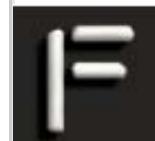


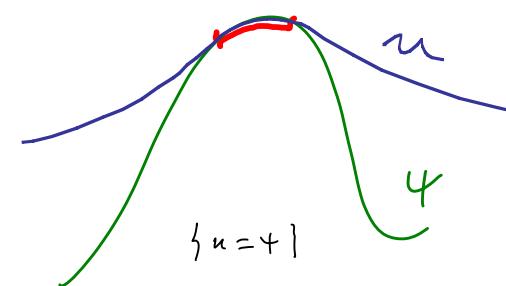
December 9, 2009
2:10 p.m.
Fields Institute, Room 210



FIELDS

"... simple, beautiful
and deep problem"

J.-L. Lions (1979)



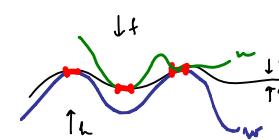
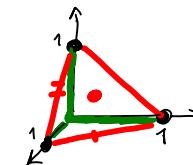
"... it seems
never die ..."

L. Caffarelli (2007)

Jose Francisco Rodrigues (University of Lisbon /
CMAF)
rodrigue@fc.ul.pt

Constrained Reaction-Diffusion and Transport Systems: the N-membrane and Multiphase Problems

We analyse vector valued diffusion and transport equations with a class of constraints of unilateral and bilateral type. Using the variational inequality approach we characterize explicitly the associated Lagrange multipliers by reducing the problems to semi-linear systems coupled through the characteristic functions of the coincident sets of the N-membranes problem, analogously to the obstacle problem. In collaboration with Lisa Santos, we obtain new results to the system associated with the Gibbs simplex for multiphase problems. We also discuss the stability of the solutions and their coincident sets, in particular, the asymptotic behaviour in time for the respective evolution problems.



Reaction-Diffusion in Gibbs Simplex

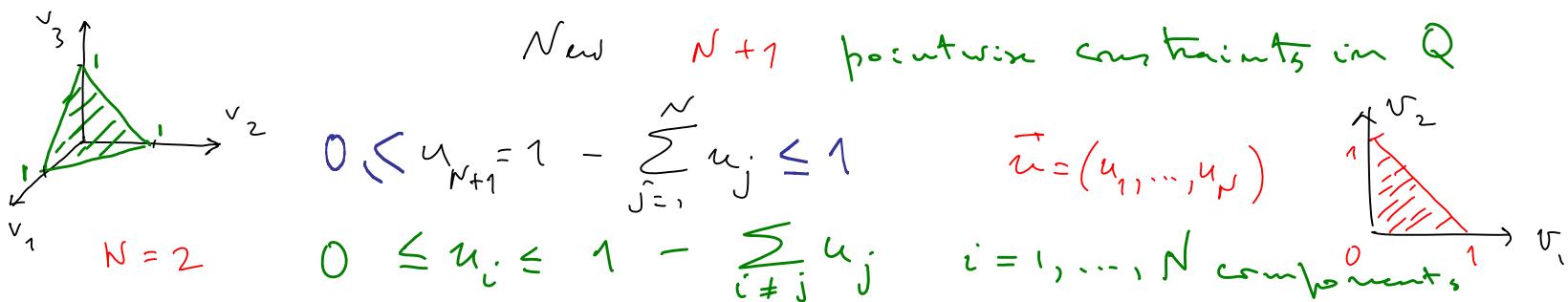
$$U = U(x, t) : Q = \Omega \times (0, T) \rightarrow \mathbb{R}^{N+1}, \quad \Omega \subset \mathbb{R}^d \text{ b.d.}$$

$\delta > 0$ $\bar{s} = \bar{\delta}$
 (with Lisz Santos)

$$\left\{ \begin{array}{l} \partial_t U - \delta \Delta U + (\vec{b} \cdot \nabla) U = F(x, t, U) \quad \in \quad Q \\ U|_{\Sigma} = 0, \quad \frac{\partial U}{\partial n}|_{\Sigma} = 0, \quad U(0) = U_0 \quad \text{in the Gibbs simplex} \end{array} \right.$$

$\Sigma_U \Sigma_V = \partial \Omega \times (0, T)$

$$\mathcal{S} = \left\{ U = (u_1, u_2, \dots, u_{N+1}) : \sum_{i=1}^{N+1} u_i = 1, \quad u_i \geq 0 \right\}$$



$U = (u_1, \dots, u_N)$ solves a system of parabolic variational inequalities

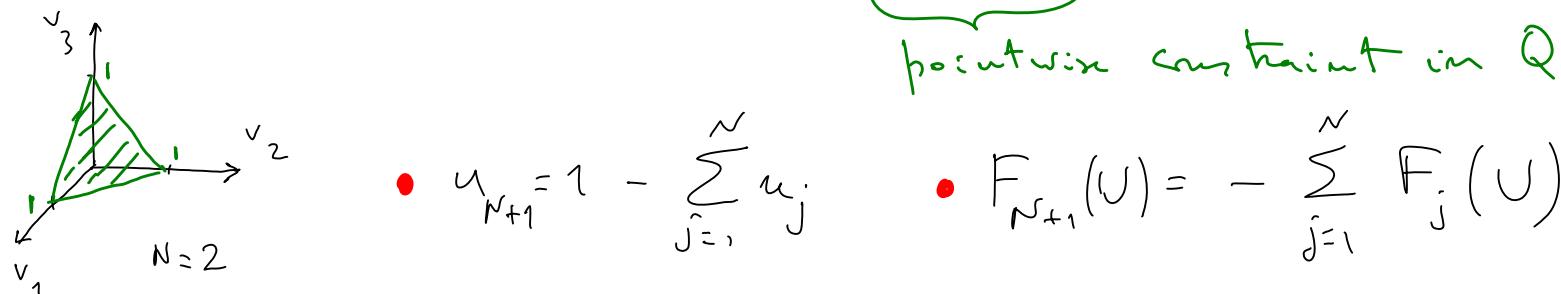
A Transport Problem on Gibbs Simplex

$$U = U(x, t) : Q = \Omega \times (0, T) \rightarrow \mathbb{R}^{N+1}$$

$$\vec{b} \cdot \vec{n} : Q \rightarrow \mathbb{R}^d, \quad \underline{\Sigma} = \left\{ \vec{b} \cdot \vec{n} < 0 \right\}_{\partial \Omega}$$

$$\begin{cases} \partial_t U + (\vec{b} \cdot \nabla) U = F(x, t, U) & \text{in } Q \\ U = 0 \text{ on } \underline{\Sigma}, \quad U(0) = U_0 \text{ in the Gibbs simplex} & \end{cases} \quad (\delta=0)$$

$$\mathcal{S} = \left\{ V = (v_1, v_2, \dots, v_{N+1}) : \sum_{i=1}^{N+1} v_i = 1, \quad v_i \geq 0 \right\}$$



- $v_{N+1} = 1 - \sum_{j=1}^N v_j$
- $F_{N+1}(U) = - \sum_{j=1}^N F_j(U)$

Replicator dynamics: $\dot{v}_i = v_i \left[\varphi_i(v) - \sum_{j=1}^{N+1} v_j \varphi_j(v) \right]$

v_i - frequency
 φ_i - fitness

Phase Field Models

$$0 \leq u_i \leq 1$$

Solids, Fluids, Chemicals, Populations

mass concentration

i -phase ($i = 1, \dots, N+1$)

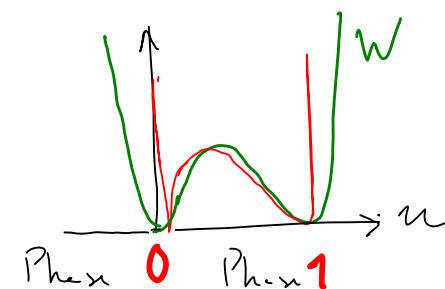
Allen-Cahn equation

$$u = u_1, \quad u_2 = 1 - u_1; \text{ scalar pb.} \quad N = 1$$

$$\partial_t u - \Delta u + \frac{1}{\varepsilon^2} W'(u) = 0$$

for the phase field variable for
interface motion between 2 phases

$$f = -\frac{1}{\varepsilon^2} W'(u)$$



$$\left\{ \begin{array}{l} u \in K = \{ u : 0 \leq u \leq 1 \text{ a.e.} \} \\ \int_Q (\partial_t u - \Delta u - f)(v - u) \geq 0, \forall v \in K \end{array} \right.$$

$\{0 < u_i < 1\}$
phase transition layer
 ε -thickness
 $u|_{\partial} = u_0 \in K \cap H^1(\partial)$

Double obstacle problem

Parabolic Variational Inequality

Phase Field Systems

$u = (u_1, \dots, u_N)$

$N \geq 2$

Each component u_i satisfies a double obstacle problem

$$0 \leq u_i \leq 1 - \sum_{j \neq i} u_j \quad \text{in } Q$$

and solves a parabolic variational inequality with $u(0) = u_0 \in K$:

$$\left\{ \begin{array}{l} u(t) \in K = \left\{ v \in \mathbb{H}^1(\Omega)^N : \sum_{j=1}^N v_j \leq 1, v_i \geq 0, i=1, \dots, N \text{ in } \Omega \right\} \\ \int \limits_{\Omega} \partial_t u \cdot (v - u) + \int \limits_{\Omega} \nabla u \cdot \nabla (v - u) \geq \int \limits_{\Omega} f(u) \cdot (v - u), \quad \forall v \in K \\ \text{a.e. } t \in (0, T) \end{array} \right.$$

and, in fact, a reaction-diffusion system with
Lagrange multipliers: $h_i(u) = F_i(v) - f_i(u) \quad (i=1, \dots, N)$

$$v \cdot v = \sum_{i=1}^N u_i v_i$$

Transport Systems $u = (u_1, \dots, u_N)$ $N \geq 2$

Each component u_i satisfies a double obstacle problem

$$0 \leq u_i \leq 1 - \sum_{j \neq i} u_j \quad \text{in } Q$$

and solves a 1st order variational inequality with $u(0) = u_0 \in \tilde{K}$:

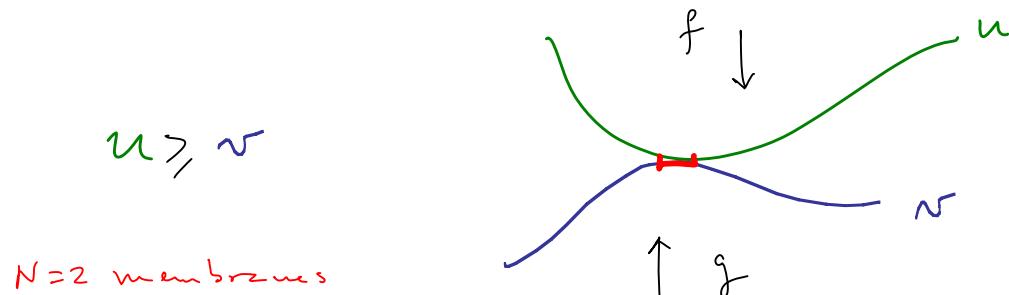
$$\left\{ \begin{array}{l} u(t) \in \tilde{K} = \left\{ v \in L^2(\Omega)^N : \sum_{j=1}^N v_j \leq 1, v_i \geq 0, i=1, \dots, N \text{ in } \Omega \right\} \\ \int_Q (\partial_t u + (\vec{b} \cdot \nabla) u) \cdot (v - u) \geq \int_Q f(u) \cdot (v - u), \quad \forall v \in \tilde{K} \\ \text{a.e. } t \in (0, T) \\ Hu \in L^2(Q)^N, \quad u|_{\Sigma} = 0 \end{array} \right.$$

and, in fact, also a transport system with

$$u \cdot v = \sum_{i=1}^N u_i v_i$$

Lagrange multipliers: $h_i(u) = F_i(v) - f_i(u) \quad (i=1, \dots, N)$

The N -membranes problem for Δ (Laplacian)



$$u \geq v$$

$$v \leq \varphi$$

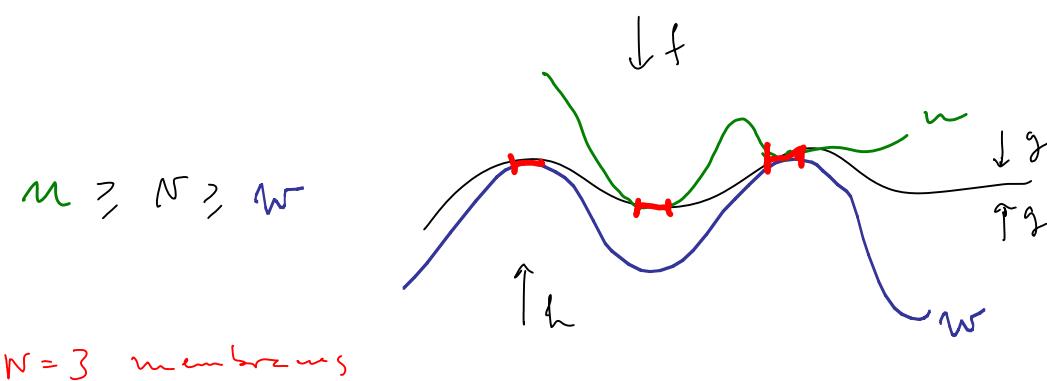
1-obstacle problem

$$U = (u, v) = (u_1, u_2)$$

$$\frac{1}{2} \int_{\Omega} |\nabla U|^2$$

Dirichlet integral

Minimization of energy associated to



$$U = (u, v, w) = (u_1, u_2, u_3)$$

$$\varphi \geq v \geq w$$

2-obstacles problem

$$u = (u_1, u_2, \dots, u_N) : u_1 \geq u_2 \geq \dots \geq u_N$$

$$P_u = \partial_t u - \Delta u$$

$$H_u = \partial_t u + (\vec{S} \cdot \vec{\nabla}) u$$

A system of elliptic, parabolic or hyperbolic v -inequalities

The Abstract Obstacle Problem

V - Banach space, reflexive, vector lattice for \leq , induced by $P = \{v \geq 0\}$

$$v = v^+ - v^- \quad v \vee w = v + (w - v)^+ \quad v \wedge w = v - (v - w)^+$$

$A: V \rightarrow V'$ nonlinear operator, hemicontinuous, coercive (strictly monotone)

$$\begin{aligned} \langle Au - Av, (u - v)^+ \rangle &\geq 0 \\ &> 0 \quad \text{if } (u - v)^+ \neq 0 \quad \text{strictly T-monotone} \end{aligned}$$

, $\forall u, v: (u - v)^+ \in V$ (Brézis-Santambrogio 1968)

$K \subset V$ convex, closed, $\neq \emptyset$ $F \in V'$, dual of V

$\exists! \quad | \quad u \in K : \langle Au - F, v - u \rangle \geq 0, \quad \forall v \in K$

Obstacles $\psi, \varphi \in V$

variational inequality

$$K_\psi = \{v \in V : v \geq \psi\} \quad K^\varphi = \{v \in V : v \leq \varphi\} \quad \underline{K_\psi^\varphi = \{\psi \leq v \leq \varphi\}}$$

$$V^* = P' - P' \subset V' \quad \text{order dual of } V, \quad P' = \{v' \in V' : \langle v', v \rangle \geq 0 \quad \forall v \in P\}$$

(Ex: $H_0^1(\Omega)^* = H^{-1}(\Omega) \cap M(\Omega)$)

Lewy-Stampacchia inequality

$$F, A\psi, A\varphi \in V^*$$

1970 for $A = -\Delta$

$$F \leq Au \leq F \vee A\psi \quad \text{if } u \geq \psi$$

1975-Henry-Joly $\frac{\partial}{\partial t} \frac{\partial u}{\partial t}$
Cherier-Tiraniello

$$\underline{F \wedge A\varphi \leq Au \leq F} \quad \text{if } u \leq \varphi$$

1976 Mosco A ,

$$\underline{\psi \leq u \leq \varphi} \quad \text{then} \quad \underline{F \wedge A\varphi \leq Au \leq F \vee A\psi}$$

2002 Polyrinos
 $H = \vec{b} \cdot \nabla + c$

The Obstacle Problem for the p -Laplacian

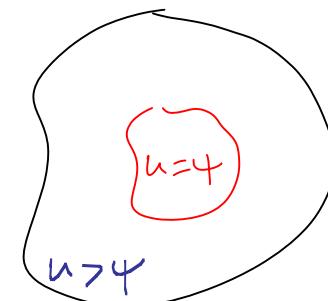
$$\Omega \subset \mathbb{R}^m, \quad \psi \in W^{1,p}(\Omega), \quad \underbrace{\psi|_{\partial\Omega} \leq 0}, \quad -\Delta_p \psi, f \in L^s(\Omega), \quad 1 < p < \infty, \quad s > m/p/(p-1)$$

$$\left\{ \begin{array}{l} u \in K_\psi = \{v \in W_0^{1,p}(\Omega) : v \geq \psi \text{ in } \Omega\} \neq \emptyset \\ \int_{\Omega} \nabla_p u \cdot \nabla(v-u) \geq \int_{\Omega} f(v-u), \quad \forall v \in K_\psi \end{array} \right. \quad (\nabla_p v = |\nabla v|^{p-2} \nabla v)$$

$\exists! u \in W_0^{1,p}(\Omega) \cap C^{1,\lambda}(\Omega)$, for some $0 < \lambda < 1$

$$f \leq -\Delta_p u \leq f + (-\Delta_p \psi - f)^+ \quad \text{in } \Omega$$

$$-\Delta_p u - (-\Delta_p \psi - f)^+ \chi_{\{u=\psi\}} = f \quad \text{a.e. in } \Omega$$



Continuous dependence of the coincidence set $\{u=\psi\}$

$$\begin{aligned} f^\nu \rightarrow f &\text{ in } L^s & \psi^\nu \rightarrow \psi &\text{ in } W^{1,p} \\ -\Delta_p \psi^\nu \rightarrow -\Delta_p \psi &\text{ in } L^1 & -\Delta_p \psi \neq f &\text{ a.e.} \end{aligned} \quad \text{then } \chi_{\{u^\nu=\psi^\nu\}} \rightarrow \chi_{\{u=\psi\}}$$

Rud. 1984
2004

Extension to $p=p(x)$ and L^1 -data (entropy solution) R-Sanchon-Urbano, 2008

Extension to the A-Laplacian $\Delta_A u = \operatorname{div}\left(\frac{a(|\nabla u|)}{|\nabla u|} \nabla u\right)$ Orlicz-Hölder Space $W^{1,A}(\Omega)$
(with S. Challal - A-Lyaghfouri)

The Obstacle Problem for 1st order operators H

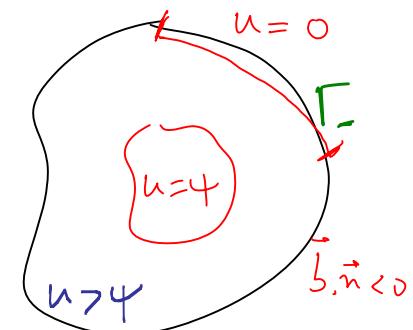
$\Omega \subset \mathbb{R}^n$, $f \in L^2(\Omega)$, $\psi \in \tilde{L}_H^2(\Omega)$, $\psi|_{\Gamma_-} \leq 0$, $H\psi = \vec{b} \cdot \nabla \psi + \beta \psi \in L^2(\Omega)$

$$\left\{ \begin{array}{l} u \in K_\psi = \{v \in \tilde{L}_H^2(\Omega) : v \geq \psi \text{ in } \Omega\} \neq \emptyset \\ \int_{\Omega} (Hu - f)(v - u) \geq 0 \quad \forall v \in \tilde{K}_\psi = \{v \in L^2(\Omega) : v \geq \psi \text{ in } \Omega\} \end{array} \right.$$

$\exists! u \in \tilde{L}_H^2(\Omega) = \{v \in L^2(\Omega) : Hv \in L^2(\Omega), v|_{\Gamma_-} \in \tilde{L}_\ell^2(\Omega)\}$ $\ell = (\vec{b}, \vec{n})$

$$f \leq Hu \leq f + (H\psi - f)^+ \text{ in } \Omega$$

$$Hu - (H\psi - f)^+ \chi_{\{u=\psi\}} = f \text{ a.e. in } \Omega$$



Continuous dependence of the coincidence set $\{u=\psi\}$

$$\begin{aligned} f' &\rightarrow f \text{ in } L^1 & \psi &\rightarrow \psi \text{ in } W^{1,p} \\ H\psi' &\rightarrow H\psi \text{ in } L^1 & H\psi \neq f \text{ a.e.} &\text{ then } \chi_{\{u'=\psi'\}} \rightarrow \chi_{\{u=\psi\}} \end{aligned}$$

Rod. 2004

Coercivity condition: $b_s - \frac{1}{2}(\vec{b} \cdot \vec{b}) \geq \beta > 0$ $\vec{b} \in W^{1,\infty}(\Omega)$, $b_s \in L^\infty(\Omega)$

Remarks on a 2-obstacles problem

$$P_u = \frac{\partial}{\partial t} u - \Delta u$$

$$H u = \frac{\partial}{\partial t} u + (\vec{b}, \nabla) u$$

$$\left\{ \begin{array}{l} u \in K_0^\varphi = \{ v \in L^2(0,T; H^1(\Omega)) : 0 \leq v \leq \varphi \text{ in } Q \}, \quad u(0) = u_0 \text{ in } \Omega \\ \int_Q (P_u - g)(v - u) \geq 0 \quad \forall v \in K_0^\varphi \end{array} \right.$$

$\varphi|_{\Sigma^-} = 0$

$\varphi \in L^2(0,T; H^1(\Omega)) \cap L^\infty(Q)$, $\varphi \geq 0$, $\partial_t \varphi \in L^2(0,T; H^1(\Omega))$, $P\varphi \in L^r(Q)$, $\frac{\partial \varphi}{\partial \nu} = 0$ $\text{at } \partial \Omega \times (0,T)$
 $g \in L^r(Q)$, $u_0 \in L^2(\Omega)$, $0 \leq u_0 \leq \varphi(0)$ in Ω . ($r > 1$)

- $\exists! u \in C([0,T]; L^2(\Omega)) \cap K_0^\varphi \cap W_r^{2,1}(Q)$, $\partial_t u \in L^r(Q) + L^2(0,T; (H^1(\Omega))^*)$ Neumann
bd. condition

• $Pu = g + g^- \chi_{\{u=0\}} - (g - P\varphi)^+ \chi_{\{u=\varphi\}}$ a.e. $Q = \Omega \times (0,T)$

Since $\{u=0\} \subset \{g \leq 0\}$ a.e. $\{u=\varphi\} \subset \{P\varphi \leq g\}$ $\bar{g} = (-g)^+$

is equivalent

$$Pu = g - g \chi_{\{u=0\}} - (g - P\varphi) \chi_{\{u=\varphi\}} \quad \text{a.e. } Q$$

or

$$Pu = 0 \quad \text{a.e. } \{u=0\}, \quad Pu = g \iff \{0 < u < \varphi\}, \quad Pu = P\varphi \quad \text{a.e. } \{u=\varphi\}$$

The 2-Membranes Problem

(Vergara-Caffarelli 1971/72
 $\beta = 2$, minimal surface operator)

$$U \in K_2 : \langle Au - F, W - U \rangle \geq 0 \quad \forall W \in K_2$$

$$\langle Au, w \rangle = \int_{\Omega} \nabla_p u \cdot \nabla w_1 + \int_{\Omega} \nabla_p v \cdot \nabla w_2$$

$$\langle F, w \rangle = \int_{\Omega} f w_1 + \int_{\Omega} g w_2$$

$$K_2 = \left\{ W = (w_1, w_2) : w_1 \geq w_2 \text{ in } \Omega, |w_i|_{\partial\Omega} = v_i, i=1,2 \right\}$$

$$1 < p < \infty$$

$$\nabla_p u = |\nabla u|^{p-2} \nabla u$$

$$f, g \in L^s(\Omega), s > np/(p-1)$$

$$v_1 \geq v_2 \text{ on } \partial\Omega$$

- $\exists! (u, v) \in [W^{1,p}(\Omega) \cap C^{1,\alpha}(\bar{\Omega})]^2, 0 < \lambda < 1$

$$\left. \begin{array}{l} f \leq -\Delta_p u \leq f \vee g \\ f \wedge g \leq -\Delta_p v \leq g \end{array} \right\} \text{a.e. in } \Omega \quad \left\{ \begin{array}{l} -\Delta_p u = f + \left(\frac{g-f}{2} \right)^+ \chi_{\{u=v\}} \\ -\Delta_p v = g - \left(\frac{g-f}{2} \right)^+ \chi_{\{u=v\}} \end{array} \right.$$

Lusin-Stieltjes inequality: $\{u=v\} \subset \{f \leq g\}$

$$f \leq -\Delta_p u \leq f \vee (-\Delta_p v) \leq f \vee g$$

$$g \wedge f \leq g \wedge (-\Delta_p u) \leq -\Delta_p v \leq g$$

$$-\Delta_p u = -\Delta_p v = \frac{f+g}{2} \text{ a.e. in } \{u=v\}$$

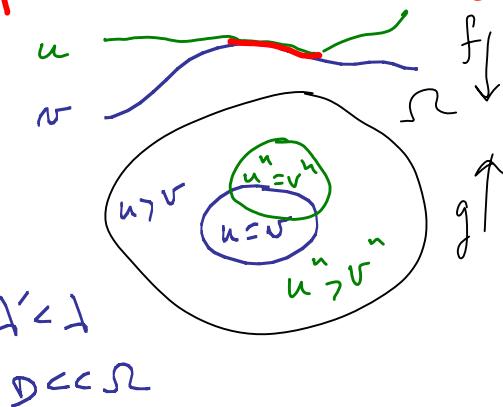
$$T_{\varphi} u \quad W = (u \pm \varphi, v \pm \varphi) \in K_2, \forall \varphi \in \mathcal{D}(\Omega)$$

Continuous dependence of the contact of two membranes

Then let $f^\gamma \rightarrow f$, $g^\gamma \rightarrow g$ in $L^s(\Omega)$, $s > n/(n-1)$

Then

$$(u^\gamma, v^\gamma) \rightarrow (u, v) \text{ in } [W^{1,1}(\Omega) \times C^{1,\lambda}(\bar{\Omega})]^2 \quad \lambda' < \lambda \quad D \ll \Omega$$



If $f \neq g$ a.e. in Ω then also
(non-degenerated limit
problem in $D \subseteq \Omega$)

$$\chi_{\{u^\gamma = v^\gamma\}} \rightarrow \chi_{\{u = v\}} \text{ in } L^q(\Omega), \forall q < \infty$$

Proof: Let $\chi^\gamma = \chi_{\{u^\gamma = v^\gamma\}} \rightarrow \chi^*$ in $L^\infty(\Omega)$ -weak*

$$-\Delta_p u^\gamma = f^\gamma + \left(\frac{g^\gamma - f^\gamma}{2}\right) \chi_{\{u^\gamma = v^\gamma\}} \text{ a.e. in } \Omega$$

$$-\Delta u = f + \left(\frac{g-f}{2}\right) \chi^* \quad \text{yields} \quad \underbrace{(g-f)(\chi_{\{u=v\}} - \chi^*)}_{\neq 0} = 0$$

so $\chi^* = \chi_{\{u=v\}}$ a.e. in Ω and weak convergence of characteristic

functions imply the strong one (no regularity is required!).

And the contact of $N \geq 3$ membranes? Azevedo, Reis Santos
2005

Variational Inequality for the N-Membranes Pb.

$$K(t) = \left\{ v \in H^1(\Omega)^N : \underbrace{v_1 \geq \dots \geq v_N}_{\text{in } \Omega}, v - h(t) \in H_0^1(\Omega)^N \right\}, t \geq 0$$

$$f \in L^2(Q)^N, h \in L^2(0,T; H^1(\Omega)^N) \cap H^1(0,T; L^2(\Omega)^N), h_1 \geq \dots \geq h_N$$

- $\exists! u \in L^2(0,T; H^1(\Omega)^N) \cap C([0,T]; L^2(\Omega)^N), \partial_t u \in L^2(Q)$

$$\begin{cases} u(t) \in K(t), \text{ a.e. } t \in (0,T) & u(0) = h(0) \text{ in } L^2(\Omega)^N \\ \int_{\Omega} \partial_t u(t) (v - u(t)) + \int_{\Omega} \nabla u(t) \cdot \nabla (v - u(t)) \geq \int_{\Omega} f(t) \cdot (v - u(t)), \forall v \in K(t), \text{ a.e. } t \in (0,T) \end{cases}$$

Lewy-Stampacchia Inequality

$$\bigwedge_{j=1}^i f_j \leq P u_i \leq \bigvee_{j=i}^N f_j \quad \text{a.e. in } Q, i=1, \dots, N \quad \begin{matrix} \wedge \inf. \\ \vee \sup. \end{matrix}$$

Regularity

- If $f, h \in L^\infty(Q)^N$ then $u, \nabla u \in C^\alpha(Q)$
then $u \in W_q^{2,1}(Q')$ $\forall \frac{1}{q} < \alpha < \infty$

$$\begin{aligned} f_N \wedge P u_{N+1} &\leq P u_N \leq f_N \\ f_i \wedge P u_{i-1} &\leq P u_i \leq f_i \vee P u_{i+1} \\ f_1 &\leq P u_1 \leq f_1 \vee P u_2 \end{aligned}$$

Evolution of N -membranes • $Pu_i = \frac{\partial}{\partial t} u_i - \Delta u_i$

$\Omega \subset \mathbb{R}^n$, bd. $Q = \Omega \times (0, T)$ $u = (u_1, \dots, u_N)$

$$\begin{cases} Pu = f + \underbrace{R(x, t, u)}_{?} & \text{in } Q \\ u = h & \text{on } \partial_p Q = \partial \Omega \times (0, T) \cup \Omega \times \{0\} \end{cases}$$

Dirichlet - Cauchy conditions

$$u_1 \geq u_2 \geq \dots \geq u_N$$

$$\chi_{j,k} = \chi_{\{u_j = \dots = u_k\}} \quad 1 \leq j < k \leq N \quad \underline{N(N-1)/2 coincidence sets}$$

$$\left\{ \begin{array}{l} \begin{aligned} P u_1 &= f_1 + \frac{1}{2} (f_2 - f_1) \chi_{1,2} & + \frac{1}{6} (2f_3 - f_2 - f_1) \chi_{1,3} \\ P u_2 &= f_2 - \frac{1}{2} (f_2 - f_1) \chi_{1,2} + \frac{1}{2} (f_3 - f_2) \chi_{2,3} & + \frac{1}{6} (2f_2 - f_1 - f_3) \chi_{1,3} \\ P u_3 &= f_3 & - \frac{1}{2} (f_3 - f_2) \chi_{2,3} + \frac{1}{6} (2f_1 - f_2 - f_3) \chi_{1,3} \end{aligned} \\ \text{N=3} \end{array} \right.$$

Suggets $f_1 \neq f_2$ $f_2 \neq f_3$ $f_1 \neq \frac{f_2 + f_3}{2}$ $f_3 \neq \frac{f_1 + f_2}{2}$ stability condition

Stationary problem $P = -\Delta$

Stability of the evolution control of N -membranes

- $P_{u_i} = f_i + \sum_{\substack{1 \leq j < k \leq N \\ j \leq i \leq k}} b_i^{j,k}[f] \chi_{j,k}$ a.e. in Ω

$$b_i^{j,k}[f] = \begin{cases} \langle f \rangle_{j,k} - \langle f \rangle_{j,k-1} & \text{if } i=j \\ \langle f \rangle_{j,k} - \langle f \rangle_{j+1,k} & \text{if } i=k \\ \frac{2}{(k-j)(k-j+1)} [\langle f \rangle_{j+1,k-1} - \frac{1}{2}(f_j + f_k)] & \text{if } j < i < k \end{cases}$$

$$\langle f \rangle_{j,k} = \frac{f_j + \dots + f_k}{k-j+1} \quad (1 \leq j \leq k \leq N)$$

< averages from j to k >

$$\chi_{j,k} = \chi_{\{u_j = \dots = u_k\}} \quad 1 \leq j < k \leq N$$

- If $\langle f \rangle_{i,j} \neq \langle f \rangle_{j+1,k}$ a.e. in $D \subset \Omega$ $\forall i, j, k \in \{1, \dots, N\}, i \leq j < k$ (#)

$$\begin{array}{c} f \xrightarrow{\gamma} f \text{ in } L^2(\Omega) \\ h \xrightarrow{\gamma} h \text{ in } L^2(\Omega) \\ h|_0 \xrightarrow{\gamma} h|_0 \text{ in } L^2(\Omega) \end{array} \left| \begin{array}{c} \text{then} \\ u^\gamma \xrightarrow{\gamma} u \end{array} \right. \chi_{j,k}^\gamma \rightarrow \chi_{j,k} \text{ in } L^1(D), \forall \gamma \in \omega$$

- $t \rightarrow +\infty \quad f(t) \rightarrow f_\infty \quad | \quad \# \infty \quad | \quad \chi_{j,k}(t) \rightarrow \chi_{j,k}^\infty \text{ in } L^1(D)$
- $h(t) \rightarrow h_\infty \quad | \quad u(t) \rightarrow u_\infty \quad | \quad D \subset \Omega$

The 2-System for $Pu = \partial_t u - \Delta u$

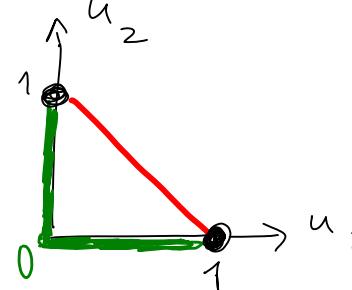
$$Hu = \partial_t u + (\vec{b} \cdot \nabla) u$$

$$u = (u_1, u_2)$$

$0 \leq u_1 \leq 1 - u_2$

$$0 \leq u_1 \leq 1 - u_2$$

$$0 \leq u_2 \leq 1 - u_1$$



$$\left\{ \begin{array}{l} P_{u_1} = f_1 + \bar{f}_1^+ \underbrace{\chi_{\{u_1=0\}}}_{\text{green}} - f_1^+ \underbrace{\chi_{\{u_1=1\}}}_{\text{blue}} - \frac{1}{2} (f_1 + f_2)^+ \underbrace{\chi_{\{u_1+u_2=1, 0 < u_1 < 1\}}}_{\text{red}} \\ P_{u_2} = f_2 + \bar{f}_2^+ \underbrace{\chi_{\{u_2=0\}}}_{\text{green}} - f_2^+ \underbrace{\chi_{\{u_2=1\}}}_{\text{blue}} - \frac{1}{2} (f_1 + f_2)^+ \underbrace{\chi_{\{u_1+u_2=1, 0 < u_2 < 1\}}}_{\text{red}} \end{array} \right.$$

- $u \in L^2(0,T; L^2(\Omega)^2 \cap \tilde{K})$, $Pu \in L^r(Q)^2$ $u(0) = u_0 \in \tilde{K} = \{v \in L^2(\Omega)^2 : v_1 + v_2 \leq 1\}$
 $0 \leq v_1, v_2 \text{ in } \Omega\}$

$$\int_Q [P\bar{u} - f(\bar{u})] \cdot (\bar{v} - \bar{u}) \geq 0 \quad \forall \bar{v} \in L^2(0,T; \tilde{K})$$

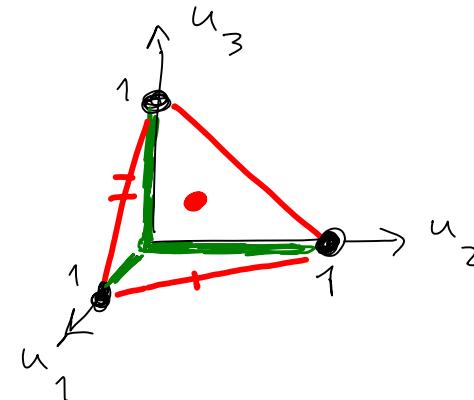
The 3 - System for $Pu = \partial_t u - \Delta u$

$$\mu u = \partial_t u + (\vec{b} \cdot \vec{\nabla}) u$$

$$0 \leq u_1 \leq 1 - u_2 - u_3$$

$$0 \leq u_2 \leq 1 - u_1 - u_3$$

$$0 \leq u_3 \leq 1 - u_1 - u_2$$



$$\begin{aligned}
 P_{u_1} &= f_1 + \underbrace{f_1^- \chi_{\{u_1=0\}}}_{+} - \underbrace{f_1^+ \chi_{\{u_1=1\}}}_{-} - \frac{1}{2} (f_1 + f_2)^+ \chi_{\left\{ \begin{array}{l} u_1 + u_2 = 1 \\ 0 < u_1 < 1 \end{array} \right\}} \\
 &\quad - \frac{1}{2} (f_1 + f_3)^+ \chi_{\left\{ \begin{array}{l} u_1 + u_3 = 1 \\ 0 < u_1 < 1 \\ 0 < u_3 < 1 \end{array} \right\}} - \frac{1}{3} (f_1 + f_2 + f_3)^+ \chi_{\left\{ \begin{array}{l} u_1 + u_2 + u_3 = 1 \\ 0 < u_1 < 1 \\ 0 < u_2 < 1 \\ 0 < u_3 < 1 \end{array} \right\}}
 \end{aligned}$$

$$P_{u_2} = \dots$$

$$P_{u_3} = \dots$$

Total number of coincidence sets

$$2^N - 1 + N$$

Characterization of the Multiphasic N-system

For $i=1, \dots, N$:

$$0 \leq u_i \leq 1 - \sum_{j \neq i} u_j = \varphi_i \quad \text{in } Q$$

$$P_{u_i} = f_i + \overline{f_i} \chi_{\{u_i=0\}} - \sum_{\substack{i \in \{i_1, \dots, i_k\} \\ 1 \leq i_1 < \dots < i_k \leq N}} \frac{1}{k} (f_{i_1} + \dots + f_{i_k})^+ \chi_{i_1 \dots i_k} \quad \text{a.e. in } Q$$

$$\chi_{i_1 \dots i_k} = \begin{cases} 0 & \text{---} \\ 1 & \text{if } \sum_{j=1}^k u_{i_j} = 1, u_{i_j} > 0, j=1 \dots k \end{cases} \quad (k=1, \dots, N)$$

$$\begin{aligned} f_i &= f_i(u) \\ u &= (u_1, \dots, u_N) \end{aligned}$$

Thus:

- $\left| \left\{ \sum_{j=1}^k f_{i_j}(u) < 0 \right\} \cap \left\{ \sum_{j=1}^k u_{i_j} = 1, u_{i_j} > 0, j=1 \dots k \right\} \right| = 0$

$$\left| \left\{ f_i(u) > 0 \right\} \cap \left\{ u_i = 0 \right\} \right| = 0 \quad i=1, \dots, N$$

$$|A| = \text{Lebesgue measure } A \subset Q \subset \mathbb{R}^{n+1}$$

- If $\sum_{j=1}^k f_{i_j}(u) \neq 0$ a.e. in $\omega \subset Q$ then

$$\begin{aligned} \chi_{\{u_i=0\}} &\rightarrow \chi_{\{u_i=0\}}^{\omega} & L^1(\omega) \\ \chi_{i_1 \dots i_k}^{\omega} &\rightarrow \chi_{i_1 \dots i_k} \end{aligned}$$

Existence of a variational solution (N -system)

$$u_0 \in \tilde{K} = \left\{ v \in L^2(\Omega)^N : \sum_{j=1}^N v_j \leq 1, v_i \geq 0, i=1, \dots, N \text{ in } \Omega \right\}$$

- i) $f = f(x, t, v) : Q \times [0, 1]^N \rightarrow \mathbb{R}^N$ continuous in v , a.e. $(x, t) \in Q$
- ii) $\exists \varphi_1 \in L^1(Q) : |f(x, t, v)| \leq \varphi_1(x, t) \quad \forall v \in [0, 1]^N, \text{ a.e. } (x, t) \in Q$

•
$$\begin{cases} \exists u \in C([0, T]; L^2(\Omega)^N \cap \tilde{K}) \cap L^2(0, T; H^1(\Omega)^N), u(0) = u_0, \partial_t u \in L^1(Q) + L^2(0, T; H^1(\Omega)^N) \\ P_u \in L^1(Q) \\ \int_Q (P_u - f(u)) \cdot (v - u) \geq 0, \forall v \in L^2(0, T; \tilde{K}) \end{cases}$$

.. If $\exists \lambda > 0 : \lambda |v - w|^2 - (f(v) - f(w)) \cdot (v - w) \geq 0$ the solution is unique!

$$\therefore f_i - \sum_{\substack{1 \leq i_1 < \dots < i_k \leq N \\ i \notin \{i_1, \dots, i_k\}}} \frac{1}{k} (f_{i_1} + \dots + f_{i_k})^+ \leq P_{u_i} \leq f_i^+ \text{ a.e. in } Q$$

$$\therefore \begin{aligned} &\text{If } \varphi_1 \in L^p(Q)^N \text{ then } u \in W_p^{2,1}(Q)^N \quad u \in C^\alpha \text{ if } p > (n+2)/2 \\ &u_0 \in K \cap W^{2-2/p, p}(\Omega)^N \quad 1 < p < \infty \quad \nabla u \in C^\alpha \text{ if } p > n+2 \end{aligned}$$

Existence of a variational solution (N -system) 1st order

$$u_0 \in \tilde{K} = \left\{ v \in L^2(\Omega)^N : \sum_{j=1}^N v_j \leq 1, v_i \geq 0, i=1, \dots, N \text{ in } \Omega \right\}$$

i) $f = f(x, t, v) : Q \times [0, 1]^N \rightarrow \mathbb{R}^N$ Lipschitz in v , a.e. $(x, t) \in Q$

ii) $\exists \varphi_1 \in L^2(Q) : |f(x, t, v)| \leq \varphi_1(x, t) \quad \forall v \in [0, 1]^N, \text{ a.e. } (x, t) \in Q$

$\exists! u \in L^2(0, T; \tilde{L}_H^2(\Omega; \mathbb{K}))$, $u|_{\Sigma} = 0$ $u(0) = u_0$

$\int_Q (Hu - fu) \cdot (v - u) \geq 0, \forall v \in L^2(0, T; \mathbb{K})$

$$Hu = \frac{\partial}{\partial t} u + (\vec{b} \cdot \nabla) u$$

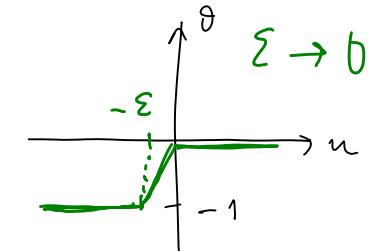
smooth $\vec{b} \in L^\infty(0, T; W^{1, \infty}(\Omega))$
 $\Sigma = \{ \vec{b}, \vec{m} < 0 \}$

$$\bullet Hu_i = f_i + f_i \chi_{\{u_i = 0\}} - \sum_{\substack{i \in \{i_1, \dots, i_k\} \\ 1 \leq i_1 < \dots < i_k \leq N}} \frac{1}{k} (f_{i_1} + \dots + f_{i_k})^+ \chi_{i_1, \dots, i_k} \quad \text{a.e. in } Q$$

$$\chi_{i_1, \dots, i_k} = \begin{cases} 0 & \text{---} \\ 1 & \text{if } \sum_{j=1}^k u_{i_j} = 1, u_{i_j} > 0, j = 1, \dots, k \end{cases} \quad (k = 1, \dots, N) \quad f_i = f_i(u)$$

Approximation by Semilinear N -systems

$$\left\{ \begin{array}{l} Hu_i^\varepsilon + f_i^+ \partial_\varepsilon(u_i^\varepsilon) - \sum_{\substack{1 \leq i_1 < \dots < i_k \leq N \\ i \in \{i_1, \dots, i_k\}}} \frac{1}{k} (f_{i_1} + \dots + f_{i_k})^+ \partial_\varepsilon(1 - u_{i_{\min}}^\varepsilon) = f_i \\ u_i^\varepsilon = 0, \text{ on } \sum_{i \in \{i_1, \dots, i_k\}} \\ u_i^\varepsilon(0) = u_{i_1} \quad \text{in } \Omega \end{array} \right. \quad \begin{array}{l} \text{in } Q = \Omega \times (0, T) \\ \text{monotone approximation} \end{array}$$



First, for $f \in L^2(Q)^N$, $u_0 \in \tilde{K}$ we find $u^\varepsilon \in \tilde{L}_H^2(Q)$

- $u_i^\varepsilon \geq -\varepsilon \quad i=1, \dots, N \quad \sum_{i=1}^N u_i^\varepsilon \leq 1 + \varepsilon \quad \text{by induction in } r=1, \dots, N$
 $u^\varepsilon \rightarrow u$ $u_{i_1 \dots i_r} \leq 1 + \varepsilon, 1 \leq i_1 < \dots < i_r \leq N$

- Energy estimate $\int_Q |u^\varepsilon - u|^2 \leq C\varepsilon \quad (\text{Homographic type})$

- Fixed point in $K = L^2(0, T; \tilde{K}) \ni w \rightarrow f(w) = g \in L^1 \rightarrow u_2 \in K$
 Strict contraction for small T



SOME RECENT REFERENCES

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On Hyperbolic Variational Inequalities of First Order and Some Applications

By

José Francisco Rodrigues

Universidade de Lisboa, Portugal

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THE NONLINEAR N -MEMBRANES EVOLUTION PROBLEM

J. F. Rodrigues,^{*} L. Santos,[†] and J. M. Urbano[‡]

Interfaces and Free Boundaries 7 (2005), 319–337

The N -membranes problem for quasilinear degenerate systems

ASSIS AZEVEDO[†]

Department of Mathematics, University of Minho,
Campus de Gualtar, 4710-057 Braga, Portugal

JOSE-FRANCISCO RODRIGUES[‡]

CMUC/University of Coimbra & University of Lisbon/CMAF,
Av. Prof. Gama Pinto, 2, 1649-003 Lisbon, Portugal

AND

LISA SANTOS[§]

CMAF/University of Lisbon & Department of Mathematics, University of Minho,
Campus de Gualtar, 4710-057 Braga, Portugal

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ON A CONSTRAINED REACTION-DIFFUSION SYSTEM RELATED TO MULTIPHASE PROBLEMS

JOSÉ-FRANCISCO RODRIGUES

Universidade de Lisboa/CMAF
Av. Prof. Gama Pinto 2
1649-003 Lisboa, Portugal

LISA SANTOS

Universidade do Minho/CMat
Campus de Gualtar
4710 - 057 Braga, Portugal