Tg$ as required.

Exercise 1. Do the m -dimensional case.

Exercise 2. Do $\mathbb{R}^m$ instead of just I^m .



Bad Mouting. I think low of PowerPoint!

- No good feelings towards Impress, Beamer, etc., either.
- Must sync with speaker, can't look back at key points, don't know what to look forward too.
- No upper bound to useless content.
- Nothing to take home.

Propaganda. I love handouts!

- I have nothing to hide and you can take what you want, forwards, backwards, here and at home.
- What doesn't fit on one sheet can't be done in one hour.
- It costs hours and pennies. The audience's worth it!
- Can put hyperbolic geometry in every talk!

Dr. Bar Mitan