

Totally free arrangements of hyperplanes

presented by

Hiroaki Terao (Hokkaido University)

Conference for Honor of PETER ORLIK
Fields Institute, Toronto, Canada
August 19, 2008

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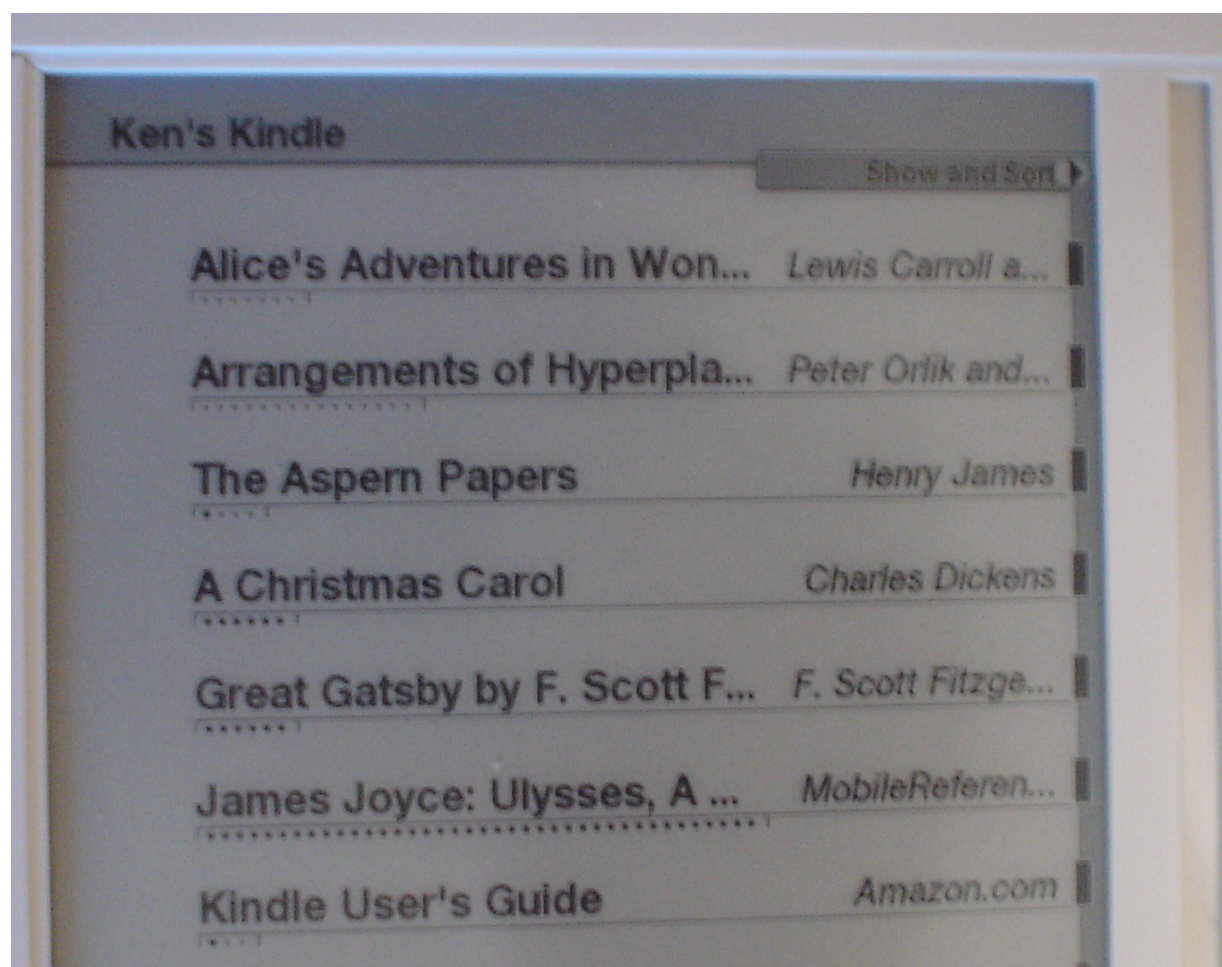
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**Peter Orlik
Hiroaki Terao**
**Arrangements
of Hyperplanes**



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$$D(\mathcal{A}, m) := \{\theta \in \text{Der}_{\mathbb{R}}(S) \mid \theta(\alpha_H) \in S \cdot \alpha_H^{m(H)} \ \forall H \in \mathcal{A}\}$$

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The freeness of multiarrangements was introduced by G. Ziegler (1989).

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For example, any arrangement in either **one-dimensional** or **two-dimensional** vector spaces is **totally free**. **Boolean arrangements** are **totally free**.

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Theorem 2.1. If $\mathcal{NFM}(\mathcal{A})$ is a finite set, then it is empty.

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Corollary 2.3 The totally freeness is closed under restriction and deletion.

3. Proof of Main Theorem - LMP vs GMP

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(\mathcal{A}, m) : a free multiarrangement

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$L(\mathcal{A})_2 := \{X \in L(\mathcal{A}) \mid \text{codim}_V(X) = 2\}$

$\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subset H\}$: **the localization** of \mathcal{A} at X

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Theorem 3.1. (Abe-T-Wakefield (2007)) If a multiarrangement (\mathcal{A}, m) is free, then

$$LMP_2(\mathcal{A}, m) = GMP_2(\mathcal{A}, m).$$

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Lemma 3.2. Let \mathcal{A} be an **irreducible** arrangement in \mathbb{K}^ℓ with $\ell \geq 2$. Then there exist $\ell + 1$ hyperplanes $H_1, H_2, \dots, H_{\ell+1}$ in \mathcal{A} satisfying the following conditions:

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$$\begin{aligned} \operatorname{codim}_V H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_p} &= p \quad (1 \leq i_1 < \cdots < i_p \leq \ell + 1, \quad 1 \leq p \leq \ell), \\ H_1 \cap H_2 \cap \cdots \cap H_{\ell+1} &= \{\mathbf{0}\}. \end{aligned}$$

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Let $\mathcal{B} := \{H_1, H_2, \dots, H_{\ell+1}\}$. Define a multiplicity m by:

$$m(H) = \begin{cases} 1 & \text{if } H \notin \mathcal{B}, \\ k & \text{if } H \in \mathcal{B}, \end{cases}$$

for every positive integer k .

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Since $\mathcal{NFM}(\mathcal{A})$ is a **finite set**, the multiarrangement (\mathcal{A}, m) is free **whenever k is sufficiently large**. Note $|L(\mathcal{B})_2| = \binom{\ell+1}{2}$.

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By the definition of LMP_2 ,

$$LMP_2(\mathcal{A}, m) \geq LMP_2(\mathcal{B}, m|_{\mathcal{B}}) = |L(\mathcal{B})_2|k^2 = \binom{\ell+1}{2}k^2.$$

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Let $|\mathcal{A}| = n$. Then

$$\sum_{d \in \exp(\mathcal{A}, m)} d = (k-1)(\ell+1) + n.$$

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$$GMP_2(\mathcal{A}, m) \leq \binom{\ell}{2} \left\{ \frac{(k-1)(\ell+1) + n}{\ell} \right\}^2 = \frac{(\ell+1)^2(\ell-1)}{2\ell} k^2 + Ak + B$$

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whenever k is sufficiently large.

This is a contradiction because

$$\binom{\ell+1}{2} - \frac{(\ell+1)^2(\ell-1)}{2\ell} = \frac{\ell+1}{2\ell} > 0. \quad \square$$

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Proof. We only show $(2) \Rightarrow (3)$ and $(4) \Rightarrow (3)$ because the other implications are easy to check.

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Since

$$D(\mathcal{A}, m) \simeq S \cdot D(\mathcal{A}_1, m|_{\mathcal{A}_1}) \oplus S \cdot D(\mathcal{A}_2, m|_{\mathcal{A}_2}) \oplus \cdots \oplus S \cdot D(\mathcal{A}_s, m|_{\mathcal{A}_s})$$

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Thus Proposition 3.3 shows that each arrangement \mathcal{A}_i is in \mathbb{K}^1 or \mathbb{K}^2 .

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Then the arrangement $\mathcal{B} = \{H_1, H_2, \dots, H_{\ell+1}\}$ is a **generic arrangement** which is known to be non-free.

This is a contradiction and thus we may conclude $\ell \leq 2$. □

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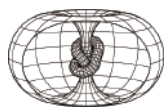
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More in general,

Question 4. What kind of restrictions does the set $\mathcal{FM}(\mathcal{A})$ (or $\mathcal{NFM}(\mathcal{A})$) impose on the original arrangement \mathcal{A} ?

I stop here.
Thank you!



The 2nd MSJ-SI

The Mathematical Society of Japan, Seasonal Institute

Arrangements of Hyperplanes



August 1 - 13, 2009

Conference Hall, Hokkaido University, Sapporo, Japan

Survey Lecturers:

Toshitake Kohno
Hal Schenck
Alexander Varchenko

Peter Orlik
Richard Stanley
Masahiko Yoshinaga

Kyoji Saito
Akimichi Takemura

Invited Speakers:

Kazuhiko Aomoto
Daniel Cohen
Alexandru Dimca
Eva Maria Feichtner
Eduard Looijenga
Mario Salvetti

Christos Athanasiadis
Frederick Cohen
Igor Dolgachev
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