Equivariant map algebras

Alistair Savage

University of Ottawa

SOGG Workshop, Toronto
October 3, 2009

Joint work with Erhard Neher and Prasad Senesi

Slides: www.mathstat.uottawa.ca/~asavag2

Full details: arXiv:0906.5189

Outline

Goal: Classify the irreducible finite-dimensional representations of a certain class of Lie algebras.

Overview:

1. Equivariant map algebras
2. Examples
3. Evaluation representations
4. Classification theorem
5. Applications
    - recover some known classifications (often in a simplified manner)
    - produce some new classifications

Terminology:

$$\text{small} = \text{irreducible finite-dimensional}$$

(Untwisted) Map algebras

Notation

k - algebraically closed field of characteristic zero

X - scheme (or algebraic variety) over k

$\Gamma = \Gamma_X = \mathcal{O}_X(X)$ - coordinate ring of X

$\mathfrak{g}$ - finite-dimensional Lie algebra over k

Definition (Untwisted map algebra)

$M(X, \mathfrak{g})$ = Lie algebra of regular maps from X to $\mathfrak{g}$

Pointwise multiplication:

$$[\alpha, \beta]_{M(X,\mathfrak{g})}(x) = [\alpha(x), \beta(x)]_{\mathfrak{g}} \text{ for } \alpha, \beta \in M(X, \mathfrak{g})$$

Note: $M(X, \mathfrak{g}) \cong \mathfrak{g} \otimes \Gamma_X$

Examples

Discrete spaces

If X is a discrete variety, then

$$M(X, \mathfrak{g}) \cong \prod_{x \in X} \mathfrak{g}, \quad \alpha \mapsto (\alpha(x))_{x \in X}, \quad \alpha \in M(X, \mathfrak{g}).$$

In particular, if $X = \{x\}$ is a point, then

$$M(X, \mathfrak{g}) \cong \mathfrak{g}, \quad \alpha \mapsto (\alpha(x)), \quad \alpha \in M(X, \mathfrak{g}).$$

The isomorphisms are given by evaluation.

Current algebras

$$X = k^n \implies \Gamma_X = k[t_1, \dots, t_n]$$

Thus, $M(X, \mathfrak{g}) \cong \mathfrak{g} \otimes k[t_1, \dots, t_n]$ is a current algebra.

Untwisted multiloop algebras

$$X = (k^\times)^n \implies \Gamma_X = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$$

Thus,

$$M(X, \mathfrak{g}) \cong \mathfrak{g} \otimes k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$$

is the untwisted multiloop algebra.

If $n = 1$, this is called the untwisted loop algebra and plays an important role in the theory of (untwisted) affine Lie algebras.

Examples

Three point algebras

$$X = k \setminus \{0,1\} = \mathbb{P}^1 \setminus \{0,1,\infty\}$$
$$\implies \Gamma_X \cong k[t, t^{-1}, (t-1)^{-1}]$$

Thus,

$$M(X, \mathfrak{sl}_2) \cong \mathfrak{sl}_2 \otimes k[t, t^{-1}, (t-1)^{-1}]$$

is the three point $\mathfrak{sl}_2$ loop algebra.

Remarks

- Removing any 2 points from k results in an isomorphic map algebra.
- $M(X, \mathfrak{sl}_2)$ is isomorphic to the tetrahedron Lie algebra and to a direct sum of 3 copies of the Onsager algebra (Hartwig-Terwilliger 2007).

Equivariant map algebras

- G - finite group
- Suppose G acts on X and $\mathfrak{g}$ by automorphisms

Definition (equivariant map algebra)

The equivariant map algebra is the Lie algebra of G-equivariant maps from X to $\mathfrak{g}$:

$$M(X,\mathfrak{g})^G = \{\alpha \in M(X,\mathfrak{g}) : \alpha(g \cdot x) = g \cdot \alpha(x) \ \forall\, x \in X,\ g \in G\}$$

Note: If X is any scheme, then $M(X,\mathfrak{g})^G \cong M(X_{\mathrm{aff}},\mathfrak{g})^G$ where $X_{\mathrm{aff}} = \operatorname{Spec} \Gamma_X$ is the affine scheme with the same coordinate ring as X. So we often assume X is affine.

Equivariant map algebras – algebraic description

- Induced action on Γ_X given by

$$(g \cdot f)(x) = f(g^{-1} \cdot x), \quad f \in \Gamma_X, \quad x \in X, \quad g \in G$$

- G acts diagonally on $\mathfrak{g} \otimes \Gamma_X$:

$$g \cdot (u \otimes f) = (g \cdot u) \otimes (g \cdot f)$$

- Then

$$M(X, \mathfrak{g})^G \cong (\mathfrak{g} \otimes \Gamma_X)^G$$

Example: Trivial G-action on $\mathfrak{g}$

If G acts trivially on $\mathfrak{g}$, then

$$M(X, \mathfrak{g})^G \cong M(X/\!/G, \mathfrak{g}) \cong \mathfrak{g} \otimes \Gamma_X^G$$

where $X/\!/G = \operatorname{Spec} \Gamma_X^G$ is the quotient of X by G.

Thus $M(X, \mathfrak{g})^G$ is isomorphic to an untwisted map algebra.

Example: multiloop algebras

$$G = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}, \quad X = (k^\times)^n$$

- For $i = 1, \ldots, n$, let ξ_i be a primitive m_i-th root of unity.
- Define action of G on X by

$$(a_1, \ldots, a_n) \cdot (z_1, \ldots, z_n) = (\xi_1^{a_1} z_1, \ldots, \xi_n^{a_n} z_n)$$

- Define action of G on $\mathfrak{g}$ by specifying commuting automorphisms σ_i, $i = 1, \ldots, n$, such that $\sigma_i^{m_i} = 1$.

Then $M(X, \mathfrak{g})^G$ is the (twisted) multiloop algebra.

If $n = 1$, this is the (twisted) loop algebra.

Affine Lie algebras

The affine Lie algebras can be constructed as central extensions of loop algebras plus a differential:

$$\widehat{\mathfrak{g}} = M(X, \mathfrak{g})^G \oplus kc \oplus kd \qquad (n = 1)$$

Example: generalized Onsager algebra

$$G = \mathbb{Z}_2 = \{1, \sigma\}, \quad X = k^\times, \quad \mathfrak{g} = \text{simple Lie algebra}$$

- G acts on X by $\sigma \cdot x = x^{-1}$
- G acts on $\mathfrak{g}$ by any involution

When G acts on $\mathfrak{g}$ by the Chevalley involution, we write

$$\mathcal{O}(\mathfrak{g}) = M(X, \mathfrak{g})^G$$

Remarks

- If $k = \mathbb{C}$, $\mathcal{O}(\mathfrak{sl}_2)$ is isomorphic to the Onsager algebra (Roan 1991)
  - Key ingredient in Onsager's original solution of the 2D Ising model
- For $k = \mathbb{C}$, $\mathcal{O}(\mathfrak{sl}_n)$ was studied by Uglov and Ivanov (1996)

Evaluation

If $\mathbf{x} = (x_1, \ldots, x_n) \subseteq X$, we have the evaluation map

$$\mathrm{ev}_{\mathbf{x}} : M(X, \mathfrak{g})^G \to \mathfrak{g}^{\oplus n}, \quad \alpha \mapsto (\alpha(x_1), \ldots, \alpha(x_n))$$

Important: This map is not surjective in general!

For $x \in X$, define

$$G_x = \{g \in G : g \cdot x = x\}$$
$$\mathfrak{g}^x = \{u \in \mathfrak{g} : G_x \cdot u = u\}$$

Lemma

For $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$, $x_i \notin G \cdot x_j$ for $i \neq j$,

$$\operatorname{im} \mathrm{ev}_{\mathbf{x}} = \mathfrak{g}^{x_1} \oplus \cdots \oplus \mathfrak{g}^{x_n}$$

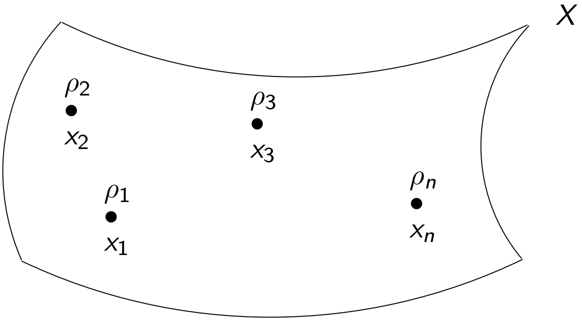
Evaluation representations

Given

- $\mathbf{x} = \{x_1, \ldots, x_n\} \subseteq X$, and
- representations $\rho_i : \mathfrak{g}^{x_i} \to \operatorname{End}_k V_i$, $i = 1, \ldots, n$

we define the (twisted) evaluation representation as the composition

$$M(X, \mathfrak{g})^G \xrightarrow{\mathrm{ev}_{\mathbf{x}}} \oplus_i \mathfrak{g}^{x_i} \xrightarrow{\otimes_i \rho_i} \operatorname{End}_k(\otimes_i V_i).$$



Important remarks

This notion of evaluation representation differs from the classical definition.

- Some authors use the term evaluation representation only for the case when evaluation is at a single point and call the general case a tensor product of evaluation representations.
- To a point $x \in X$, we associate a representation of $\mathfrak{g}^x$ instead of $\mathfrak{g}$. If G acts freely, this coincides with the usual definition.
- Recall that (when $\mathfrak{g}^x \subsetneq \mathfrak{g}$) not all reps of $\mathfrak{g}^x$ extend to reps of $\mathfrak{g}$ – so the new definition is more general.
- We do not require the representations ρ_i to be faithful.

We will see that the more general definition allows for a more uniform classification of representations.

Evaluation representations

$$\mathcal{R}_x = \{\text{isomorphism classes of small reps of } \mathfrak{g}^x\}$$

$$\mathcal{R}_X = \bigsqcup_{x \in X} \mathcal{R}_x$$

Since $G_{g \cdot x} = g G_x g^{-1}$, we have

$$g \cdot \mathfrak{g}^x = \mathfrak{g}^{g \cdot x}$$

We have an action of G on $\mathcal{R}_X$: if $[\rho] \in \mathcal{R}_x$, then

$$g \cdot [\rho] = [\rho \circ g^{-1}] \in \mathcal{R}_{g \cdot x},$$

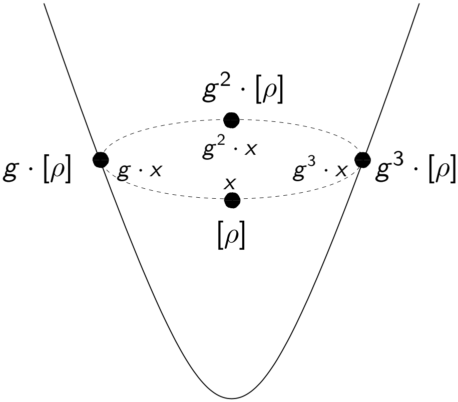
Definition ($\mathcal{F}$)

$\mathcal{F}$ is set of all $\Psi : X \to \mathcal{R}_X$ such that

1. Ψ is G-equivariant,
2. $\Psi(x) \in \mathcal{R}_x$ for all $x \in X$, and
3. $\operatorname{supp} \Psi = \{x \in X : \Psi(x) \neq 0\}$ is finite.

Evaluation representations

We think of $\Psi \in \mathcal{F}$ as assigning a finite number of (isom classes of) reps of $\mathfrak{g}^x$ to points $x \in X$ in a G-equivariant way.



Evaluation representations

For each $\Psi \in \mathcal{F}$, define

$$\mathrm{ev}_\Psi = \mathrm{ev}_{\mathbf{x}}(\Psi(x_i))_{i=1}^n = \mathrm{ev}_{x_1} \Psi(x_1) \otimes \cdots \otimes \mathrm{ev}_{x_n} \Psi(x_n)$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ is an n-tuple of points of X containing one point from each G-orbit in $\operatorname{supp} \Psi$ (the isom class is independent of this choice).

For $\Psi \in \mathcal{F}$, ev_Ψ is the isomorphism class of a small representation of $M(X, \mathfrak{g})^G$.

Proposition

The map

$$\mathcal{F} \longrightarrow \{\textit{isom classes of small reps of } M(X, \mathfrak{g})^G\}, \quad \Psi \mapsto \mathrm{ev}_\Psi$$

is injective.

One-dimensional representations

Recall: Any 1-dimensional rep of a Lie algebra L corresponds to a linear map $\lambda : L \to k$ such that $\lambda([L, L]) = 0$.

We identify such 1-dimensional reps with elements

$$\lambda \in (L/[L, L])^*$$

Two 1-dimensional reps are isomorphic if and only if they are equal as elements of $(L/[L, L])^*$.

Classification Theorem

Theorem (Neher-S.-Senesi 2009)

Suppose G is a finite group acting on an affine scheme (or variety) X and a finite-dimensional Lie algebra $\mathfrak{g}$. Let $\mathfrak{M} = M(X, \mathfrak{g})^G$.

Then the map

$$(\lambda, \Psi) \mapsto \lambda \otimes \mathrm{ev}_\Psi, \quad \lambda \in (\mathfrak{M}/[\mathfrak{M}, \mathfrak{M}])^*, \quad \Psi \in \mathcal{F}$$

gives a surjection

$$(\mathfrak{M}/[\mathfrak{M}, \mathfrak{M}])^* \times \mathcal{F} \twoheadrightarrow \{\text{isom classes of small representations of } \mathfrak{M}\}$$

In particular, all small representations are of the form

$$(\text{1-dim rep}) \otimes (\text{evaluation rep}).$$

Classification – Remarks

$$(\lambda, \Psi) \mapsto \lambda \otimes \mathrm{ev}_\Psi, \quad \lambda \in (\mathfrak{M}/[\mathfrak{M}, \mathfrak{M}])^*, \quad \Psi \in \mathcal{F}$$

1. This map is not injective in general since we can have nontrivial evaluation reps which are 1-dimensional. This happens when $\mathfrak{g}^x$ is not perfect (e.g. reductive but not semisimple).

   Example: $\mathfrak{g} = \mathfrak{sl}_2$, $G = \mathbb{Z}_2$, $X = k = \mathbb{C}$
   - G acts on $\mathfrak{g}$ by the Chevalley involution.
   - G acts on X by multiplication by -1.
   - Then $\mathfrak{g}^0 = \mathfrak{g}^G$ is one-dimensional and so has nontrivial 1-dim reps.

2. However, we can specify precisely when $\lambda \otimes \mathrm{ev}_\Psi \cong \lambda' \otimes \mathrm{ev}_{\Psi'}$.

3. The restriction of the map to either factor is injective.

Classification

$$(\lambda, \Psi) \mapsto \lambda \otimes \mathrm{ev}_\Psi, \quad \lambda \in (\mathfrak{M}/[\mathfrak{M},\mathfrak{M}])^*, \quad \Psi \in \mathcal{F}$$

Corollary

1. *If $\mathfrak{M}$ is perfect (i.e. $\mathfrak{M} = [\mathfrak{M},\mathfrak{M}]$), then we have a bijection*

   $$\mathcal{F} \leftrightarrow \{\textit{isom classes of small reps}\}, \quad \Psi \mapsto \mathrm{ev}_\Psi .$$

   In particular, all small reps are evaluation reps.

2. *If $[\mathfrak{g}^G, \mathfrak{g}] = \mathfrak{g}$, then $\mathfrak{M}$ is perfect and the above bijection holds.*

3. *If G acts on $\mathfrak{g}$ by diagram automorphisms, then $[\mathfrak{g}^G, \mathfrak{g}] = \mathfrak{g}$ and the above bijection holds.*

Note: Being perfect is not a necessary condition for the all small reps to be evaluation reps (as we will see).

Application: untwisted map algebras

If G is trivial, then

$$M(X, \mathfrak{g})^G = M(X, \mathfrak{g}), \quad \mathfrak{g}^G = \mathfrak{g}$$

Thus, if $\mathfrak{g}$ is perfect,

$$[\mathfrak{g}^G, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$$

and so all small reps are evaluation reps.

Application: multiloop algebras

Corollary

If $\mathfrak{M}$ is a (twisted) multiloop algebra, then $\mathfrak{M}$ is perfect and so we have a bijection

$$\mathcal{F} \leftrightarrow \{\textit{isom classes of small reps}\}, \quad \Psi \mapsto \mathrm{ev}_\Psi .$$

In particular, all small reps are evaluation reps.

Remarks

1. This recovers results of Chari-Pressley (for loop algebras) and Batra, Lau (multiloop algebras), but with a different description.
2. The description given above (in terms of $\mathcal{F}$) gives a simple and uniform description of the somewhat technical conditions appearing in previous classifications.
3. Action of G on X is free and so $\mathfrak{g}^x = \mathfrak{g}$ for all $x \in X$. So the more general notion of evaluation rep does not play a role.

Application: generalized Onsager algebra

$$G = \mathbb{Z}_2 = \{1, \sigma\}, \quad X = k^\times, \quad \mathfrak{g} = \text{simple Lie algebra}$$

- G acts on X by $\sigma \cdot x = x^{-1}$
- G acts on $\mathfrak{g}$ by any involution

Corollary

With G, X, $\mathfrak{g}$ as above, we have a bijection

$$\mathcal{F} \leftrightarrow \{\textit{isom classes of small reps}\}, \quad \Psi \mapsto \mathrm{ev}_\Psi .$$

In particular, all small reps are evaluation reps.

Remarks – generalized Onsager algebra

- There are two types of points of X:
  - $x \in \{\pm 1\} \implies G_x = G = \mathbb{Z}_2$, $\mathfrak{g}^x = \mathfrak{g}^G$
  - $x \notin \{\pm 1\} \implies G_x = \{1\}$, $\mathfrak{g}^x = \mathfrak{g}$
- $\mathfrak{g}^G$ can be semisimple or have a one-dimensional center

When $\mathfrak{g}^G$ has a one-dimensional center:

- the generalized Onsager algebra is not perfect
- we can place (nontrivial) one-dim reps of $\mathfrak{g}^G$ at the points ± 1
- under our more general definition of evaluation rep, all small reps are evaluation reps
- under classical notion of evaluation rep, there are small reps which are not evaluation reps

Moral: The more general definition of evaluation rep allows for a more uniform classification.

Special case: Onsager algebra

- When $k = \mathbb{C}$ and G acts on $\mathfrak{g} = \mathfrak{sl}_2$ by the Chevalley involution, then

$$\mathcal{O}(\mathfrak{sl}_2) \stackrel{\text{def}}{=} M(X, \mathfrak{sl}_2)^G$$

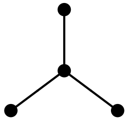
  is the Onsager algebra

- $\mathfrak{g}^{\{\pm 1\}}$ is one-dimensional abelian and $\mathcal{O}(\mathfrak{sl}_2)$ is not perfect.
- Small reps of $\mathcal{O}(\mathfrak{sl}_2)$ were classified previously (Date-Roan 2000)
  - classical definition of evaluation rep was used
  - not all small reps were evaluation reps
  - this necessitated the introduction of the type of a representation

Note: For the other cases, the classification seems to be new.

Application: a nonabelian example

$$G = S_3, \quad X = \mathbb{P}^1 \setminus \{0, 1, \infty\}, \quad \mathfrak{g} = \mathfrak{so}_8 \quad \text{(type } D_4\text{)}$$



D_4

- symmetry group of Dynkin diagram of $\mathfrak{g}$ is S_3
- so G acts on $\mathfrak{g}$ by diagram automorphisms
- for any permutation of the set $\{0, 1, \infty\}$, $\exists!$ homography of $\mathbb{P}^1$ inducing that permutation
- so G acts naturally on X

Thus we can form the equivariant map algebra $M(X, \mathfrak{g})^G$ and show that it is perfect.

Our classification tells us all small reps are eval reps and gives a bijection between these and the set $\mathcal{F}$.

Application: a nonabelian example

It is straightforward to find the points with nontrivial stabilizer:

x	G_x	Type of $\mathfrak{g}^x$
-1	$\{\mathrm{Id}, (0\,\infty)\} \cong \mathbb{Z}_2$	B_3
2	$\{\mathrm{Id}, (1\,\infty)\} \cong \mathbb{Z}_2$	B_3
$\frac{1}{2}$	$\{\mathrm{Id}, (0\,1)\} \cong \mathbb{Z}_2$	B_3
$e^{\pm\pi i/3}$	$\{\mathrm{Id}, (0\,1\,\infty), (0\,\infty\,1)\} \cong \mathbb{Z}_3$	G_2

The sets

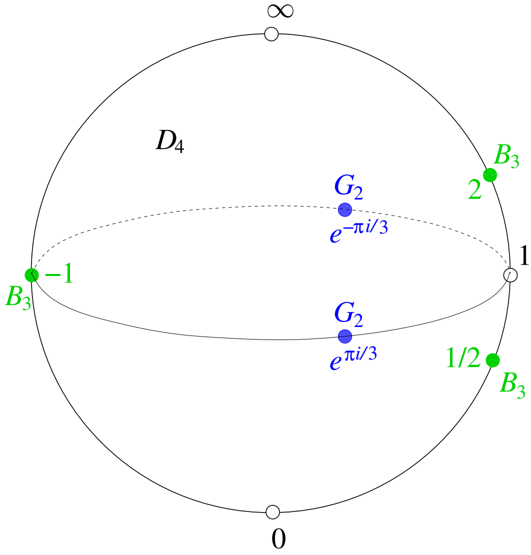
$$\left\{-1, 2, \frac{1}{2}\right\} \quad \text{and} \quad \left\{e^{\pi i/3}, e^{-\pi i/3}\right\}$$

are G-orbits.

So elements of $\mathcal{F}$ can assign

- irreps of type B_3 to the 3-element orbit
- irreps of type G_2 to the 2-element orbit
- irreps of type D_4 to the other points (6-element orbits)

Application: a nonabelian example



Question

Question

When are all small representations evaluation representations?

We have seen

- perfect (i.e. $\mathfrak{M} = [\mathfrak{M}, \mathfrak{M}]$) $\implies$ all small reps are eval reps
- the converse is not true (e.g. the Onsager algebra)

Reduction

Since all reps are of the form

$$(\text{1-dim rep}) \otimes (\text{eval rep})$$

it suffices to know when there are one-dimensional reps that are not evaluation reps.

When are all small reps eval reps?

Definition

$$\tilde{X} = \{x \in X \mid \mathfrak{g}^x \neq [\mathfrak{g}^x, \mathfrak{g}^x]\}$$

Recall

$$\mathfrak{g}^x = [\mathfrak{g}^x, \mathfrak{g}^x] \iff \text{all one-dimensional reps of } \mathfrak{g}^x \text{ are trivial}$$

Thus, $\tilde{X}$ is precisely the set of points where we can place nontrivial one-dimensional evaluation representations.

Proposition (Neher-S-Senesi 2009)

If X is a Noetherian scheme (i.e. Γ is finitely generated) and $|\tilde{X}| = \infty$, then $M(X, \mathfrak{g})^G$ has one-dimensional representations that are not evaluation representations.

When are all small reps eval reps?

Example

$$G = \mathbb{Z}_2 = \{1, \sigma\}, \quad X = k^2, \quad \mathfrak{g} = \mathfrak{sl}_2(k)$$

- $\sigma \cdot (x_1, x_2) = (x_1, -x_2)$, $(x_1, x_2) \in k^2$
- σ acts as Chevalley involution on $\mathfrak{g}$

Then

- $x = (x_1, x_2) \in k^2,\ x_2 \neq 0 \implies G_x = \{1\} \implies \mathfrak{g}^x = \mathfrak{g}$
- $x = (x_1, 0) \in k^2 \implies G_x = G \implies \mathfrak{g}^x = \mathbb{C}$ (abelian)

Thus

$$\tilde{X} = \{(x_1, 0) \mid x_1 \in k\} \quad \text{and so} \quad |\tilde{X}| = \infty.$$

Therefore $M(X, \mathfrak{g})^G$ has one-dimensional reps that are not eval reps.

When are all small reps eval reps?

Question

$|\tilde{X}| < \infty$ is a necessary condition for all small reps to be eval reps.

Is it sufficient?

Answer: NO

When are all small reps eval reps?

Let $\mathbf{x}$ be a finite subset of X and consider the commutative diagram:

$$\begin{tikzcd}
\mathfrak{M} \arrow[r, two heads, "\mathrm{ev}_{\mathbf{x}}"] \arrow[dr, two heads] & \bigoplus_{x \in \mathbf{x}} \mathfrak{g}^x \arrow[r, two heads] & \bigoplus_{x \in \mathbf{x}} \mathfrak{g}^x / [\mathfrak{g}^x, \mathfrak{g}^x] \\
 & \mathfrak{M}/[\mathfrak{M}, \mathfrak{M}] \arrow[ur, two heads, "\gamma"'] &
\end{tikzcd}$$

Theorem

If $|\tilde{X}| < \infty$, *then*

$$(\lambda, \Psi) \mapsto \lambda \otimes \mathrm{ev}_\Psi, \quad \lambda \in (\ker \gamma)^*, \quad \Psi \in \mathcal{F},$$

is a bijection

$$(\ker \gamma)^* \times \mathcal{F} \longleftrightarrow \{\textit{isom classes of small reps}\}$$

Corollary

If $|\tilde{X}| < \infty$*, all small reps are eval reps if and only if* $\ker \gamma = 0$*. This is true if and only if*

$$[\mathfrak{M}, \mathfrak{M}] = \mathfrak{M}^d := \{\alpha \in \mathfrak{M} \mid \alpha(x) = [\mathfrak{g}^x, \mathfrak{g}^x] \ \forall\, x \in X\}.$$

Note: $[\mathfrak{M}, \mathfrak{M}] \subseteq \mathfrak{M}^d$ is always true.

Example: $|\tilde{X}| < \infty$ with small reps that are not eval reps

- $\mathfrak{g} = \mathfrak{sl}_2(k)$
- $X = Z(x^2 - y^3) = \{(x, y) \mid x^2 = y^3\} \subseteq k^2$
- So $\Gamma = k[x, y]/(x^2 - y^3)$
- $G = \mathbb{Z}_2 = \{1, \sigma\}$
- $\sigma \cdot (x, y) = (-x, y)$
- This action fixes $x^2 - y^3$ and so induces an action of G on X.
- Only fixed point is the origin.
- Thus $\tilde{X} = \{0\}$ and so $|\tilde{X}| < \infty$

Then one can easily show that

$$\mathfrak{M}^d/[\mathfrak{M}, \mathfrak{M}] \cong yk[y]/(y^3) \neq 0$$

Thus $[\mathfrak{M}, \mathfrak{M}] \subsetneq \mathfrak{M}^d$ and so $\mathfrak{M}$ has small reps that are not eval reps.

Note: X has a singularity at 0.

Further directions (work in progress)

- The category of finite-dimensional representations of an equivariant map algebra is not semisimple in general.
- Can one describe the finite-dimensional representations (not necessarily irreducible)?
  - (twisted) Weyl modules (untwisted case considered by Chari-Fourier-Khandai 2009)
  - block decompositions
    - untwisted loop (Chari-Moura)
    - twisted loop (Senesi)
    - untwisted map algebras (Kodera)
- Replace equivariant map algebras by G-equivariant sheaves of Lie algebras

Happy Birthday Allen!!