Bar complex of Orlik-Solomon algebra and rational universal holonomy maps

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Plan of Talk

(1) Preliminaries on iterated integrals

(2) Bar complex of Orlik-Solomon algebra

(3) Finite type invariants for braids

(4) Universal holonomy map defined over $\mathbf{Q}$

1 Iterated integrals

$\omega_1, \cdots, \omega_k$: differential forms on M

$\Delta_k = \{(t_1, \cdots, t_k) \in \mathbf{R}^k \, ; \, 0 \leq t_1 \leq \cdots \leq t_k \leq 1\}$

$$\varphi : \Delta_k \times \Omega M \to \underbrace{M \times \cdots \times M}_{k}$$

defined by $\varphi(t_1, \cdots, t_k; \gamma) = (\gamma(t_1), \cdots, \gamma(t_k))$
The iterated integral of $\omega_1, \cdots, \omega_k$ is defined as

$$\int \omega_1 \cdots \omega_k = \int_{\Delta_k} \varphi^*(\omega_1 \times \cdots \times \omega_k)$$

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is the integration along fiber with respect to the projection $p : \Delta_k \times \Omega M \to \Omega M$.
It is a differential form on the loop space ΩM with degree $p_1 + \cdots + p_k - k$, where

$$\deg \omega_j = p_j.$$

Proposition (Chen complex)

As a differential form on the loop space $d\int \omega_1 \cdots \omega_k$ is

$$\sum_{j=1}^{k} (-1)^{\nu_{j-1}+1} \int \omega_1 \cdots \omega_{j-1} d\omega_j \; \omega_{j+1} \cdots \omega_k$$

$$+ \sum_{j=1}^{k-1} (-1)^{\nu_j+1} \int \omega_1 \cdots \omega_{j-1} (\omega_j \wedge \omega_{j+1}) \omega_{j+2} \cdots \omega_k$$

where $\nu_j = \deg \omega_1 + \cdots + \deg \omega_j - j$.

Bar complex $B^*(M)$ is a total complex associated with Chen complex

$$B^{-k,p}(M) = \left[\otimes^k \mathcal{E}^*(M)[1]\right]^{p-k}$$

The iterated integral induces a map

$$I : B^*(M) \longrightarrow \mathcal{E}^*(\Omega M)$$

which induces an isomorphism on cohomology if M is simply connected.

Example

In the case $m \geq 3$, there is an isomorphism

$$H^*(B\mathrm{Conf}_n(\mathbf{R}^m)) \cong H^*_{DR}(\Omega\mathrm{Conf}_n(\mathbf{R}^m))$$

In the case $m = 2$, $H^0(B\mathrm{Conf}_n(\mathbf{R}^2))$ is isomorphic to the space of finite type invariants for P_n.

In the case $m \geq 3$, the de Rham cohomology

$$H_{DR}^{k(m-2)}(\Omega \mathrm{Conf}_n(\mathbf{R}^m))$$

is represented by iterated integrals of the form

$$\sum a_{i_1 j_1 \cdots i_k j_k} \int \omega_{i_1 j_1} \cdots \omega_{i_k j_k}$$

$$+(\text{iterated integrals of length} < k)$$

Here the coefficients $a_{i_1 j_1 \cdots i_k j_k} \in \mathbf{Z}$ satisfy the graded infinitesimal pure braid relations.

2 Bar complex of Orlik-Solomon algebra

Let $\{H_j\}$, $1 \leq j \leq r$, be a family of complex hyperplanes in $\mathbf{C}^m$ and f_j be a linear form defining the hyperplane H_j.

$$M = \mathbf{C}^m \setminus \bigcup_{1 \leq j \leq r} H_j$$

We consider the logarithmic differential forms

$$\omega_j = \frac{1}{2\pi\sqrt{-1}} \frac{df_j}{f_j}, \quad 1 \le j \le r.$$

The Orlik-Solomon algebra A is the $\mathbf{Z}$ subalgebra of $\mathcal{E}^*(M)$ generated by the logarithmic forms ω_j, $1 \le j \le r$.

Proposition The inclusion

$$i : B^*(A) \longrightarrow B^*(M)$$

induces an isomorphism of algebras

$$H^*(B^*(A)) \otimes \mathbf{C} \cong H^*(B^*(M)).$$

3 Acyclicity of the bar complex

The following vanishing holds for the cohomology of the bar complex of the Orlik-Solomon algebra for fiber type arrangements.

$$H^j(B^*(A)) \cong 0, \quad j \neq 0.$$

4 Finite type invariants

There is an increasing sequence of modules:

$$V_0(P_n) \subset V_1(P_n) \subset \cdots \subset V_k(P_n) \subset \cdots$$

$$V_k(P_n) = \mathrm{Hom}(\mathbf{Z}P_n/I^{k+1}, \mathbf{Z})$$

We set $V(B_n) = \bigcup_{k\geq 0} V_k(B_n)$: the space of finite type invariants for B_n.

A : Orlik-Solomon algebra for braid arrangement

$$\mathcal{F}^{-k}B^*(A) = \bigoplus_{q\leq k} B^{-q,p}(A), \quad k = 0, 1, 2, \cdots$$

This induces a filtration $\mathcal{F}^{-k}H^0(B^*(A))$, $k \geq 0$, on the cohomology of the bar complex.

The above iterated integral defines a map

$$\iota : H^0(B^*(A)) \longrightarrow \mathrm{Hom}(\mathbf{Z}P_n, \mathbf{C}).$$

The iterated integral map ι gives the isomopshisms

$$\mathcal{F}^{-k}H^0(B^*(A)) \otimes \mathbf{C} \cong V_k(P_n)_{\mathbf{C}},$$
$$H^0(B^*(A)) \otimes \mathbf{C} \cong V(P_n)_{\mathbf{C}}.$$

We shall explain that these isomorphisms actually hold over $\mathbf{Q}$.

The iterated integrals give multivalued functions on M.

$$I : \mathcal{F}^{-k} H^0(B^*(A)) \longrightarrow F_k(M)$$

$F_k(M)$ is the space of order k hyperlogarithms.

$$F_k(M) \subset F_{k+1}(M)$$

$$dF_{k+1}(M) \subset F_k(M) \otimes A^1$$

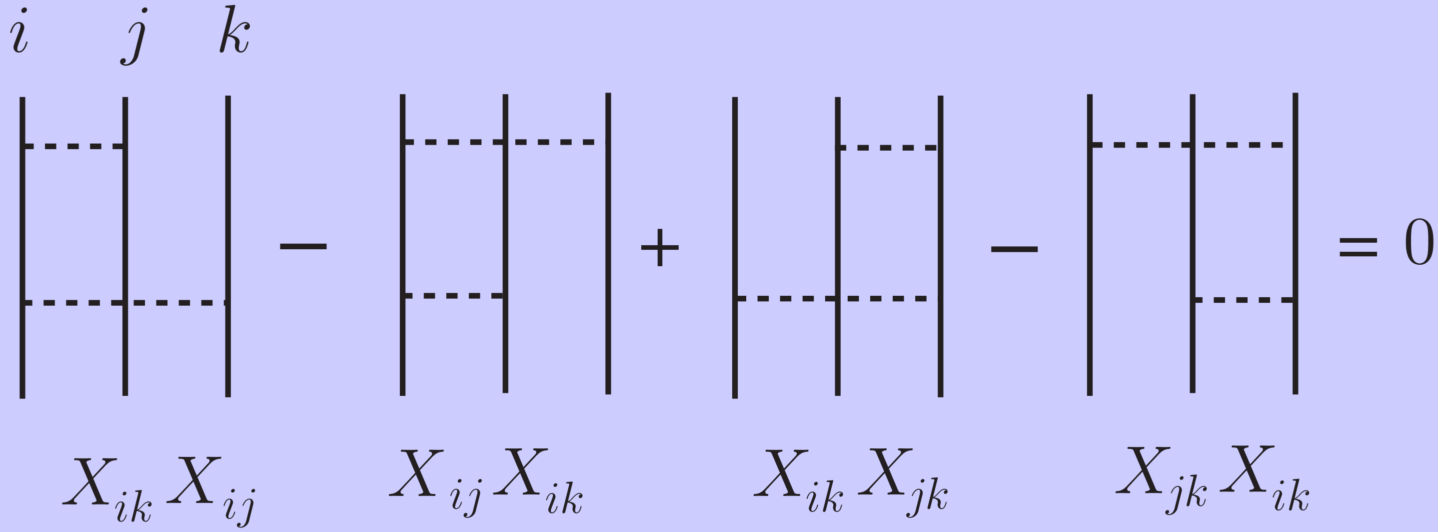
5 Drinfel'd associator and holonomy of braid groups

We denote by $\mathcal{A}_n$ the algebra over $\mathbf{Z}$ generated by X_{ij}, $1 \leq i \neq j \leq n$, with the relations :

$$X_{ij} = X_{ji} \tag{1}$$

$$[X_{ik}, X_{ij} + X_{jk}] = 0 \quad i, j, k \;\text{ distinct}, \tag{2}$$

$$[X_{ij}, X_{k\ell}] = 0 \quad i, j, k, \ell \;\text{ distinct}. \tag{3}$$



We define the semi-direct product $\mathcal{A}_n \rtimes \mathbf{Z}S_n$ by the relation

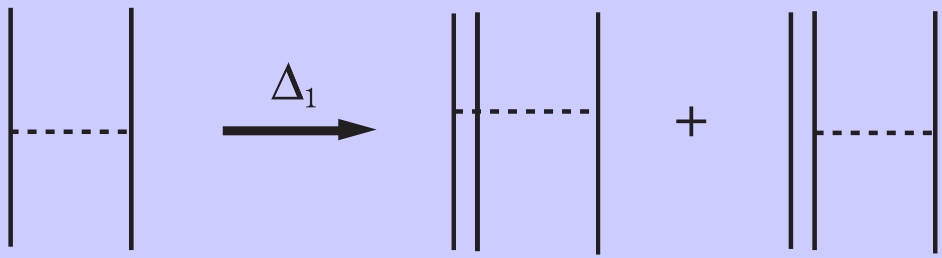
$$X_{ij} \cdot \sigma = \sigma \cdot X_{\sigma(i)\sigma(j)}$$

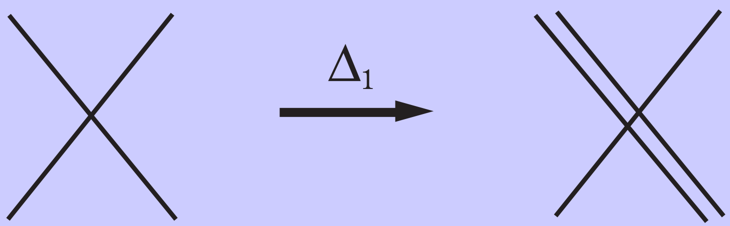
for $\sigma \in S_n$.

We define the operation of doubling the i-th vertical strand

$$\Delta_i : \mathcal{A}_n \longrightarrow \mathcal{A}_{n+1}$$

by the correspondence





The map $\varepsilon_i : \mathcal{A}_n \longrightarrow \mathcal{A}_{n-1}$, $1 \leq i \leq n$, is defined by setting $\varepsilon_i(X)$ to be represented by the chord diagram obtained by deleting the i-th vertical strand if there is no horizontal chord on the i-th vertical strand in $X \in \mathcal{A}_n$ and to be 0 otherwise.

$t_{ij} \in S_n$: permutation of i-th and j-th letters.

The element R is defined by

$$R = t_{12} \exp\left(\frac{1}{2} X_{12}\right).$$

A Drinfel'd associator Φ is an element of $\widehat{\mathcal{A}}_3 \otimes \mathbf{C}$ satisfying the following properties.

- (strong invertibility)

$$\varepsilon_1(\Phi) = \varepsilon_2(\Phi) = \varepsilon_3(\Phi) = 1$$

- (skew symmetry)

$$\Phi^{-1} = t_{13} \cdot \Phi \cdot t_{13}$$

- (pentagon relation)

$$(\Phi \otimes id) \cdot (\Delta_2 \Phi) \cdot (id \otimes \Phi) = (\Delta_1 \Phi) \cdot (\Delta_3 \Phi)$$

$$\begin{tikzcd}
 & ((ab)c)d \arrow[dl, "\Phi^{123}"'] \arrow[dr, "\Delta_1\Phi"] & \\
(a(bc))d \arrow[d, "\Delta_2\Phi"'] & & (ab)(cd) \arrow[d, "\Delta_3\Phi"] \\
a((bc)d) \arrow[rr, "\Phi^{234}"] & & a(b(cd))
\end{tikzcd}$$

- (hexagon relation)

$$\Phi \cdot (\Delta_2 R) \cdot \Phi = (R \otimes id) \cdot \Phi \cdot (id \otimes R)$$

$$\begin{tikzcd}
 & (ab)c \arrow[dl, "\Phi"'] \arrow[dr, "R^{12}"] & \\
a(bc) \arrow[d, "\Delta_2 R"'] & & (ba)c \arrow[d, "\Phi"] \\
(bc)a \arrow[dr, "\Phi"'] & & b(ac) \arrow[dl, "R^{23}"] \\
 & b(ca) &
\end{tikzcd}$$

The original Drinfel'd associator is an element in the ring of non-commutative formal power series $\mathbf{C}[[X, Y]]$ describing a relation of the solutions $G_0(z)$ and $G_1(z)$ of the differential equation

$$G'(z) = \left(\frac{X}{z} + \frac{Y}{z-1}\right) G(z) \tag{4}$$

with the asymptotic behavior

$$G_0(z) \sim z^X, \quad z \longrightarrow 0$$
$$G_1(z) \sim (1-z)^Y, \quad z \longrightarrow 1.$$

We set

$$G_0(z) = G_1(z)\Phi_{KZ}(X, Y)$$

and it can be shown that $\Phi_{KZ}(X_{12}, X_{23})$ satisfies the above properties for an associator. Drinfel'd shows that there exists an associator with coefficients in $\mathbf{Q}$ unique upto "gauge equivalence".

An explicit rational associator up to degree 4 terms is of the form

$$\Phi(X,Y) = 1 - \frac{\zeta(2)}{(2\pi i)^2}[X,Y]$$

$$- \frac{\zeta(4)}{(2\pi i)^4}[X,[X,[X,Y]]] - \frac{\zeta(4)}{(2\pi i)^4}[Y,[Y,[X,Y]]]$$

$$- \frac{\zeta(3,1)}{(2\pi i)^4}[X,[Y,[X,Y]]] + \frac{1}{2}\frac{\zeta(2)^2}{(2\pi i)^4}[X,Y]^2 + \cdots$$

with $X = X_{12}$, $Y = X_{23}$, where $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$ and $\zeta(3,1) = \pi^4/360$.

We set

$$R_{j,j+1} = t_{j,j+1} \exp\left(\frac{1}{2} X_{j,j+1}\right).$$

For the generators σ_j, $1 \leq j \leq n-1$, of the braid group B_n we put

$$\Theta(\sigma_j) = \Phi_j \cdot R_{j,j+1} \cdot \Phi_j^{-1}, \quad 1 \leq j \leq n-1$$

Φ_j is defined by means of a rational Drinfel'd associator by the formulae

$$\Phi_j = \Phi\left(\sum_{i=1}^{j-1} X_{ij},\ X_{j,j+1}\right), \quad j > 1$$

and $\Phi_1 = 1$.

Theorem

Θ defines an injective homomorphism

$$\Theta : B_n \longrightarrow (\widehat{A}_n \otimes \mathbf{Q}) \rtimes \mathbf{Z}S_n.$$

The map Θ is called a universal holonomy homomorphism of the braid group over $\mathbf{Q}$.

Theorem

The universal holonomy homomorphism Θ is equivalent to the holonomy map

$$B_n \longrightarrow (\widehat{\mathcal{A}}_n \otimes \mathbf{C}) \rtimes \mathbf{Z}S_n.$$

defined by

$$\gamma \mapsto 1 + \int_\gamma \omega + \int_\gamma \omega\omega + \cdots$$

$$\omega = \sum_{i,j} \omega_{ij} X_{ij}$$

Theorem

We have the following isomorpshisms for finite type invariants over the field of rational numbers.

$$V(P_n)_{\mathbf{Q}} \cong \mathrm{Hom}(\widehat{\mathcal{A}}_n, \mathbf{Q})$$

$$V(B_n)_{\mathbf{Q}} \cong \mathrm{Hom}(\widehat{\mathcal{A}}_n \rtimes \mathbf{Z}S_n, \mathbf{Q})$$