Fields Institute, U of T

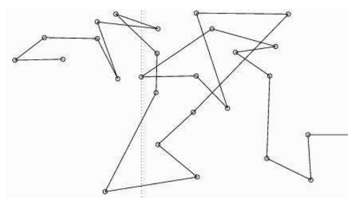
DASF's 85th

30 April 2010

Trajectories: some applications and some theory

David R. Brillinger

Statistics, UC Berkeley



Reminiscences / U of T days.

Teachers.

Year 1 R. Wormleighton (Princeton 55, Tukey)

Year 2 A. P.Dempster (Putnam winner, Princeton 56, Tukey)

Year 3 D. A. S. Fraser (Putnam winner, Princeton 48, Tukey)

"Statistics: an Introduction"

Year 4 D. B. DeLury, D. A. S. Fraser

"Nonparametric Methods in Statistics"

Fisher "Statistical Methods and Scientific Inference"

¿ Probability? Measure theory - Weber

DASF and APD each took roads less travelled

- role models for the students

Claude Bissell "Don't be well-rounded. Be angular."

Does anyone here not recognize this handwriting?

Other characteristics: a nonacademic life, daughters, canoes, lakes, cottages, travelling, swimming

Nonlinear analyses, writings, supervising theses, talks, books, research

NONPARAMETRIC TOLERANCE REGIONS

BY D. A. S. FRASER

University of Toronto

1. Summary. Nonparametric tolerance regions can be constructed from statistically equivalent blocks using published graphs by Murphy [8]. In this paper the procedure for obtaining the statistically equivalent blocks is generalized. The n 'cuts' used to form the $n + 1$ blocks need not cut off one block at a time, but at each stage may cut off a group of blocks, the group to be further divided at a later stage by a different type of cut in general. An example is given which indicates possible applications.

The results are also interpreted for discontinuous distributions by indicating the necessary modifications to the corresponding theorem in [7].

2. Introduction. The generality with which nonparametric tolerance regions can be formed has been successively treated by Wilks, Wald, Scheffé, Tukey, and others in a series of papers [1], [2], [3], [4], [5], [6], [7]. In each case the sample space for n observations from a continuous distribution is divided by these observations into $n + 1$ regions or blocks. Subject to mild restrictions on the procedure used to divide the sample space, the proportions of the population contained in these regions have an elementary distribution, a uniform distribution over a set prescribed by simple inequalities. Furthermore the marginal distribution of the proportion of the population which lies in a group of these regions has the Beta distribution. This enables the statistician to choose enough regions to make a probability statement such as the following: "In repeated sampling

Trajectory - a concept basic to engineering and science

- an old word for a random process

Structure of talk.

I. Track data $\{(t_k, \mathbf{r}(t_k)), k=1,...,K\}$ - six examples

II. Some formalism

III. Some probability

IV. Stochastic modelling

V. Data analysis of the examples

VI. Inference tools

VII. Summary and discussion

I. Track data

Example 1. *Argentina*

med ex-frun Claudia och dottern **Giannina**, se det argentinska landslaget göra inte mindre än sex makalösa mål på Serbien och Montenegro i ödesmatchen i ”Dödens grupp”.

Efter varje mål skrek

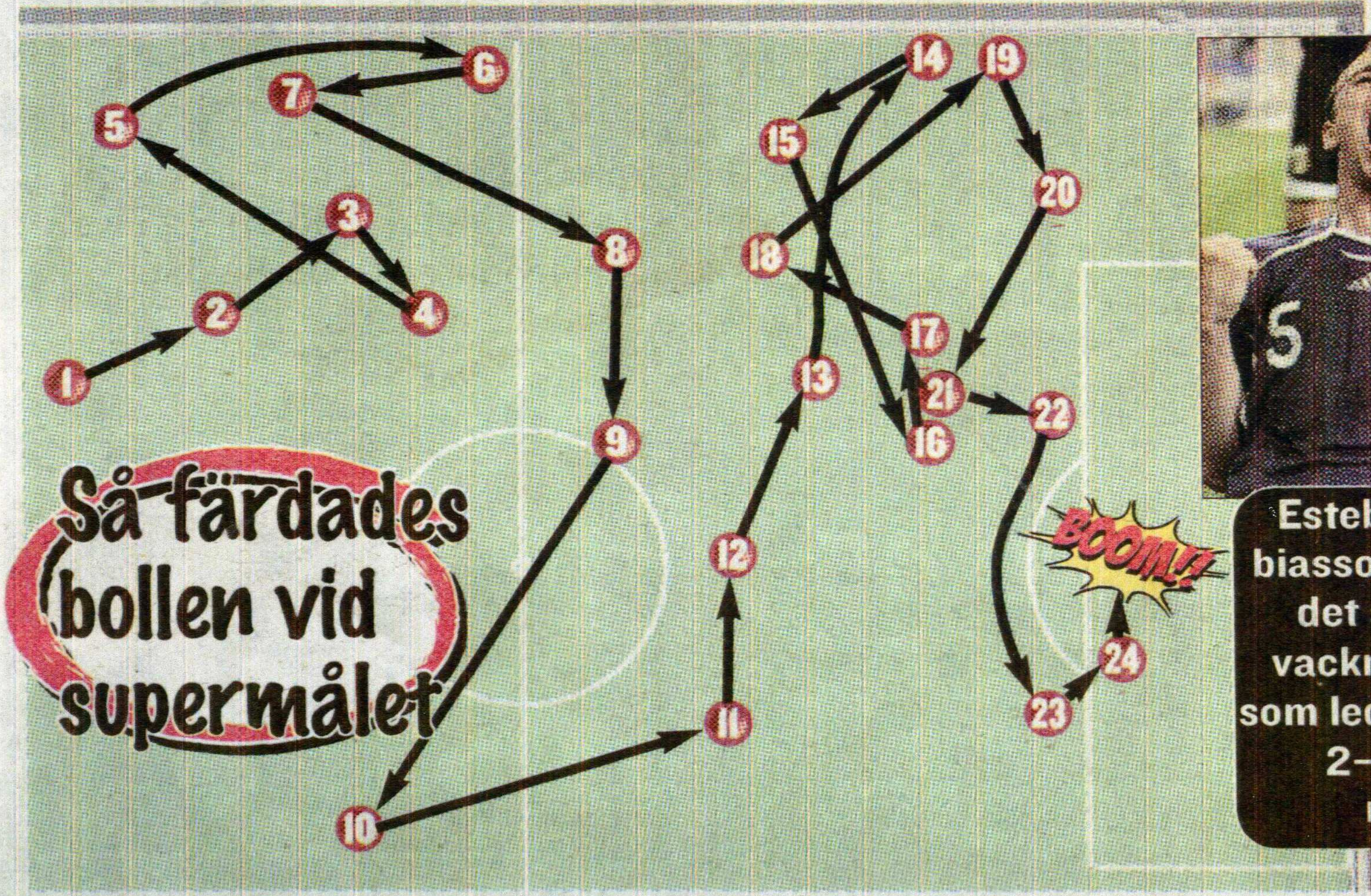
Jonas Esbjörnsson

jonas.esbjornsson @expressen.se

ten José Pekerman instämde:

– Vi gjorde en fantastisk match.

Efter åtta mål på två matcher är Argentina nu plötsligt storfavorit till guldet.



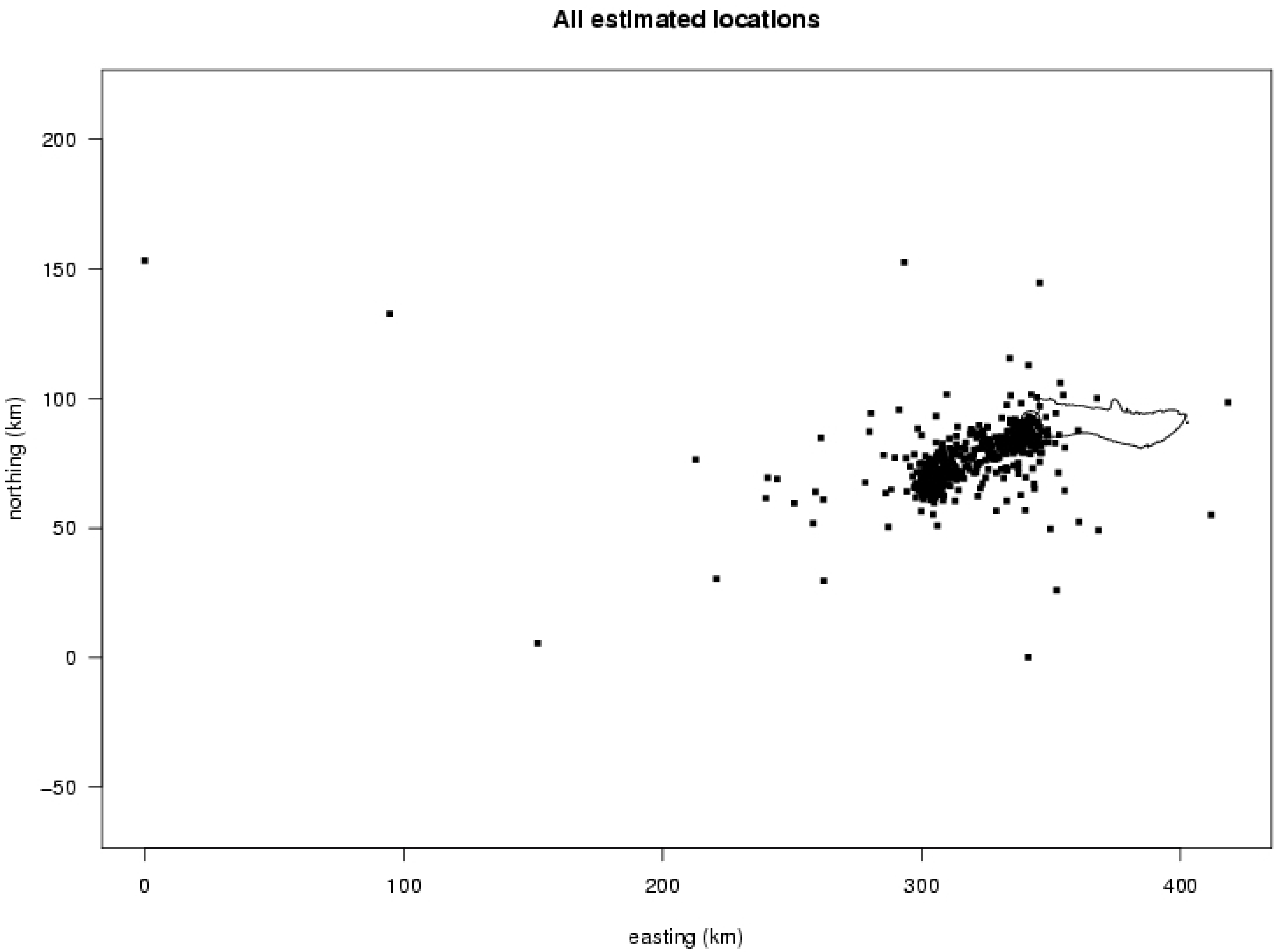
Esteban Cambiasso avslutade det sanslöst vackra anfallet som ledde fram till 2–0 i går.

Foto: AP

Example 2. *Marine biology*. Hawaiian monk seal

Most endangered marine mammal in US waters, $n \approx 1300$
Live 30 yrs. Male 230 kg, female 270 kg
Motivation: management purposes, where forage geographically and vertically?

GPS locations, $\{\mathbf{r}(t_k)\}$



Brillinger, Stewart and Littnan (2008)

Example 3. *Chandler wobble*.Variation of latitude due to nutation

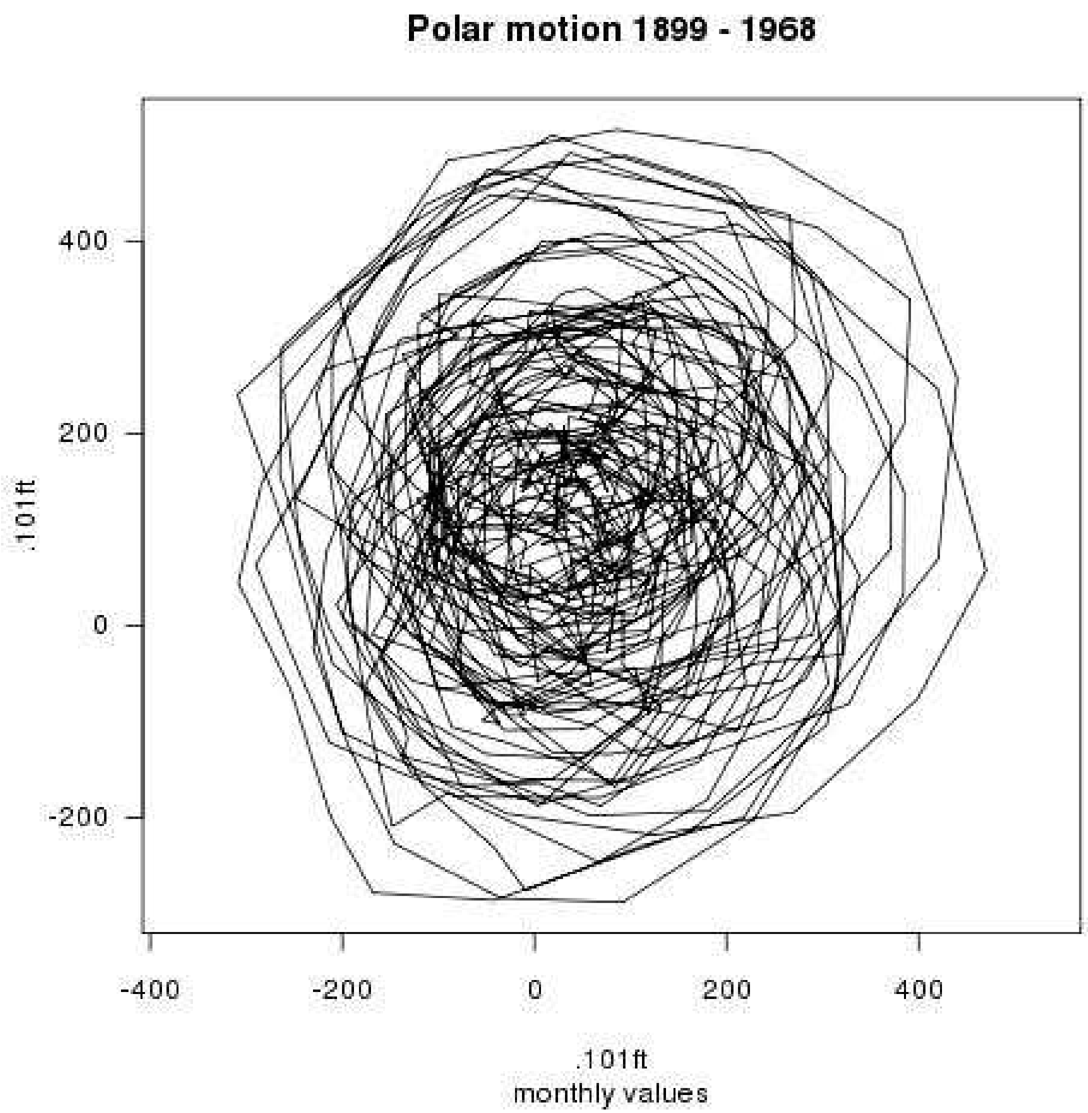
Predicted by Euler. Period 428 days. Observed by Chandler (actuary)

L. Euler

1707-1783

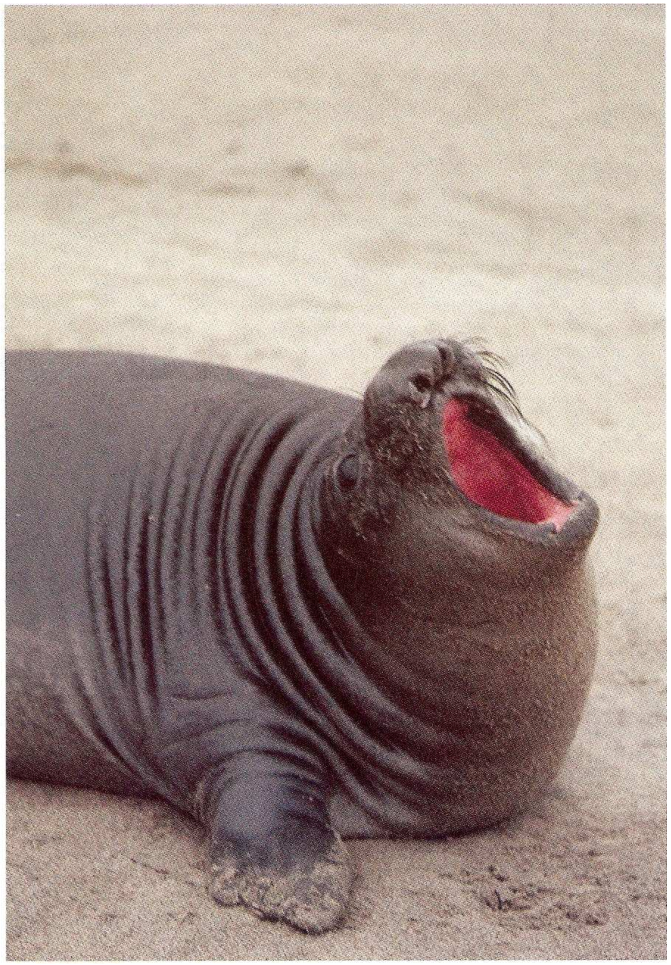
S. Chandler, actuary

1846-1913



D. R. Brillinger (1973)

Example 4. *Elephant seal*.

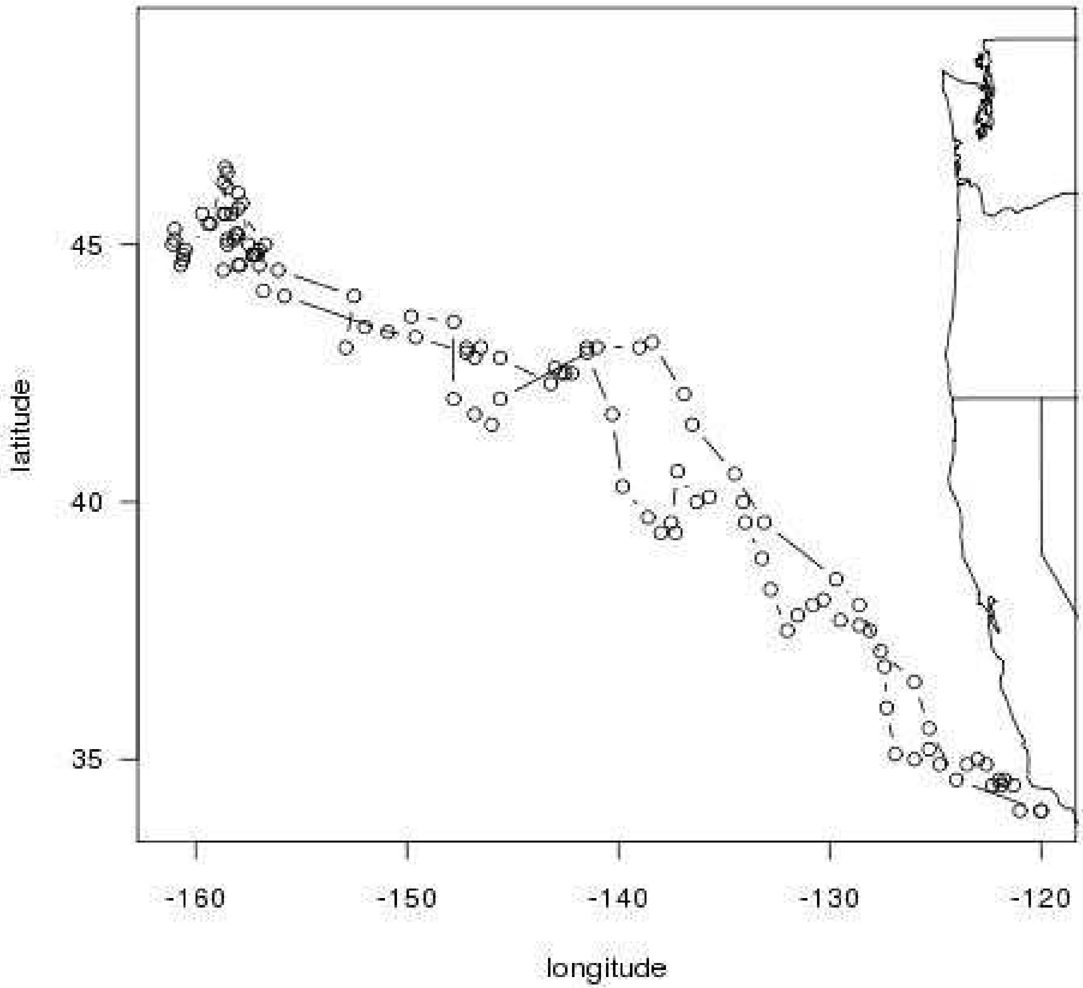


Were endangered, now 150000

Females: 600-800kg   Males: 2300kg

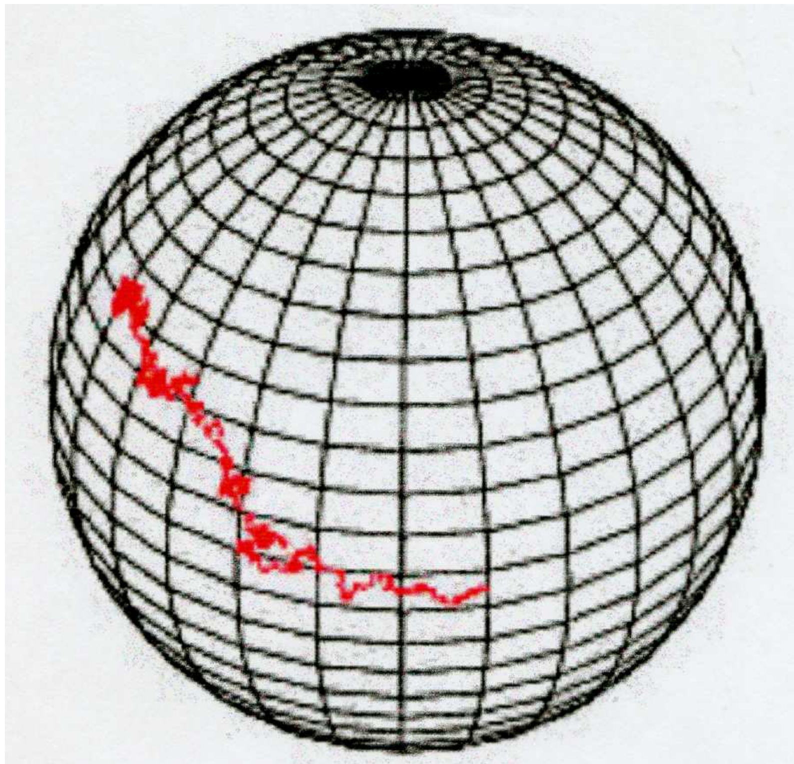
Females: live 16-18 yrs: Males: 12-14

One journey



D. R. Brillinger and B. S. Stewart (1998)

Surface of sphere



Example 5. *Whale shark*.

Slow moving filter feeder.

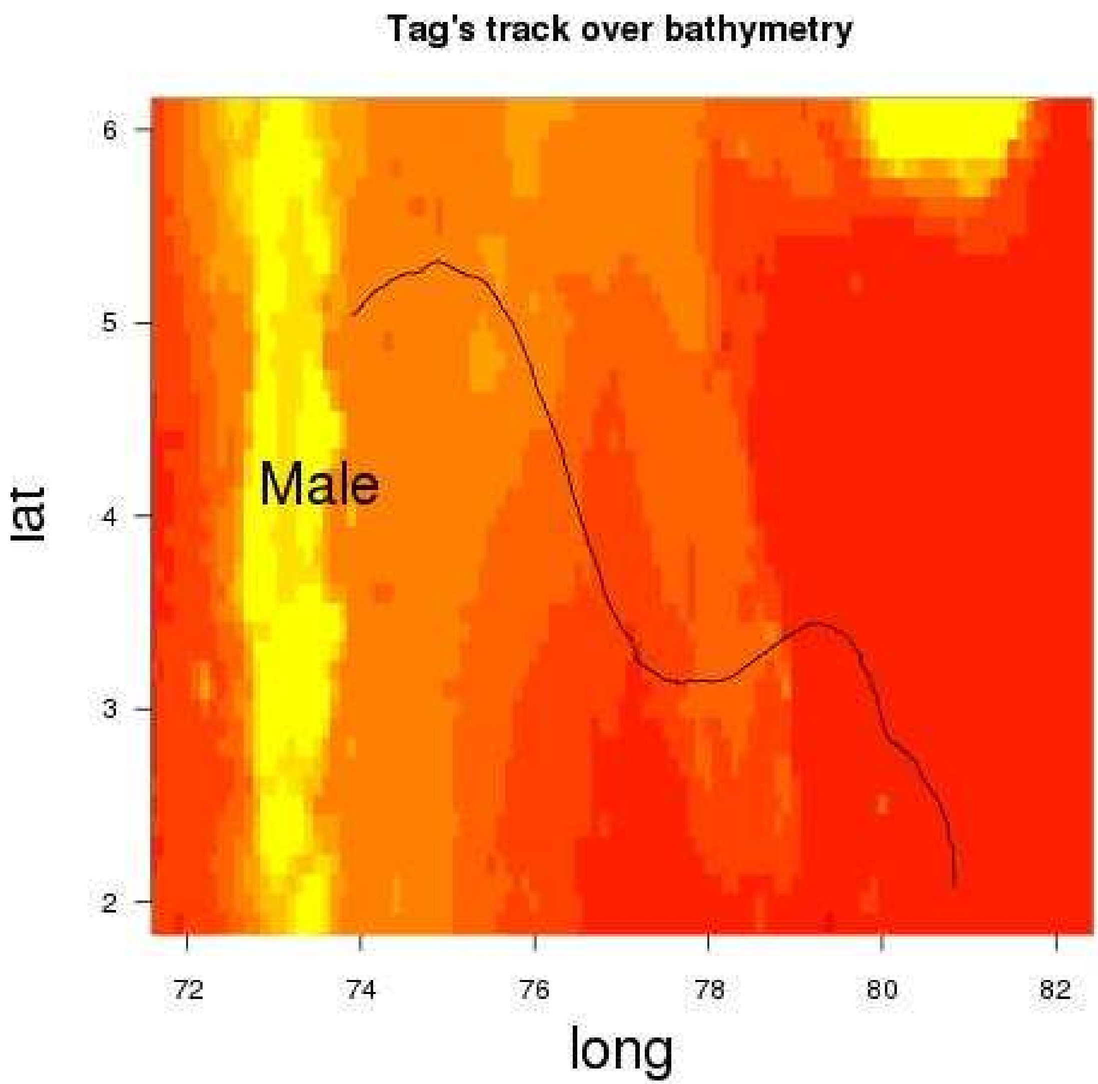
Largest living fish species.

Can grow up to 60 ft in length and can weigh up to 15 tons

Brent

Popup tag provides locations via GPS

Whale shark's tag trajectory after (unplanned) release



D. R. Brillinger and B. S. Stewart (2009)

Example 6.

Starkey Reserve, Oregon

Designed to answer management questions, ...

Can elk, deer, cows, bikers, hikers, riders, hunters coexist?

Foraging strategies, habitat preferences, dynamics of population densities?

Photo by F. Vanni

Summer

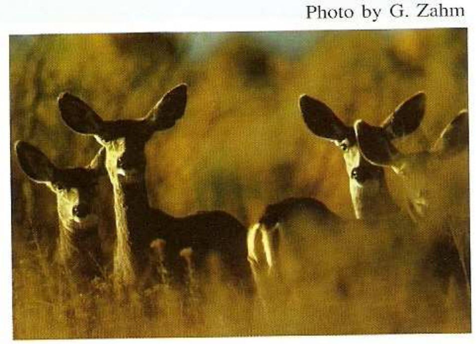
Photo by M. Wisdom

Winter

Photo by R. Cook

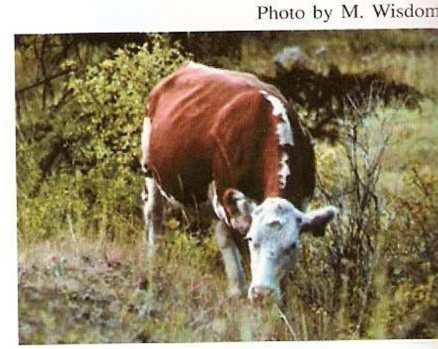
Elk

Photo by G. Zahm

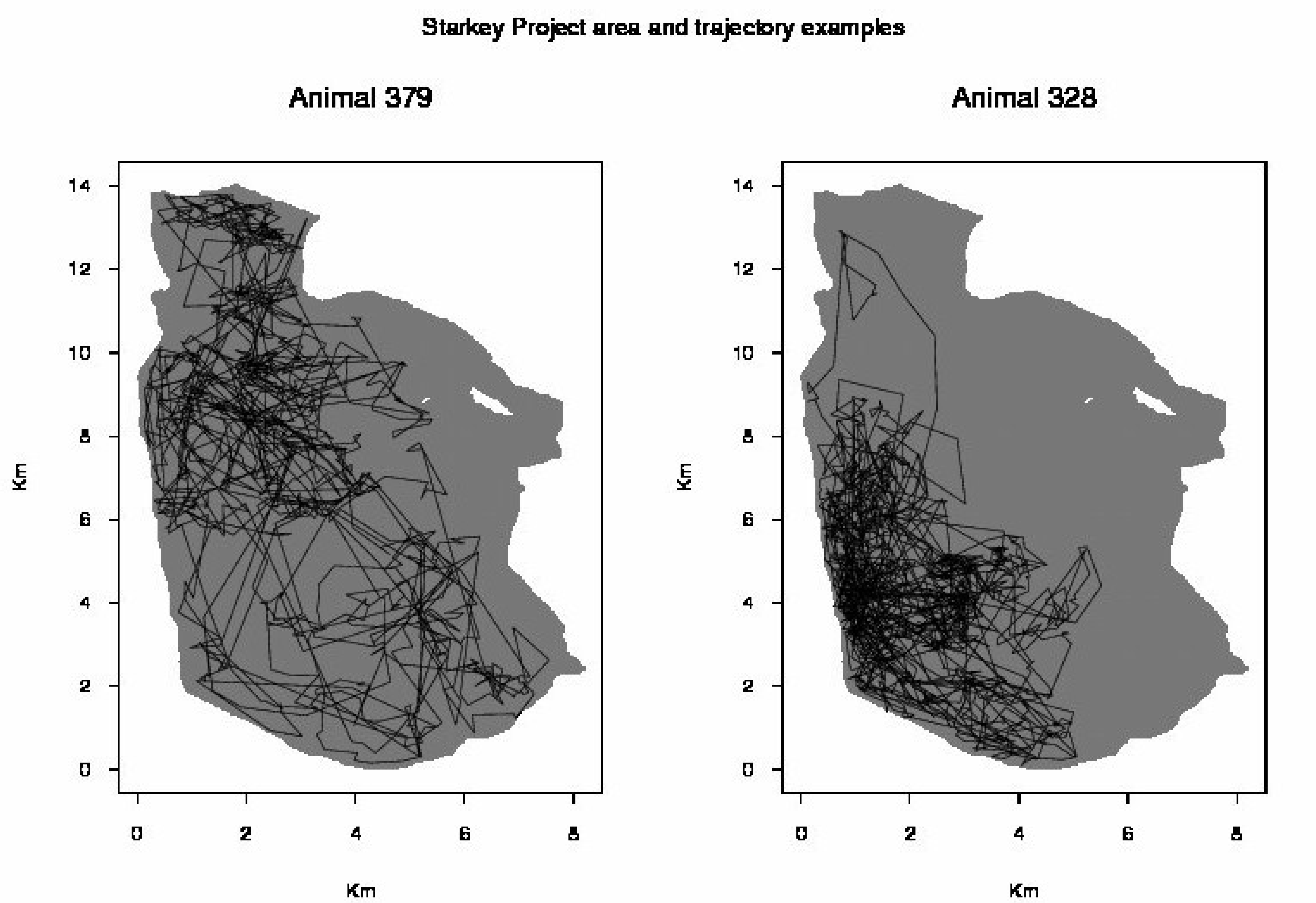


Mule deer

Photo by M. Wisdom



Cattle


Starkey Project area and trajectory examples
Animal 379
Animal 328
Km
Km
Km
Km
0
2
4
6
8
10
12
14

II. Some formalism.

Differential equations

(t, $\mathbf{r}(t)$) t: time, **r**: location

Deterministic case

$d\mathbf{r}(t)/dt = \mathbf{v}(t)$ OR $d\mathbf{r} = \mathbf{v}dt$, **v**: velocity

G. Leibniz

1646-1716

Newton's second law, $F = ma$

Scalar-valued potential function, H

Planar case, location $\mathbf{r} = (x,y)$, time t

An example $\quad d\mathbf{r}(t)/dt = \mathbf{v}(t)$

$$d\mathbf{v}(t)/dt = -\beta\, \mathbf{v}(t) - \beta \text{ grad } H(\mathbf{r}(t),t)$$

$\mathbf{v}$: velocity $\quad \beta$: damping (friction)

becomes $d\mathbf{r}/dt = -\text{ grad } H(\mathbf{r},t) = \boldsymbol{\mu}(\mathbf{r},t)$, for β large

Useful heuristic

Particular potential functions.

Attraction

To point a, $H(r) = \alpha|\mathbf{r}\text{-}\mathbf{a}|^2$, $\alpha > 0$

$\tfrac{1}{2}\sigma^2 \log |\mathbf{r}\text{-}\mathbf{a}| - \delta|\mathbf{r}\text{-}\mathbf{a}|$ bird motion

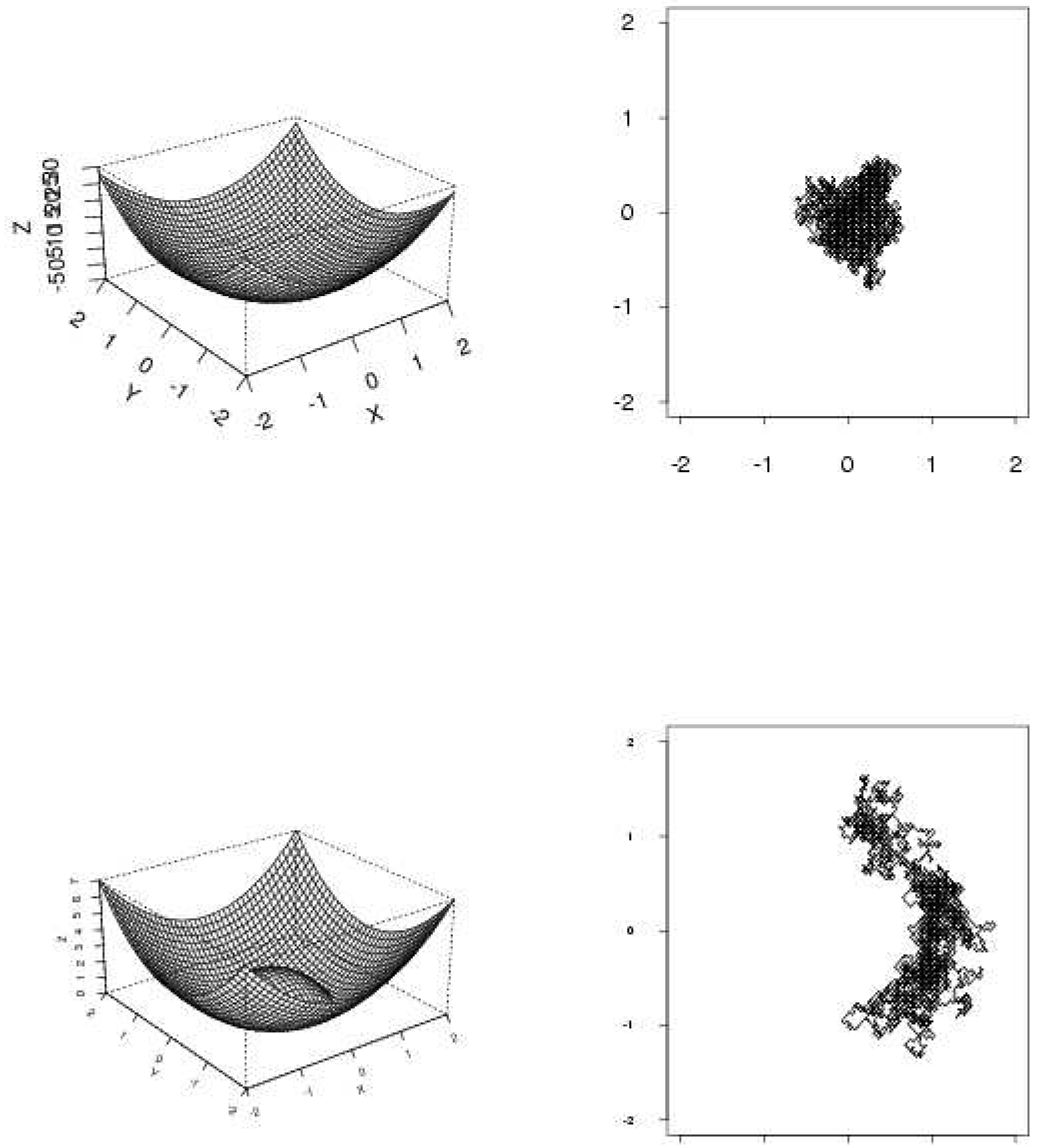
To region, **a** nearest point

Repulsion

From point, $H(\mathbf{r}) = |\mathbf{r}\text{-}\mathbf{a}|^{-2}$

From region, **a** nearest point

Potential functions and simulated trajectories



Attraction and repulsion

$$H(\mathbf{r}) = \alpha(1/r^{12} - 1/r^{6})$$

Quadratic

$$H(\mathbf{r}) = \beta_{10}x + \beta_{01}y + \beta_{20}x^2 + \beta_{11}xy + \beta_{02}y^2$$

Wavelets, local regression, tensor spline expansion

$$H(x,y) = \sum_{j,k} \beta_{jk}\, \Phi_j(x)\, \Psi_k(y), \quad \text{tensor product}$$

Moving attractor/repellor

$$H(\mathbf{r},t) = \beta(|\mathbf{r}-\mathbf{a}(t)|)$$

III. Some probability.

Probability space, (Ω, F, P)

Sample space Ω, σ-field F, probability measure, P

Random variable X,

$\{\omega \text{ in } \Omega: X(\omega) \le x\}$ in F for x in R

Vector-valued case - on same probability space

Filtration $\{F_n\}$, sequence of increasing σ-fields each in F

$\{Y_n\}$ adapted to F, Y_n is F_n measureable for all n

Random function, $\{\mathbf{B}(t;\omega)\}$ ω a r.v

Brownian motion, $\mathbf{B}(t)$, values in R^p

position of particle at time t

form of random walk in continous time

Disjoint increments are independent and such that $\mathbf{B}(t+s)-\mathbf{B}(t)$ is $N_p(\mathbf{0},s\mathbf{I}_p)$ $s > 0$

$d\mathbf{B}(t)=\mathbf{B}(t+dt)-\mathbf{B}(t)$ is $N_p(0,dt\mathbf{I}_p)$, dt small > 0

Almost surely: continuous, nowhere differentiable, unbounded variation

Stochastic integral.

Assume $\phi(t)$ is F_t -measureable and

$$E \int_0^T \phi(t)^2 dt < \infty$$

A sequence of predictable step functions $\{\phi_n\}$ exists with

$$\lim_n E \int_0^T |\phi_n(t)-\phi(t)|^2 dt = 0$$

$$I(\phi_n) = \int_0^T \phi_n(t)dB(t)$$

$$I(\phi) = \int_0^T \phi(t)dB(t) = \lim_n I(\phi_n)$$

($\{I(\phi_n)\}$ is a Cauchy sequence in $L^2(P)$, limit exists)

$I(\phi)$ may be approximated by $I(\phi_n)$

Stochastic differential equations (SDEs).

$$d\mathbf{r}(t) = \boldsymbol{\mu}(\mathbf{r}(t),t)dt + \boldsymbol{\sigma}(\mathbf{r}(t),t)d\mathbf{B}(t)$$

$$\mathbf{r}(t) - \mathbf{r}(0) = \int_0^t d\mathbf{r}(s) = \int_0^t \boldsymbol{\mu}(\mathbf{r}(s),s)ds + \int_0^t \boldsymbol{\sigma}(\mathbf{r}(s),s)d\mathbf{B}(s)$$

$\boldsymbol{\mu}$: drift, $\boldsymbol{\sigma}$: diffusion, $\{\mathbf{B}(t)\}$: Brownian

Interpretations

$$E\{d\mathbf{r}(t)|\mathbf{r}(u), u \le t\} = \mu(\mathbf{r}(t),t)dt$$

$$Var\{d\mathbf{r}(t)|\mathbf{r}(u), u \le t\} = \sigma(\mathbf{r}(t))\sigma(\mathbf{r}(t))'dt$$

$\mu(\mathbf{r},t) = - \nabla H(\mathbf{r},t)$ H real-valued $\nabla = (\partial/\partial x, \partial/\partial y)$

IV. Stochastic modelling.

Available data, $\{t_j, \mathbf{r}(t_j)\}$

Euler scheme approximant to SDE is

$$(\mathbf{r}(t_{i+1})-\mathbf{r}(t_i))/(t_{i+1}-t_i) = \boldsymbol{\mu}(\mathbf{r}(t_i),t_i) + \boldsymbol{\sigma}(\mathbf{r}(t_i),t_i)\mathbf{Z_{i+1}}/\sqrt{(t_{i+1}-t_i)} \quad t_{i+1}>t_i)$$

$\mathbf{Z}_i$: independent standard vector normals
Values for t between t_i and t_{i+1} obtained by simple interpolation.

Ignoring an initial term log likelihood:

$$-\tfrac{1}{2} \Sigma_i (\log 2\pi + \log |\boldsymbol{\sigma}_i \boldsymbol{\sigma}_i'| + \mathrm{tr}\{(\mathbf{r}_{i+1} - \mathbf{r}_i - \boldsymbol{\mu}_i)(\boldsymbol{\sigma}_i \boldsymbol{\sigma}_i^{')-1}(\mathbf{r}_{i+1}-\mathbf{r}_i-\boldsymbol{\mu}_i)'\})$$

V. Data analyses.

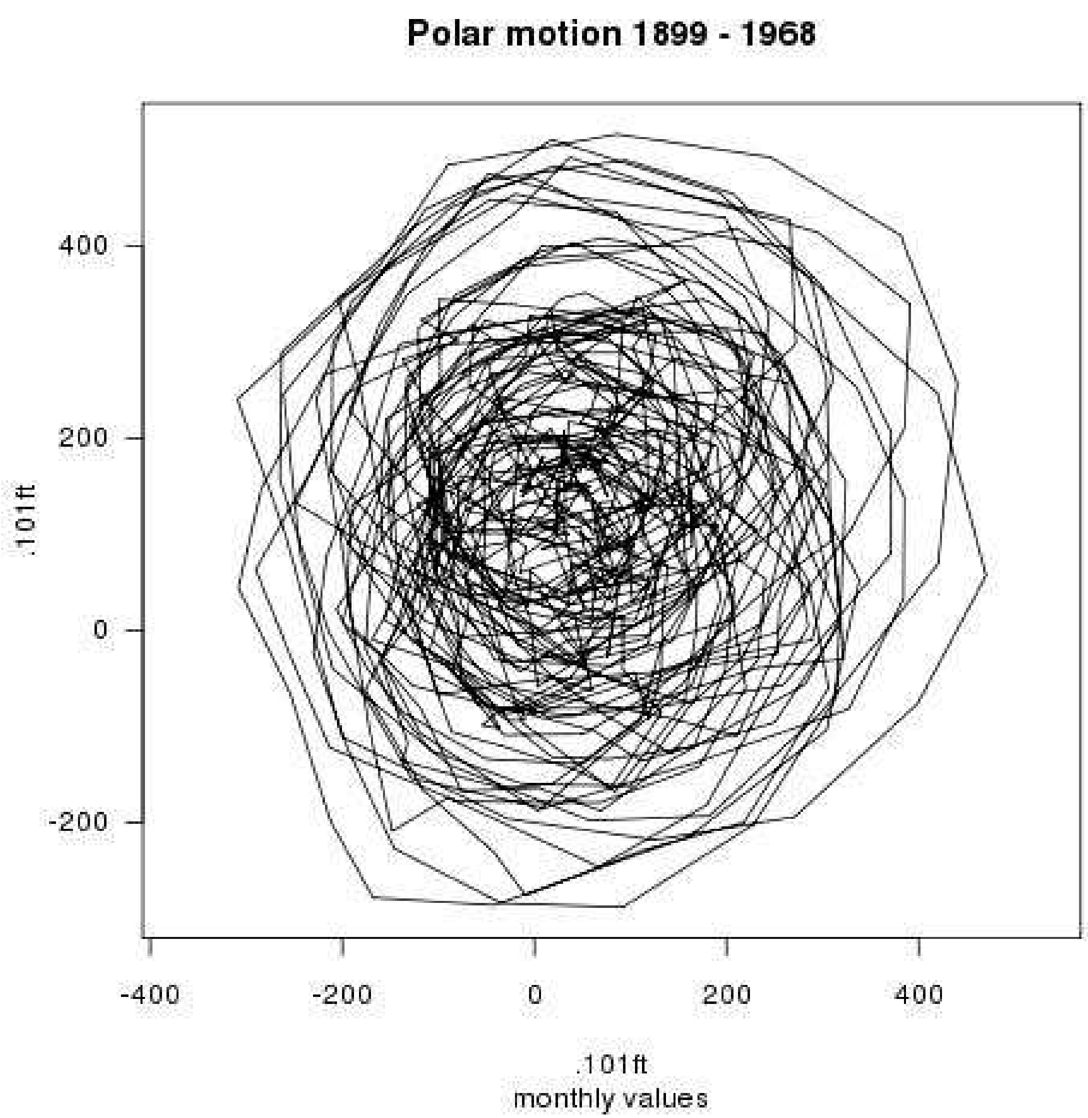
Example 3. *Chandler wobble*.

Chandler inferred presence of 12 and approx 14 months components in wobble.

Serious concern to scientists at end of 1800s

Data would provide information on interior structure of Earth

Network of stations set up to collect North Star coordinates


Polar motion 1899 - 1968
400
200
0
-200
-400
-200
0
200
400
.101ft
.101ft
monthly values

Monthly derived data, $\Delta t = 1$ month.

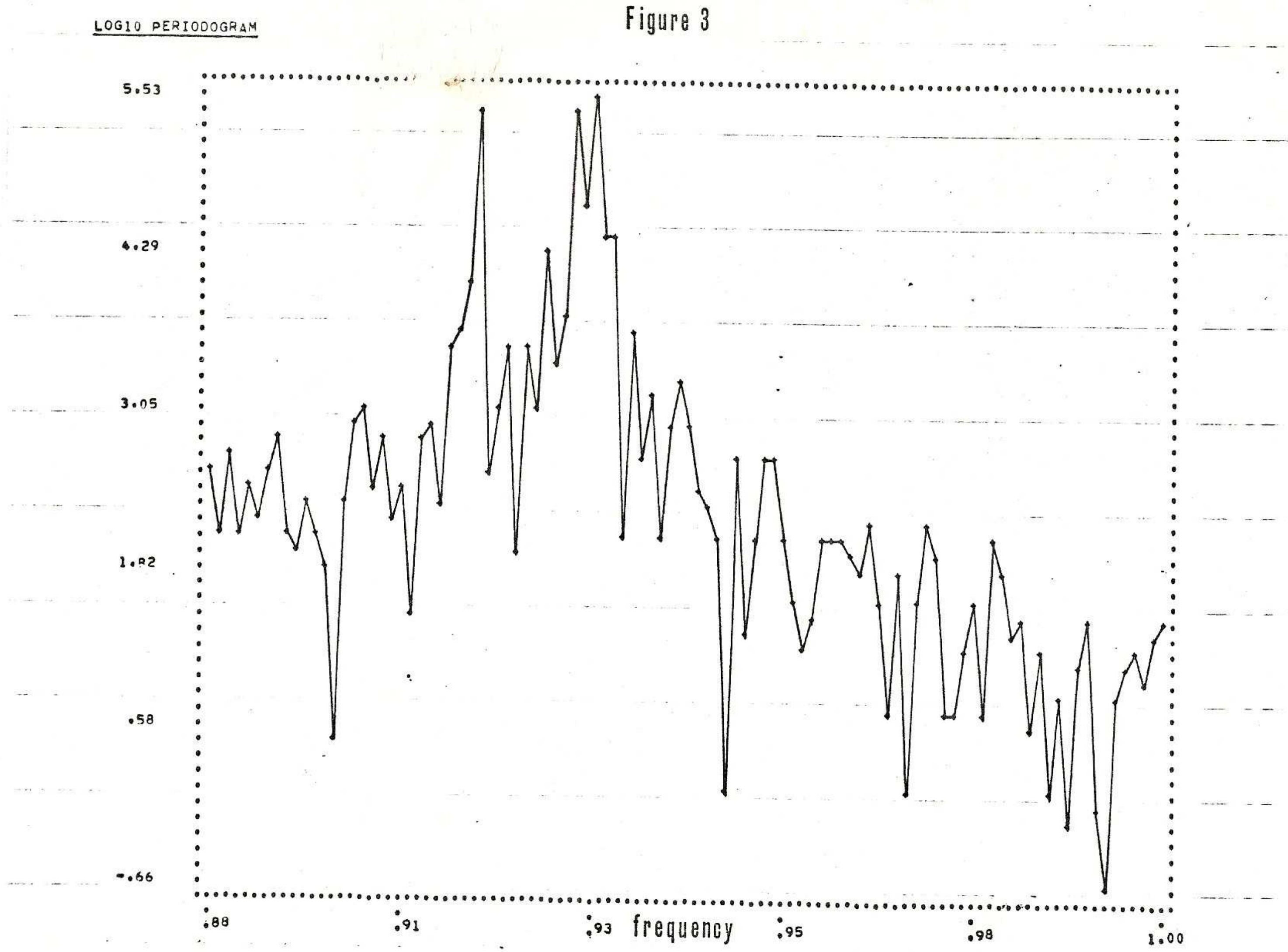
Work with complex-values, $Z(t) = X(t) + iY(t)$.

Compute the location differences, $\Delta Z(t)$, and then the finite FT

$$d_Z^T(\lambda) = \Sigma_{t=0}^{T-1} \exp\{-i\lambda t\}[Z(t+1)-Z(t)]$$

Periodogram

$$I_{ZZ}^T(\lambda) = (2\pi T)^{-1}|d_Z^T(\lambda)|^2 \qquad\qquad 0 \le \lambda < 2\pi$$



periodogram - 1972 graphics!

Model.

Arato, Kolmogorov, Sinai, (1962) set down SDE

$$dX = -\psi\, Xdt - \omega\, Ydt + \sigma\, dB$$

$$dY = \quad \omega\, Xdt - \psi\, Ydt + \sigma\, dC$$

$$Z = X + iY \qquad\qquad \Phi = B + iC$$

General stimulus

$$dZ = -\gamma\, Zdt + \sigma\, d\Phi \qquad \gamma = \psi - i\,\omega$$

Adding measurement noise, power spectrum is

$$|i\lambda + \gamma|^{-2} f_{\Phi\Phi}(\lambda) + \sigma^2 |1 - \exp\{-i\lambda\}|^2 / 2\pi$$

But what is source of B, C? Of 12 mo, 14 mo?

Stationary-mixing series, periodograms, I_s at $\lambda = 2\pi s/T$ approx independent exponentials parameter f_s

Suggesting estimation criterion (quasi-likelihood)

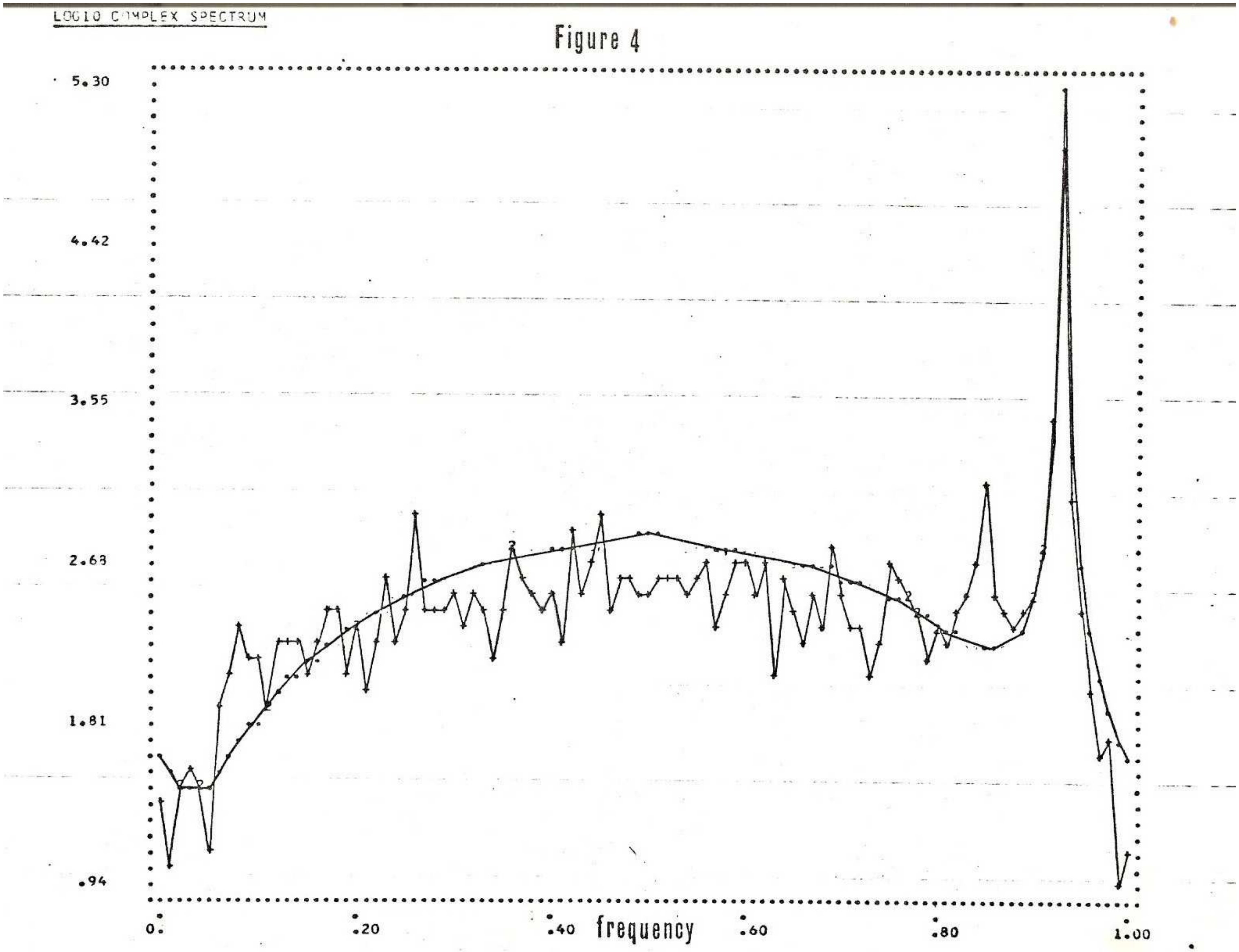
$$L = \Pi_s f_s^{-1} \exp\{-I_s/f_s\}$$

and approximate standard errors

Gaussian estimation, Whittle method

Removed seasonal

Figure 4


LOG10 COMPLEX SPECTRUM
5.30
4.42
3.55
2.68
1.81
.94
0.
.20
.40
.60
.80
1.00
frequency

Discussion. A spectral based solution

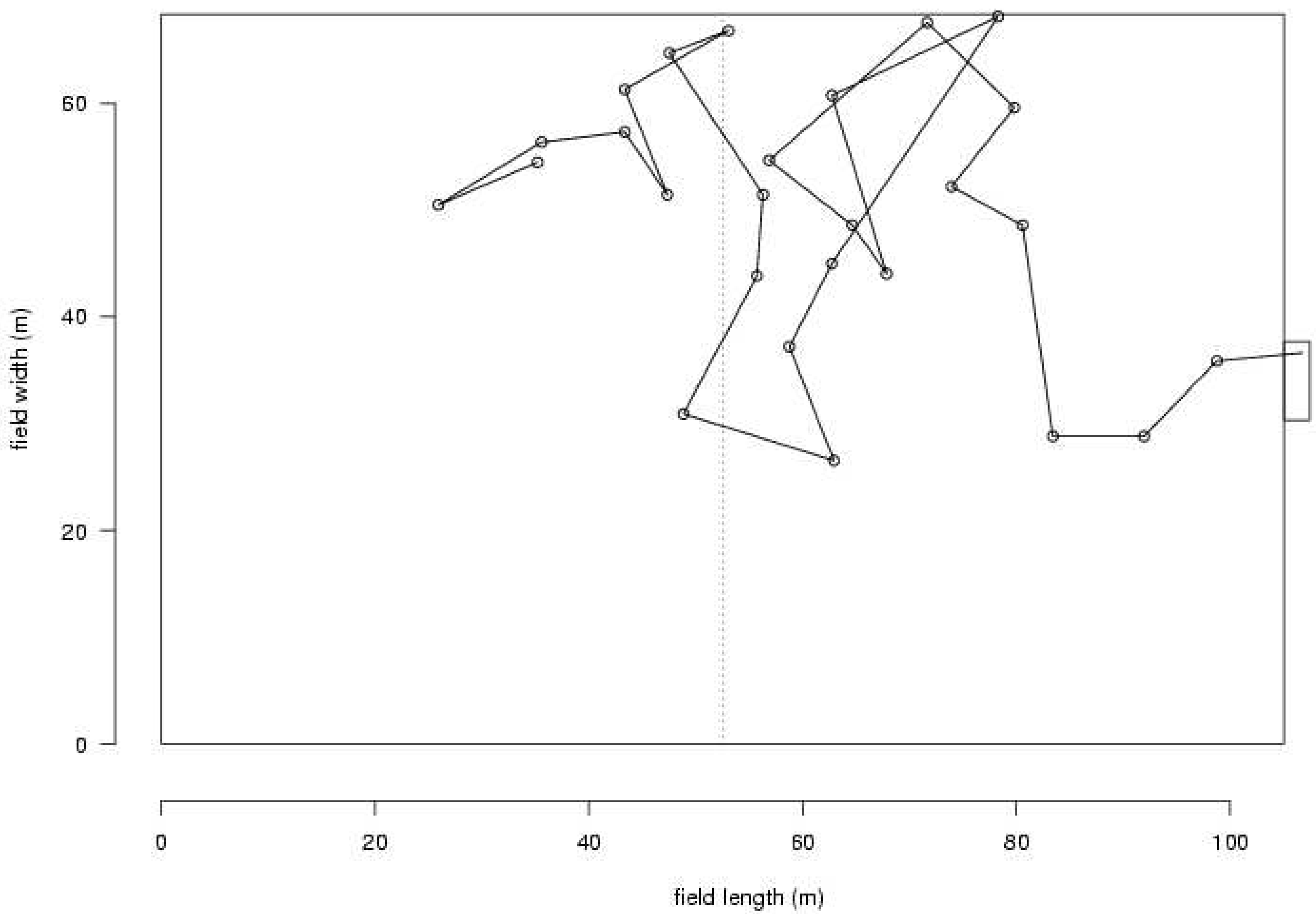
Looked for association with earthquakes, atmospheric pressure by filtering at Chandler frequency

Mystery "solved" by modern data and models.

Using 1985 to 1996 data, R. S. Gross (NASA) concluded two thirds of wobble caused by changes in ocean-bottom water pressure, one-third by changes in atmospheric pressure.

Earth's rotation changes are a big source of uncertainty in navigating interplanetary spacecraft

Example 1. *25-pass goal*. 2006 Argentina vs. Serbia-Montenegro



$H(\mathbf{r}) = \alpha \log |\mathbf{r}| + \beta |\mathbf{r}| + \gamma_1 x + \gamma_2 y + \gamma_{11} x^2 + \gamma_{12} xy + \gamma_{22} y^2$

$\mathbf{r} = (x,y)$

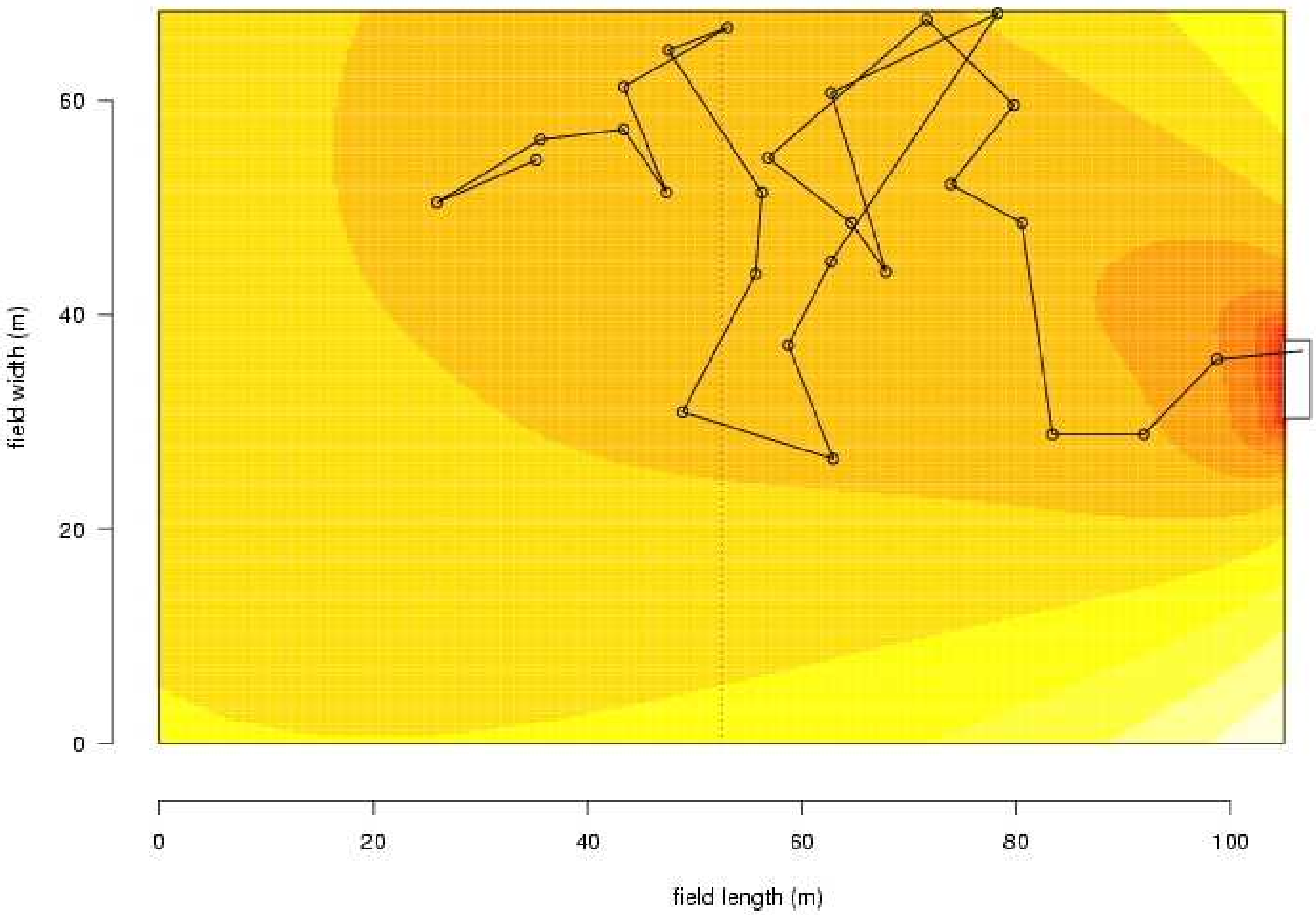
attraction (goalmouth) plus smooth

$|\mathbf{r} - \mathbf{a}_0|$, $\mathbf{a}_0$ closest point of goalmouth

$(\mathbf{r}(t_{i+1})-\mathbf{r}(t_i))/ (t_{i+1}-t_i) = \boldsymbol{\mu}(\mathbf{r}(t_i)) + \boldsymbol{\sigma}\, \mathbf{Z}_{\mathbf{i+1}}/\sqrt{}(t_{i+1}-t_i)$

$\boldsymbol{\mu} = -\nabla H$, stack, WLS

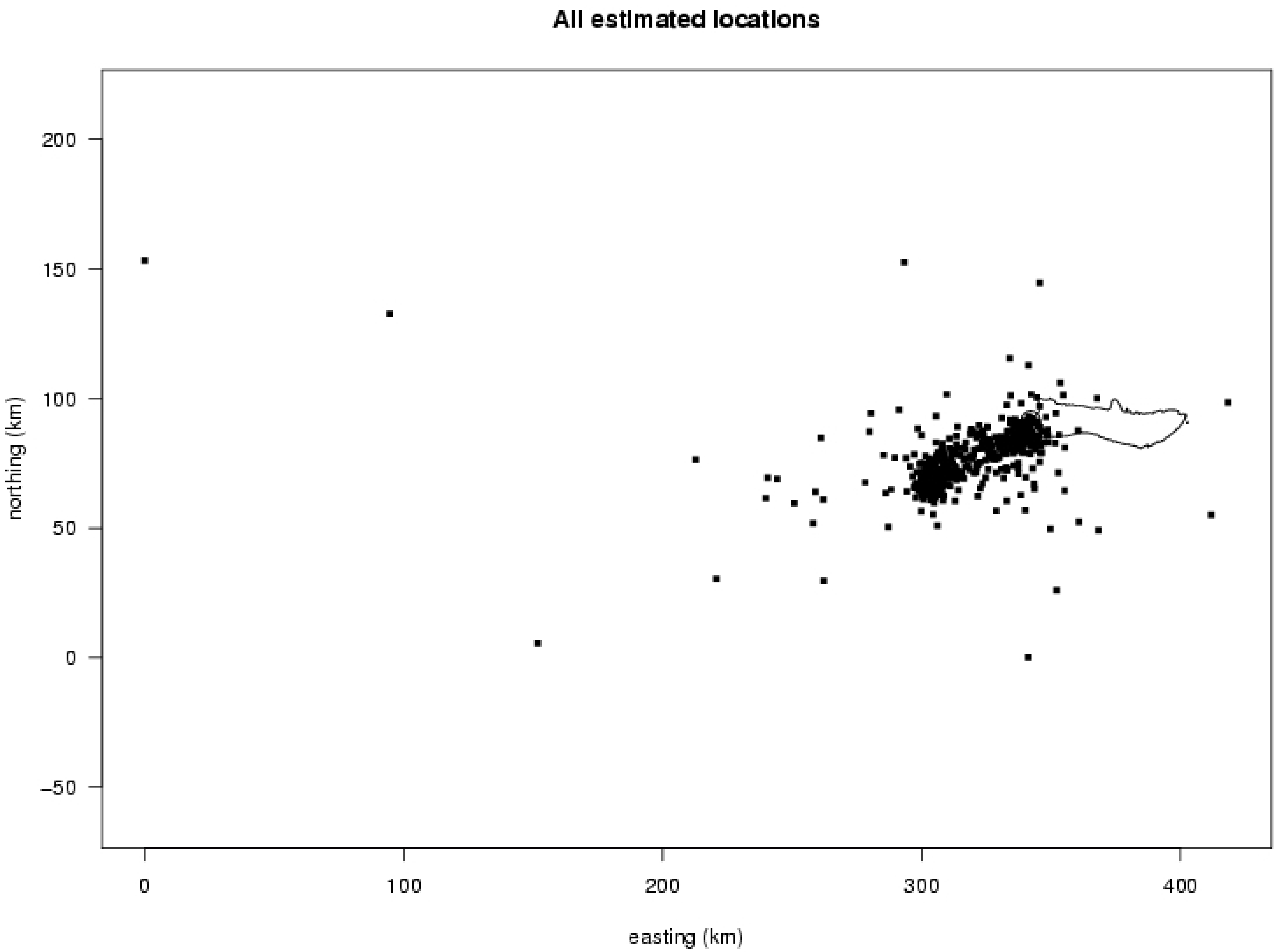
Estimate of H



Example 2.

Hawaiian monk seal

Photo © James Watt, courtesy of NOAA/Dept of Commerce.



Brillinger, Stewart and Littnan (2008)

Bagplot. EDA tool

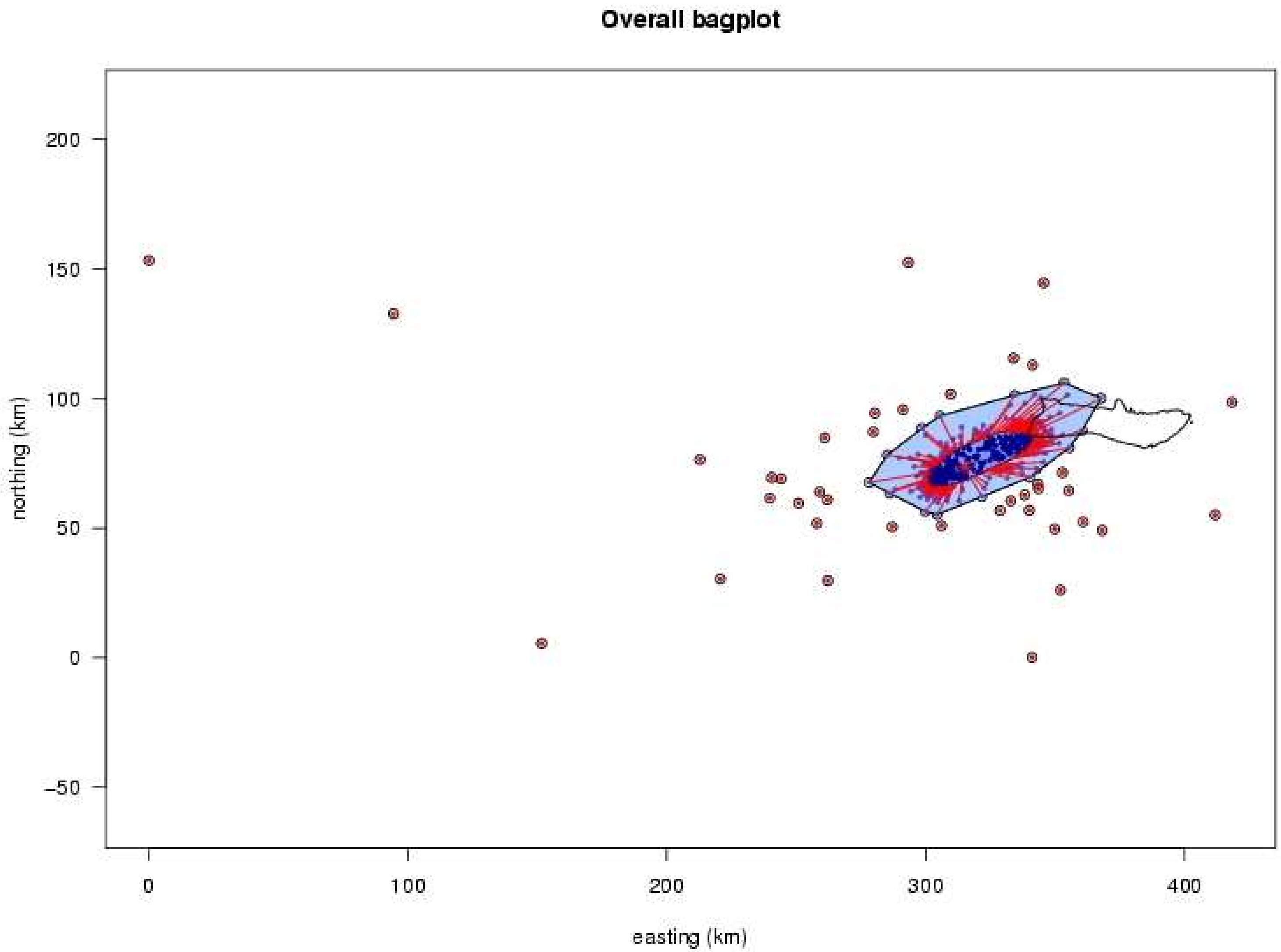
Multi-d generalization of boxplot (example later)

Center is multi-d median

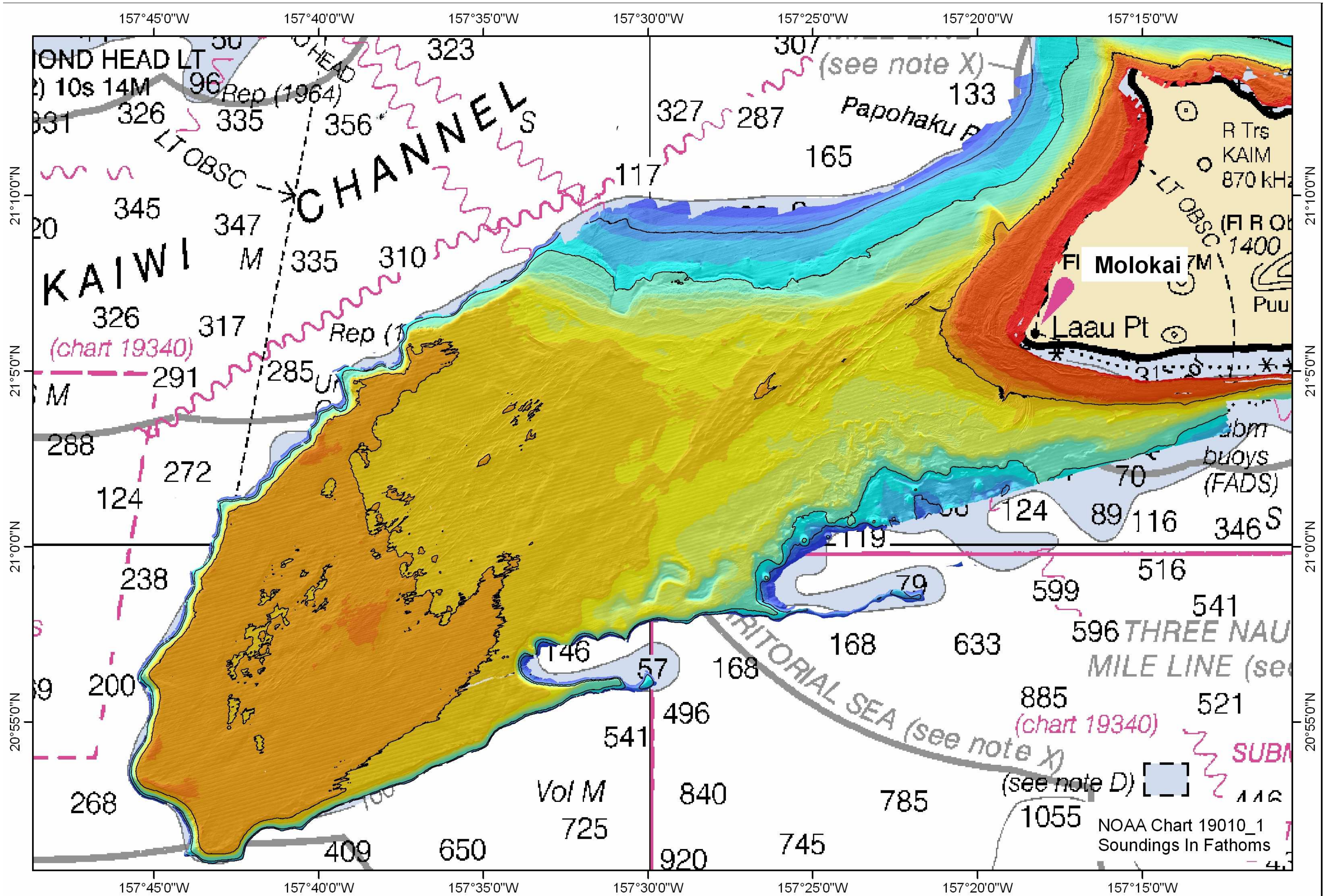
Bag contains 50% of observations with greatest depth (ordering based on halfspaces)

Fence separates inliers from outliers – inflates bag by factor of 3

Robust/resistant



Penguin Bank!



Penguin Bank Bathymetry
5 m grid cell size
NOAA Ship Hiialakai Simrad em3002d (300 kHz) and R/V AHI
Reson 8101 (240 kHz) multibeam and SHOALS LIDAR Bathymetry
NOAA Coral Reef Ecosystem Division
Not For Navigation

nautical miles: 0, 5, 10
kilometers: 0, 5, 10, 15
UTM Projection: Zone 04N Ellipsoid: WGS-84

Depth (m): 0, 40, 50, 60, 70, 80, 200
Grid Cell Size: 5 meters
Contour Interval: 50 meters

$H(\mathbf{r},t)$ - two points of attraction, one offshore, one onshore

Potential function

$\frac{1}{2}\sigma^2 \log |\mathbf{r}\text{-}\mathbf{a}| \; - \; \delta|\mathbf{r}\text{-}\mathbf{a}|$

cp. Kendall bird navigation SDE

a(t) changes

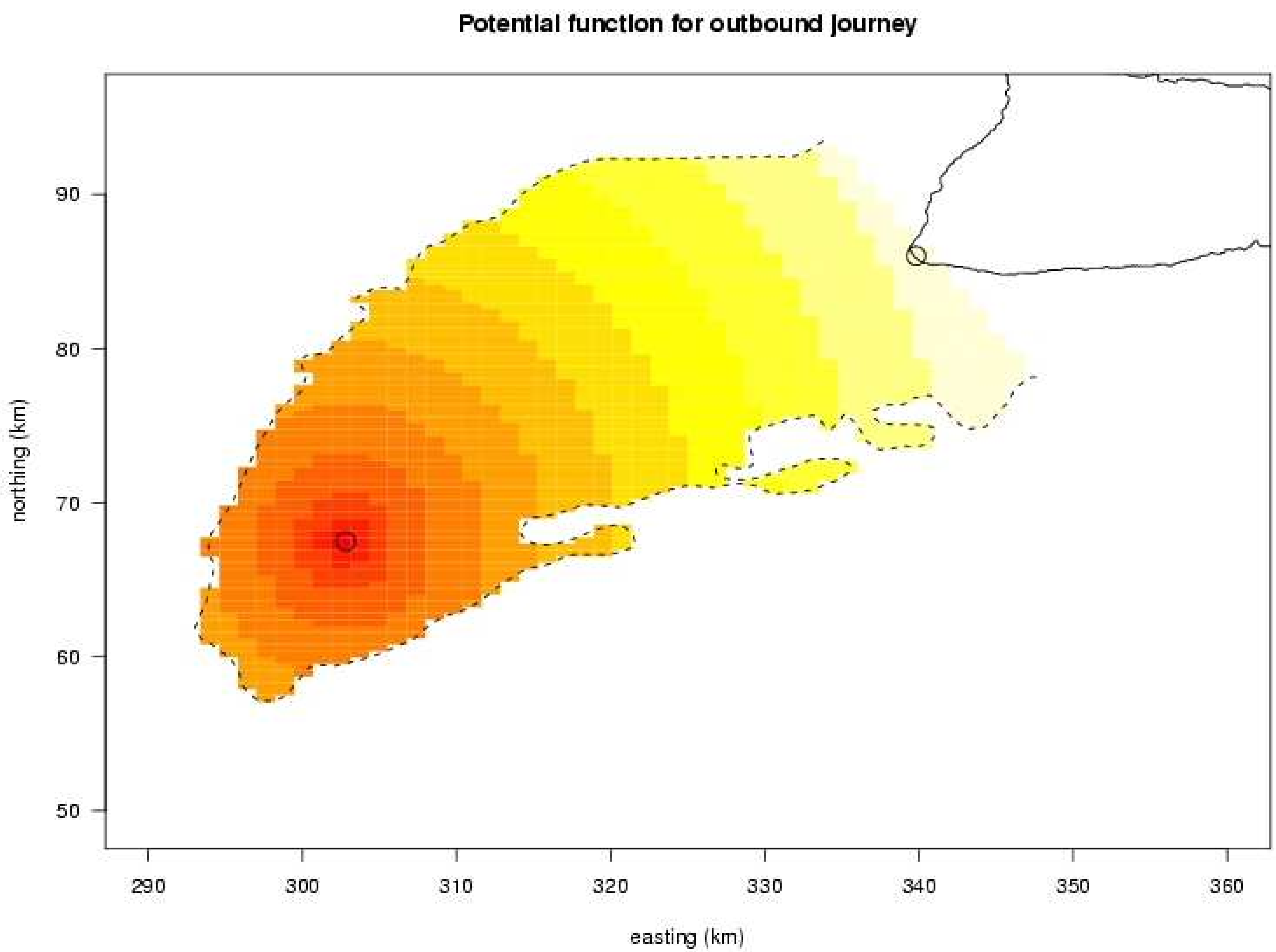
Parametric $\boldsymbol{\mu} = -\text{grad } H$

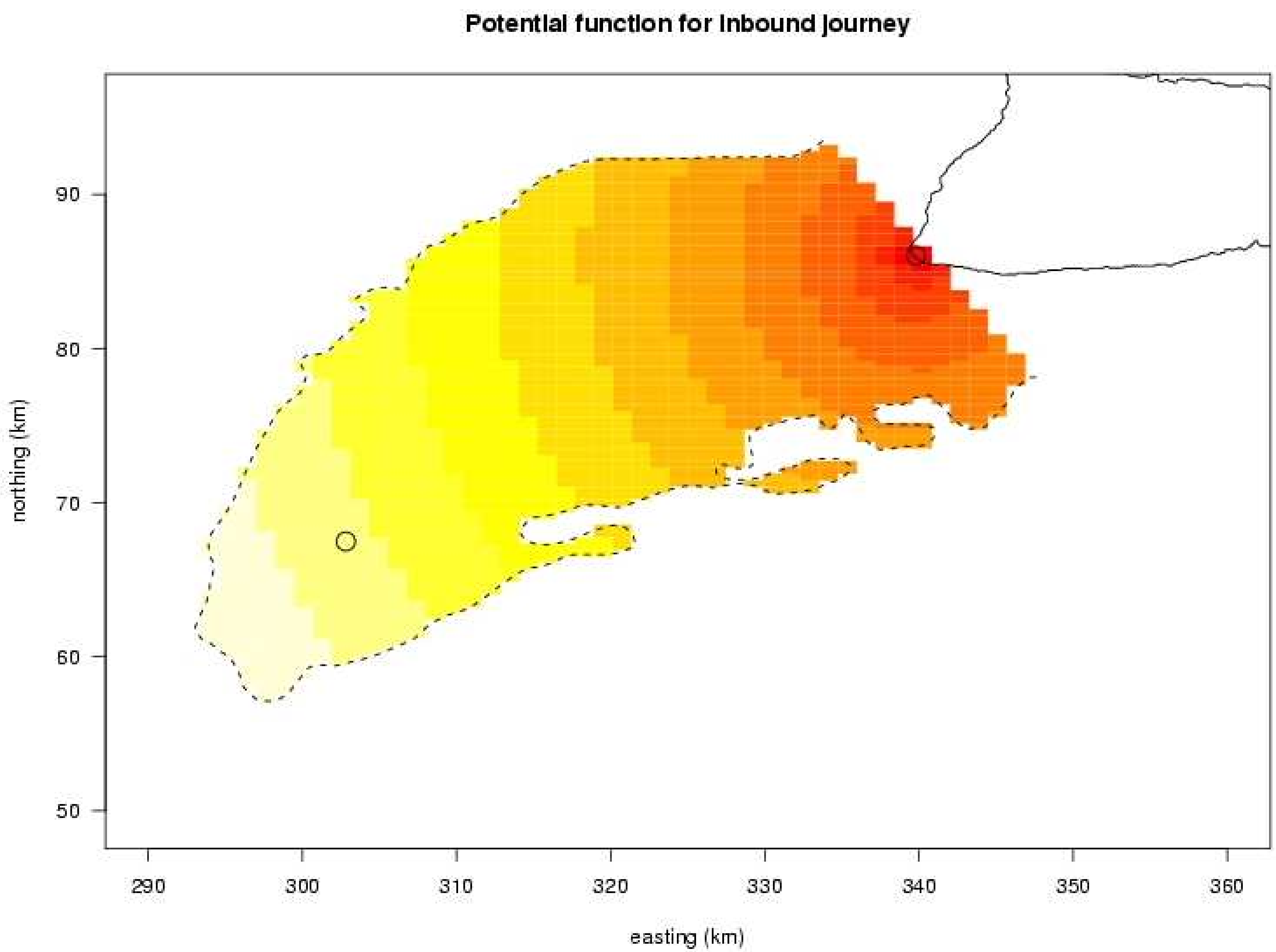
Approximate likelihood from

$$(\mathbf{r}(t_{i+1})-\mathbf{r}(t_i))/\,(t_{i+1}-t_i) = \boldsymbol{\mu}(\mathbf{r}(t_i)) + \boldsymbol{\sigma}\, \mathbf{Z}_{\mathbf{i+1}}/\sqrt{(t_{i+1}-t_i)}$$

Robust/resistant WLS

Estimate σ^2 from mean squared error


Potential function for outbound journey
northing (km)
easting (km)
50
60
70
80
90
290
300
310
320
330
340
350
360

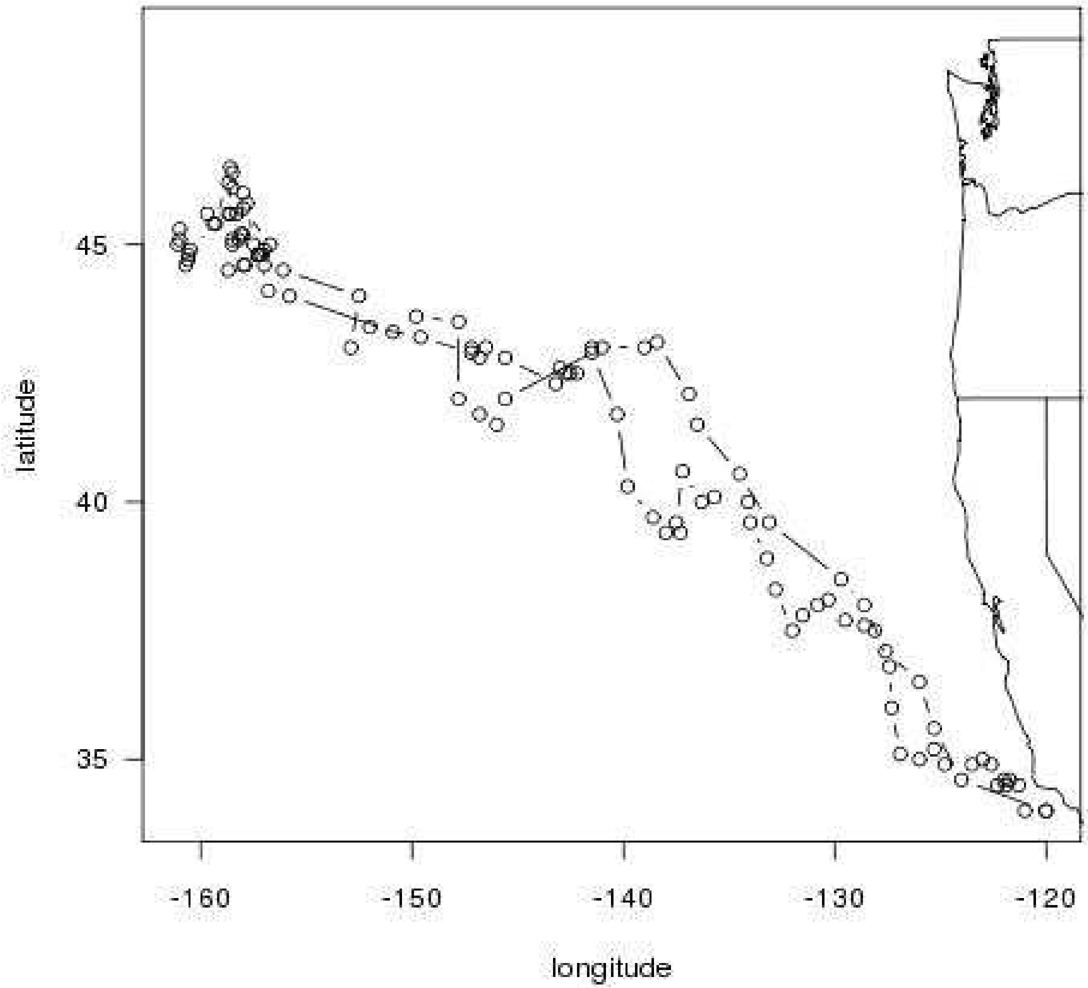


Potential function for inbound journey
northing (km)
easting (km)
90
80
70
60
50
290
300
310
320
330
340
350
360

Example 4. *Elephant seal journey*.

Were endangered

One journey



D. R. Brillinger and B. S. Stewart (1998)

"Particle" heading towards North Pole on the Earth's surface

bivariate Brownian disturbance (U,V). σ s.d.

(θ,ϕ): longitude, colatitude

$$d\theta_t = \sigma\, dU_t + (\sigma^2/2\tan\theta_t - \delta)dt$$

$$d\phi_t = (\sigma/\sin\theta_t)\, dV_t$$

Brownian with drift, δ, on a sphere

Parameter estimation.

Discrete approximation

$$\theta_{t+1} - \theta_t = \sigma^2/(2 \tan \theta_t) - \delta + \sigma\, \varepsilon_{t+1}$$

$$\varphi_{t+1} - \varphi_t = \sigma / (\sin \theta_t)\eta_{t+1}$$

Measurement error

$$\theta_t' = \theta_t + \tau\, \varepsilon_t'$$

$$\varphi_t' = \varphi_t + \tau / (\sin \theta_t')\eta_t$$

Results. Likelihood by simulation

No measurement error

parameter	estimate	s.e.
δ	.0112 rad/day	.001
σ	.00805	

Measurement error   Likelihood by simulation

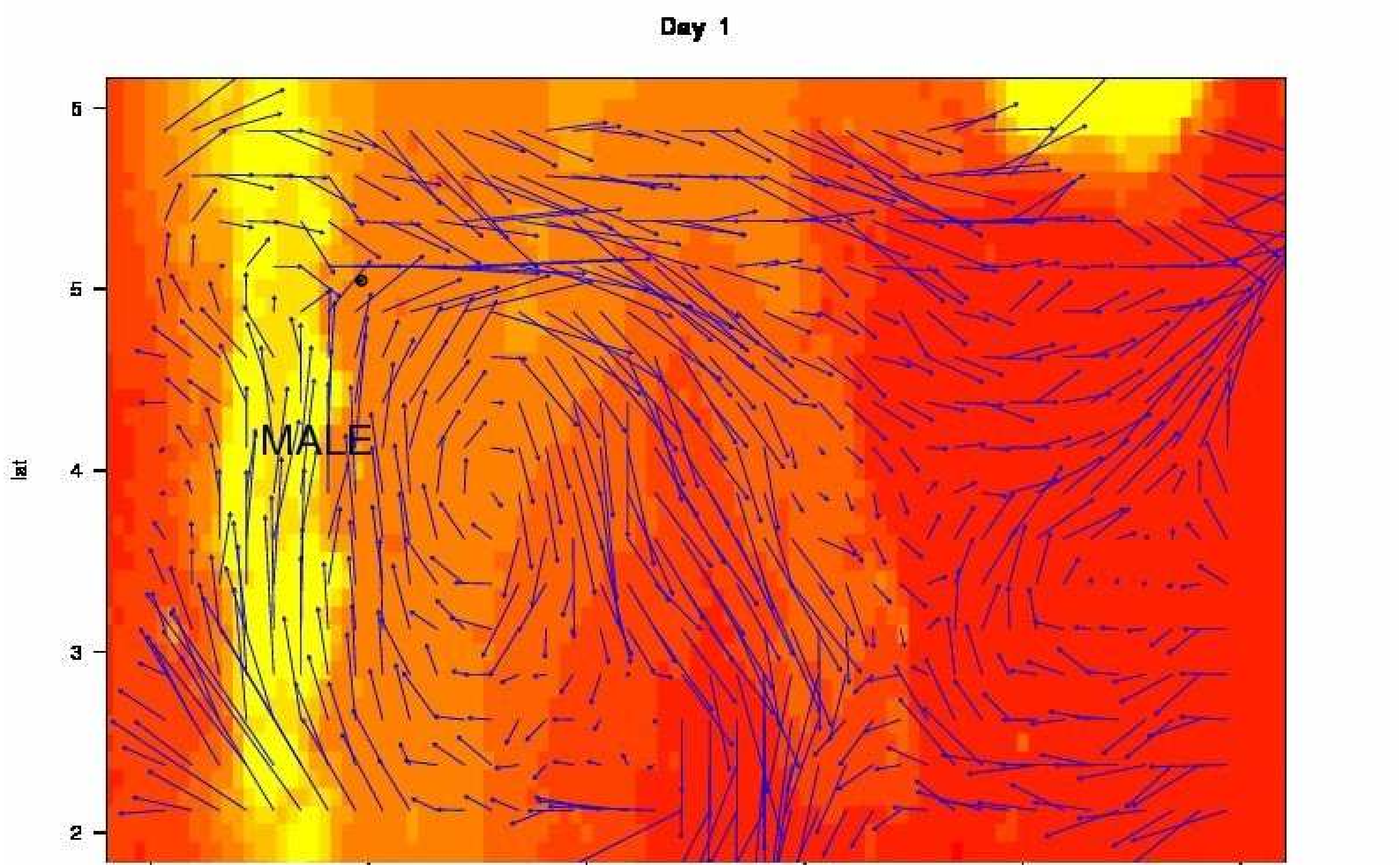
out δ	.0126	.0001
in δ	.0109	.0001
σ	.000489	.0000004
τ	.0175	.0011

Example 5. *Tag's trajectory*

10 day composite geostrophic currents, 2008/06/29

Sri Lanka, Maldives

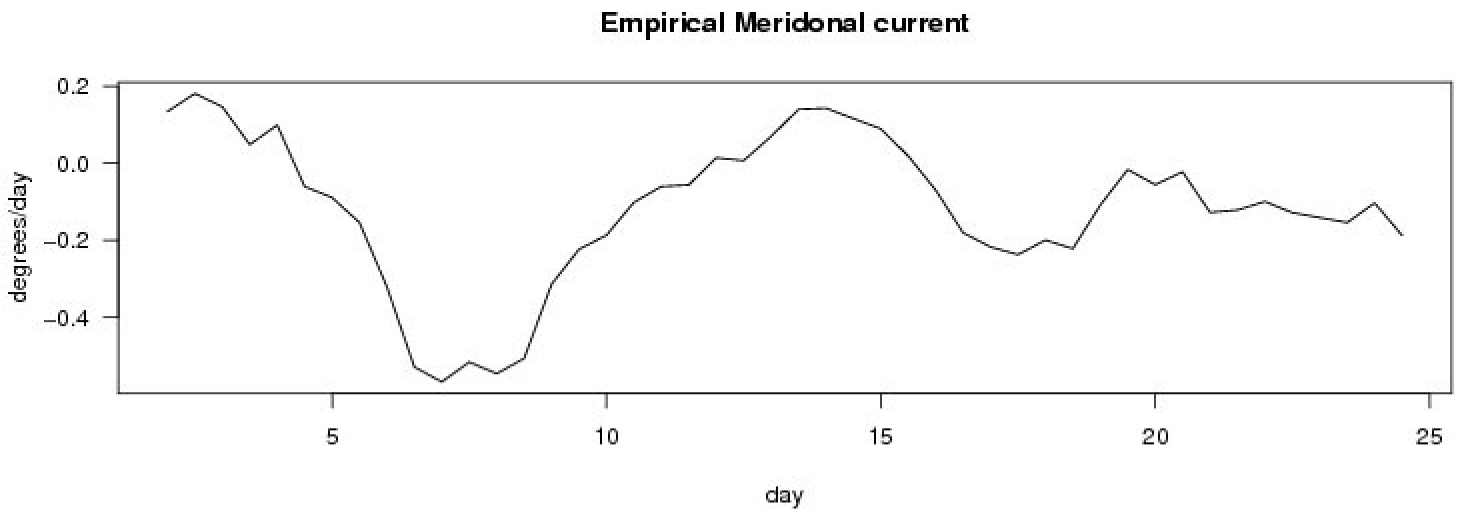
Backgound bathymetry - yellow is highest/shallowest



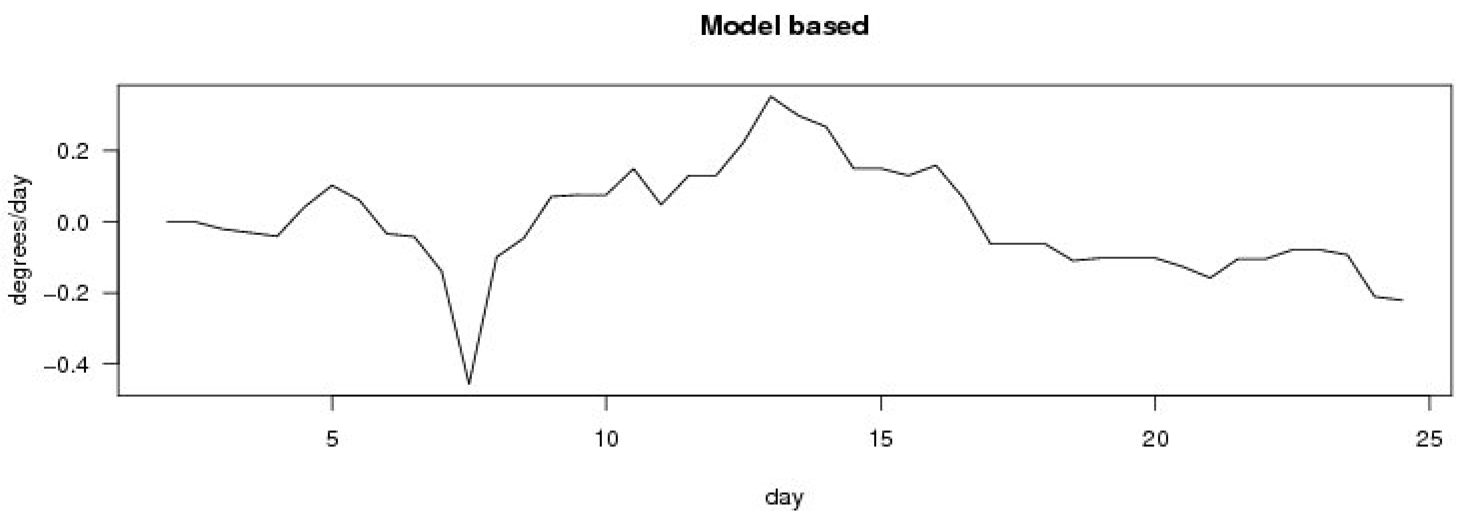
tag "movie", slides80176.pdf

Brent's interpretation.

"Looks like the drifter starts out behaving according to the driving forces of surface current.The odd and interesting event is when it moves south into that small apparently weak gyre towards the end. It then goes back to moving under influence of current heading south when it comes out of gyre, but this is in opposite trajectory that it would have followed if it had followed dominant flow before it had entered gyre. This seems to be a key change in state of expected movement from the null prediction."



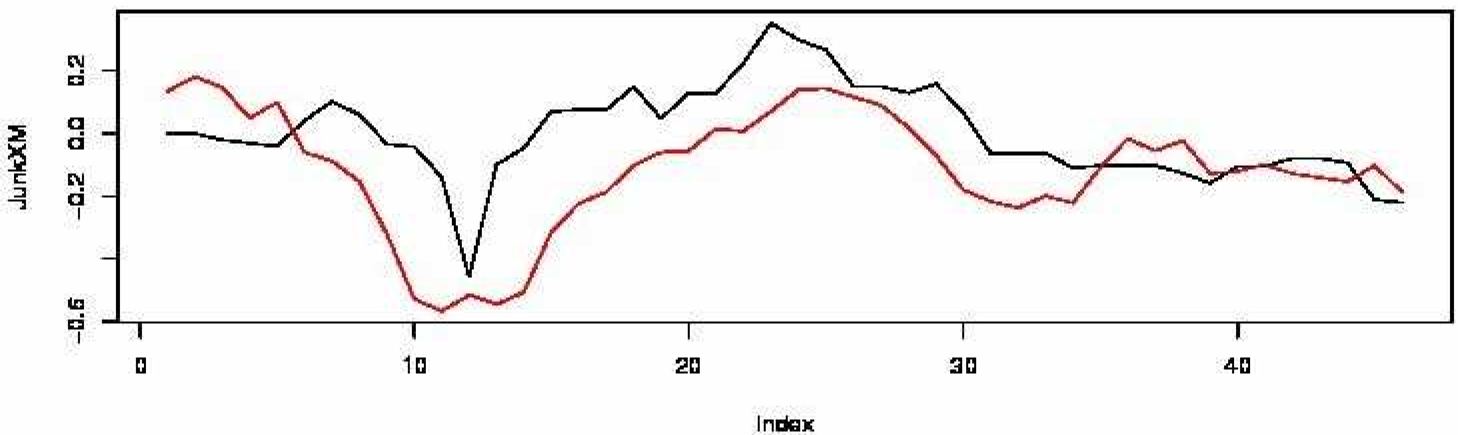
from
tag



from
NOAA
values

On same plot


JunkXZ
0.0
0.2
0.4
0.6
0
10
20
30
40
Index


JunkXM
-0.6
-0.2
0.0
0.2
0
10
20
30
40
Index

Functional stochastic differential equation (FSDE). cp ar(p)

$$d\mathbf{r}(t) = \boldsymbol{\mu}(H(t),t)dt + \boldsymbol{\sigma}(H(t),t)d\mathbf{B}(t)$$

H(t): history based on past, $\{\mathbf{r}(s),\ s \le t\}$

Process is Markov when $H(t) = \{\mathbf{r}(t)\}$

Interpretation

$$\mathbf{r}(t) - \mathbf{r}(0) = \int_0^t \boldsymbol{\mu}(H(s),s)ds + \int_0^t \boldsymbol{\sigma}(H(s),s)d\mathbf{B}(s)$$

Incorporating currents, winds, past locations

Regression model, tag velocity

$$(\mathbf{r}(t_{i+1})-\mathbf{r}(t_i))/(t_{i+1}-t_i) = \mu\,(H(t_i),t_i) + \boldsymbol{\alpha} + \boldsymbol{\beta}_C\mathbf{X}_C(\mathbf{r}(t_i),t_i) + \boldsymbol{\beta}_V\mathbf{X}_V(\mathbf{r}(t_i),t_i) + \boldsymbol{\sigma}\mathbf{Z}_{i+1}/\surd(t_{i+1}-t_i)$$

where

$$\mu(H(t),t) = \boldsymbol{\gamma}\int_{t-1}^{t} \mathbf{r}(s)dM(s)$$

$$M(t) = \#\{t_i \le t\},$$ counting function

regression coeficients, $n = 206$

zonal case

γ	0.742828	0.051252	14.494
β_C	0.201452	0.039224	5.136
β_V	-0.009115	0.003862	-2.360

$R^2 = 0.804$

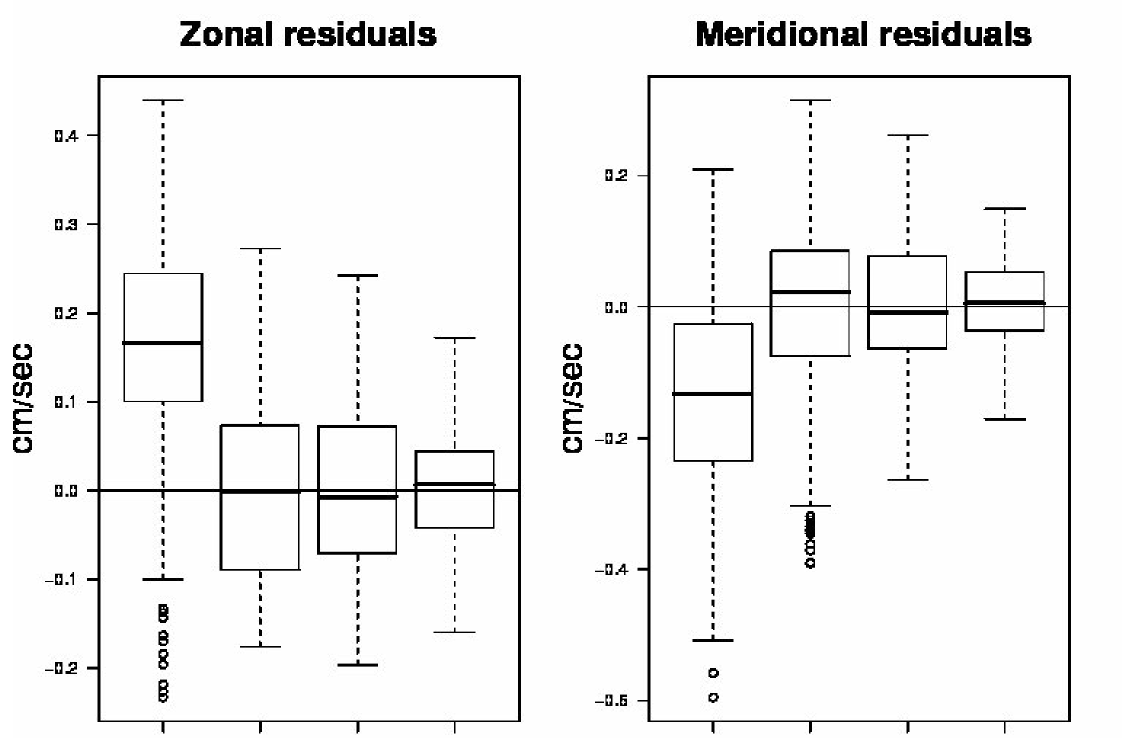
meridional case

γ	0.708062	0.041549	17.042
β_C	0.240575	0.039707	6.059
β_V	0.025608	0.005453	4.696

$R^2 = 0.854$

Residuals introducing variables successively

parallel boxplots



$\alpha = 0, \beta = 1$; β_C; β_C, β_V; β_C, β_V, γ

Discusssion of tag analysis.

Use NOAA values with some caution

motivations - SDE, FSDE

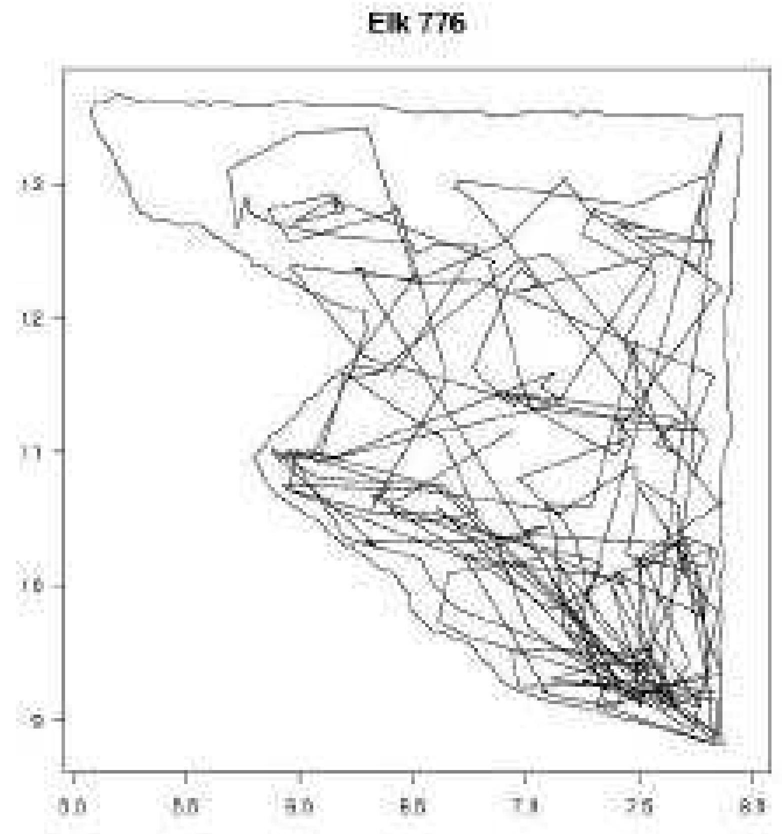
Can use SDE result for simulation

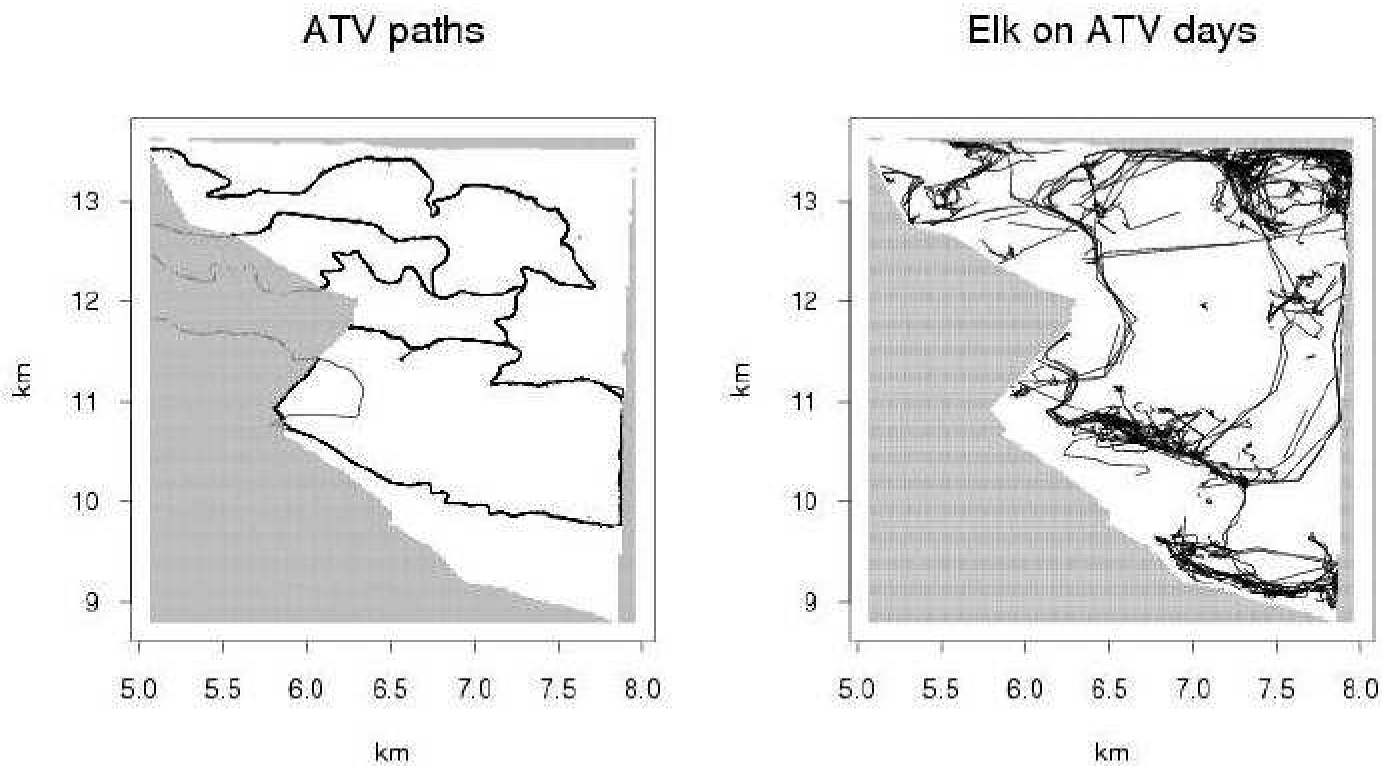
Residuals to discover things

robust/resistant smoothing is basic

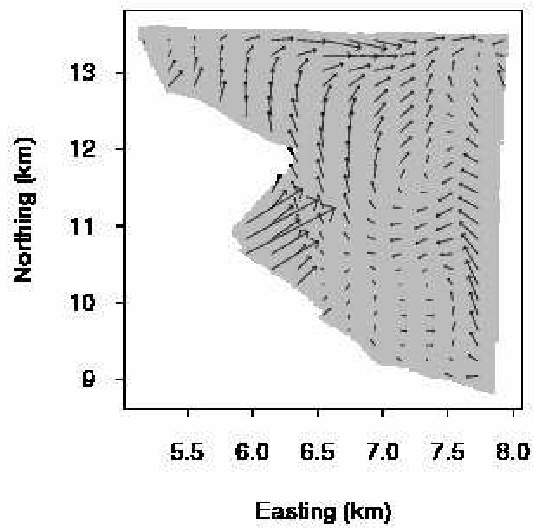
Example 6. *Rocky Mountain elk*.

Starkey Reserve, Oregon, NE pasture, April-October 2003

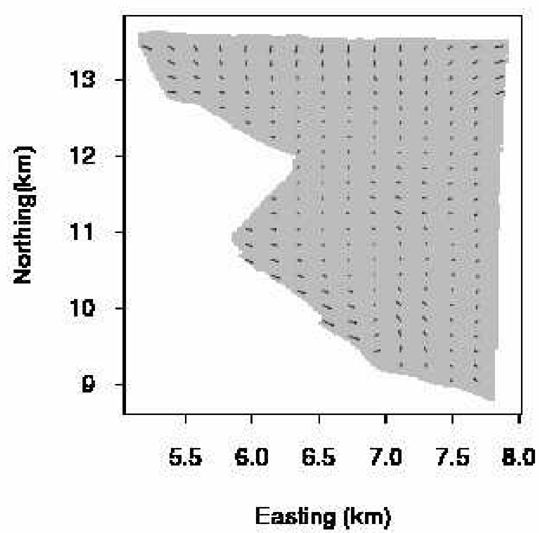



ATV paths
Elk on ATV days
km
9
10
11
12
13
5.0
5.5
6.0
6.5
7.0
7.5
8.0

Velocity field - ATV periods



Velocity field - control periods



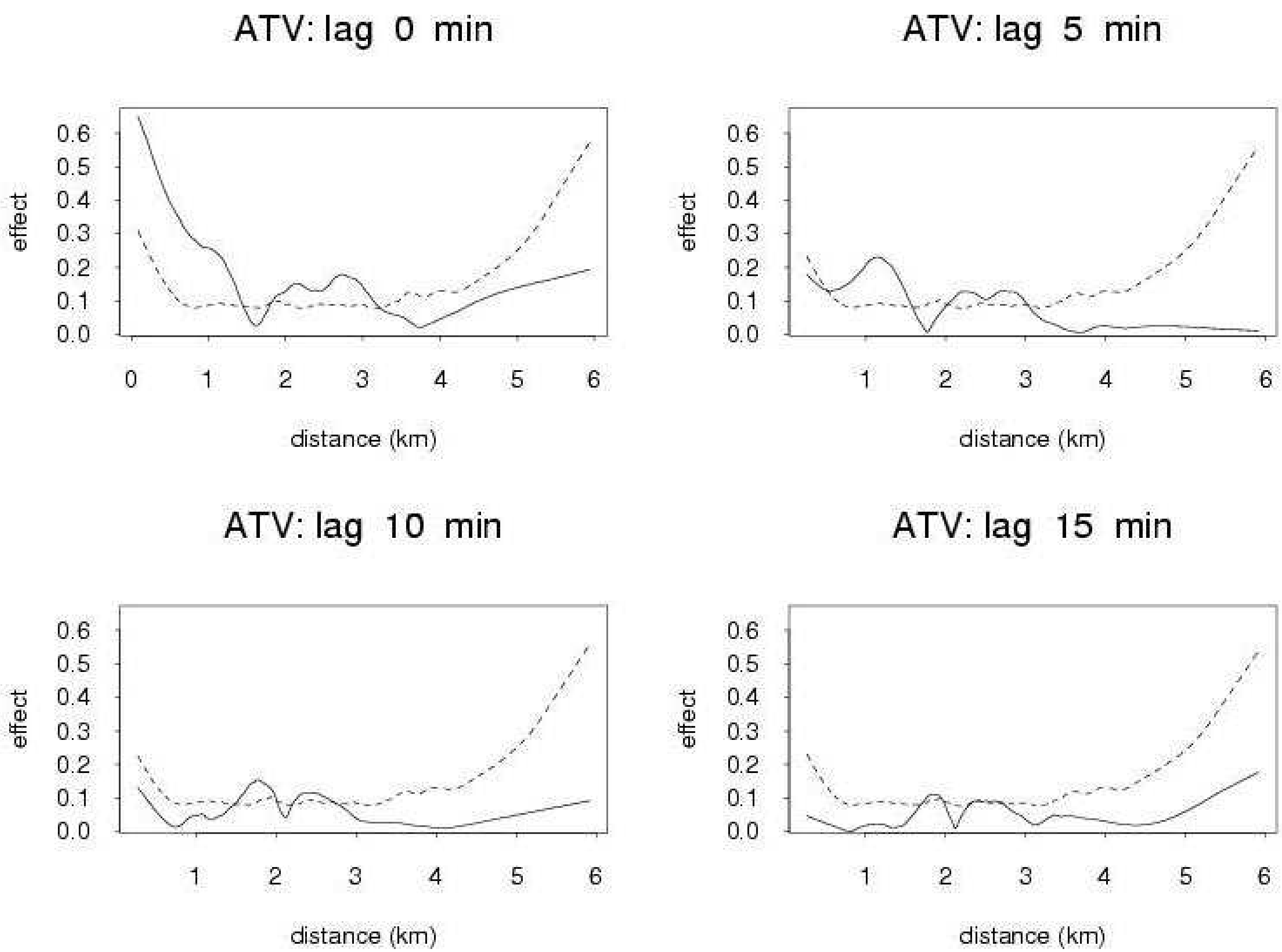
Model.

$$d\mathbf{r}(t) = \boldsymbol{\mu}(\mathbf{r}(t))dt + \boldsymbol{\upsilon}(|\mathbf{r}(t)-\mathbf{x}(t-\tau)|)dt + \boldsymbol{\sigma}d\mathbf{B}(t)$$

$\mathbf{x}(t)$: location of ATV at time t

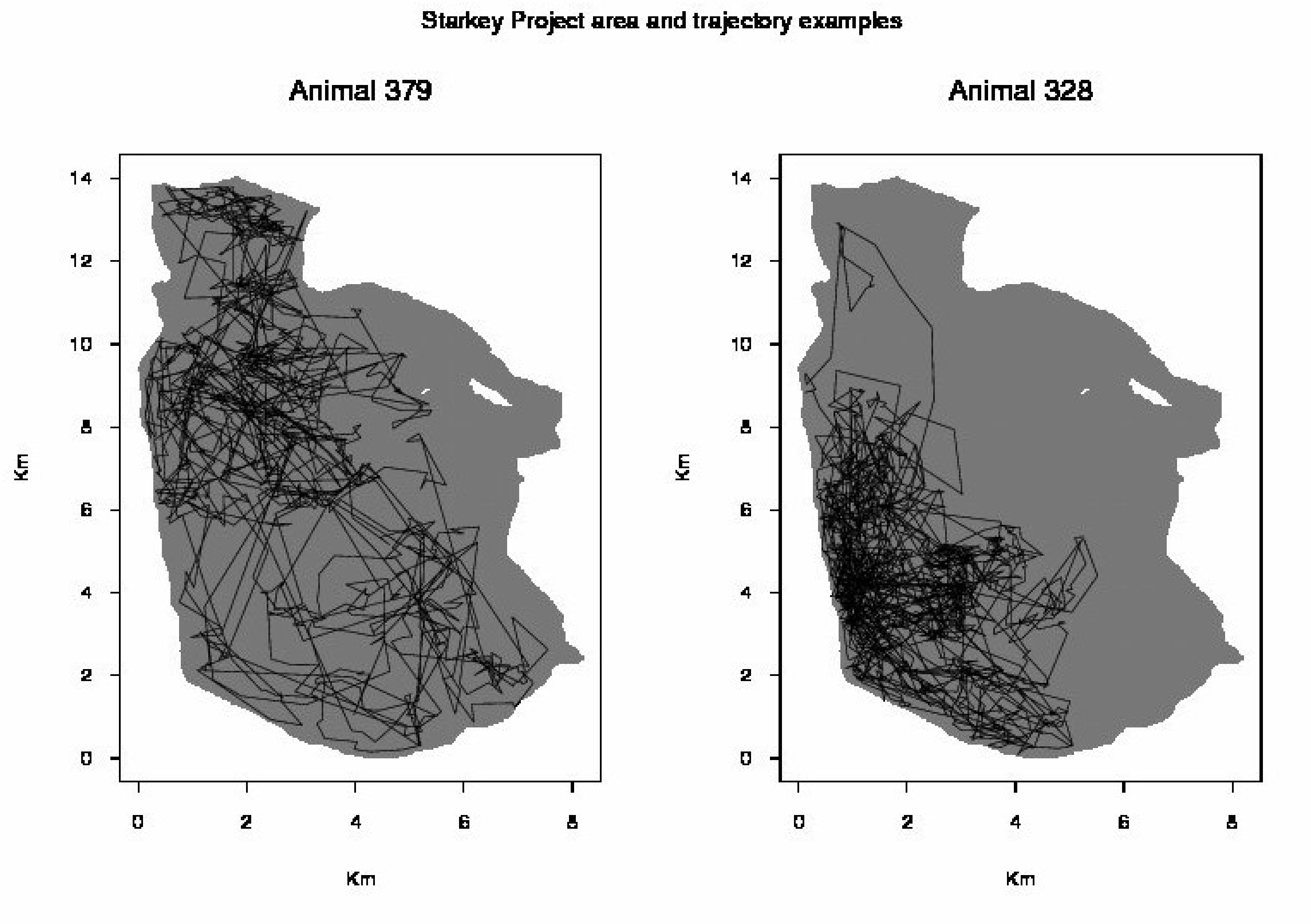
τ: time lag

Plots of $|\boldsymbol{\upsilon}_\tau|$ vs. distance $|\mathbf{r} - \mathbf{x}_\tau|$



ATV: lag 0 min
ATV: lag 5 min
ATV: lag 10 min
ATV: lag 15 min
effect
distance (km)

There are a number of elk



Particle system. Several moving particles

Are particles interacting? If yes, how?

$d\mathbf{r}_i(t) = - \nabla U(\mathbf{r}_i(t))dt - \nabla\Sigma_{j\neq i} V(||\mathbf{r}_i(t) - \mathbf{r}_j(t)||)dt + \boldsymbol{\sigma} d\mathbf{B}_i(t)$

Different times and spacings for different particles i

Pairwise interaction tensor model, i = 1,...,p

$U(x,y) = \Sigma \beta_{jk} \Phi_j(x)\Psi_k(y)$: environment

$V(||\mathbf{s}||) = \Sigma \gamma_j \Omega_j(||\mathbf{s}||)$: other particles

$\nabla U(x,y) = (\Sigma \beta_{jk} \varphi_j(x)\Psi_k(y), \Sigma \beta_{jk} \Phi_j(x)\psi_k(y))$

$\nabla V(||\mathbf{s}||) = \Sigma \gamma_l \omega_l(||\mathbf{s}||)\mathbf{s}/||\mathbf{s}||$

Get estimates of the β and the $\gamma\mathbf{s}/||\mathbf{s}||$

Use to form estimates of U and V

VI. Inference tools.

Martingale difference series.

$$E\{X_{n+1}|\{X_0,...,X_n\}) = 0$$

With the basic model

$$(\mathbf{r}(t_{i+1})-\mathbf{r}(t_i))/\,(t_{i+1}-t_i) = \boldsymbol{\mu}(\mathbf{r}(t_i),t_i) + \boldsymbol{\sigma}\,\mathbf{Z}_{i+1}/\surd(t_{i+1}-t_i)$$

one has

$$\mathbf{r}(t_{i+1}) - E\{\mathbf{r}(t_{i+1}) \mid F_i\}$$

is martingale difference series with respect to the σ-field F_i generated by $\{\mathbf{r}(t_1),\ldots,\mathbf{r}(t_i)\}$

Theorem. Lai and Wei (1982) supposing

$$\mathbf{y}_i = \mathbf{x}_i^T\boldsymbol{\beta} + \boldsymbol{\varepsilon}_i,\ i=1,2,\ldots$$

with $\{\boldsymbol{\varepsilon}_i\}$ martingale differences wrt increasing sequence of σ-fields $\{F_n\}$ show

$$(\mathbf{X}_n^T\mathbf{X}_n)^{1/2}(\mathbf{b}\text{-}\boldsymbol{\beta}) \rightarrow N(0, \sigma^2\mathbf{I}), \text{ in distribution as } n\rightarrow\infty.$$

Also show

$$((\boldsymbol{\varphi}(\boldsymbol{r})(\boldsymbol{X}_n^T\boldsymbol{X}_n)^{-1}\boldsymbol{\varphi}(\boldsymbol{r})^T)^{-1/2}\boldsymbol{\varphi}(\boldsymbol{r})^T\ (\boldsymbol{b}\text{-}\boldsymbol{\beta})/s_n \rightarrow N(0,1)$$

in distribution as $n\rightarrow\infty$ with

$$s_n = ((n-1)p)^{-1}\mathrm{RSS}$$

Chang and Chin (1995) consider nonlinear least squares including

$$\mathrm{var}\{\varepsilon_n | F_{n-1}) = \mathrm{g}(\mathbf{z}_\mathrm{n};\, \theta)$$

$\mathbf{z}_\mathrm{n}$, is observable and F_{n-1} measureable

SDE benefits.
conceptual, extendable, simulation, analytic results, prediction, effective

Potential function real-valued

Motivates parametric and nonparametric estimates

difficulties: enforcing boundary

Uncertainties - haven't focused on

general methods: jackknife and bootstrap

Order of approximation

Unequal spacings/times

Crossings - trajectories heading into regions (eg. soccer ball, debris)

General discussion.

Trajectories basic to science for many centuries

DEs ==> SDEs ==> FSDEs

SDEs motivate statistical models

Potential approach: advantages - real-valued, unifying

Fits useful: description, summary, comparison, simulation, prediction, appraisal, bootstrapping, derived quantities

Acknowledgements.

Aager, Dewitt, Don, Foley, Guckenheimer, Kie, Oster, Preisler, Stewart, Littnan, Mendolssohn, Lovett, Spector

The End

DASF:

Born on road adjacent to the campus

vs. Wellesley Hospital

My actuarial career

vs. statistician

Surfing

vs. windsurfing

Navy

stoker vs. cadet

Temagami, skiing in, Christmas tree