Approximating Submodular Functions Everywhere

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Submodular Functions

- Definition

  $f : 2^{[n]} \rightarrow \mathbb{R}$ is submodular if, for all $A, B \subseteq [n]$:

  $$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

  - Discrete analogue of convex functions [Lovász '83]
  - Arise in combinatorial optimization, probability, economics (diminishing returns), geometry, etc.
  - Typically given by an oracle

- Fundamental Examples

  Rank function of a matroid, cut function of a graph, ...

Optimizing Submodular Functions in Oracle Model

- Minimum (lattice of minima) of a submodular function f can be obtained with polynomially many oracle calls [GLS], [Schrijver '01], [Iwata, Fleischer, Fujishige '01], ...
  - Example of submodular fctn minimization. [Edmonds '70]. Given 2 matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$:

$$\max\{|I| : I \in \mathcal{I}_1 \cap \mathcal{I}_2\} = \min\{r_1(S) + r_2(E \setminus S) : S \subseteq E\}$$

  - Replace r_1, r_2 by general monotone submodular functions $\longrightarrow$ polymatroid intersection

- Maximum of a nonnegative submodular function can be approximated within $2/5$ [Feige et al. '07]

Submodular Max-Min Fair Allocation Problem

Making kids smile

Problem

Consider m buyers and set $[n]$ of items. Buyer j has a monotone submodular function $f_j : 2^{[n]} \rightarrow \mathbb{R}$. Goal:

$$\max_{\textit{partitions } (P_1, P_2, \cdots, P_m) \textit{ of } [n]} \min_j f_j(P_j).$$

- [Golovin '05]: $(n - m + 1)$-approximation algorithm
- [Khot and Ponnuswami '07]: $(2m - 1)$-approx. alg.

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Santa Claus problem:

Special case when f_j's are modular functions: $f_j(S) = \sum_{i \in S} c_{ij}$

- Santa Claus problem NP-hard even for $m = 2$ kids
- [Asadpour and Saberi '07]: $O(\sqrt{m} \log^3 m)$-approximation algorithm

Problem

Given oracle for a monotone, submodular f, construct (implicitly) in P-time a function g such that, for all $S \subseteq V$:

$$g(S) \leq f(S) \leq \alpha(n)g(S).$$

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Our Positive Result

Construct in deterministic P-time a (submodular) function

$g(S) = \sqrt{\sum_{i \in S} c_i}$ with

- $\alpha(n) = \sqrt{n+1}$ for matroid rank functions f, or
- $\alpha(n) = O(\sqrt{n} \log n)$ for general monotone submodular f

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Our Negative Result

With polynomially many oracle calls, $\alpha(n) = \Omega(\sqrt{n}/\log n)$ (even for randomized algs)

Improved to $\alpha(n) = \Omega(\sqrt{n/\log n})$ by Svitkina and Fleischer, 2008.

Polymatroid

Definition

Given submodular f, polymatroid

$$P_f = \left\{ x \in \mathbb{R}^n_+ : x(S) := \sum_{i \in S} x_i \leq f(S) \text{ for all } S \subseteq [n] \right\}$$

A few properties [Edmonds '70]:

- Can optimize over P_f with greedy algorithm
- Vertices of P_f. Take permutation $\sigma \in S_n$, and define x^σ
  $x^\sigma_{\sigma(i)} = f(\{\sigma(1), \sigma(2), \cdots, \sigma(i)\}) - f(\{\sigma(1), \cdots, \sigma(i-1)\})$
- Separation problem for P_f is submodular fctn minimization
- For monotone f, can reconstruct f:

$$f(S) = \max_{x \in P_f} \langle 1_S, x \rangle$$

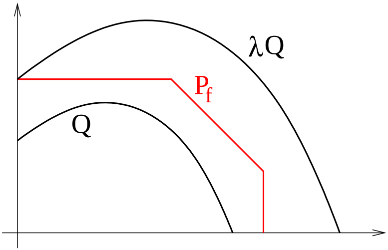
Basic Approach

If $Q \subseteq P_f \subseteq \lambda Q$ then

$$g(S) \le f(S) \le \lambda g(S)$$

where

$$g(S) = \max_{x \in Q} \langle 1_S, x \rangle$$

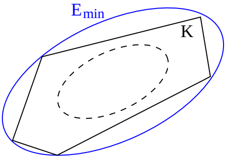


Theorem

Let K be a convex body in $\mathbb{R}^n$. Let E_{min} (or Löwner ellipsoid) be min volume ellipsoid circumscribed to K ($E_{min} \supseteq K$). Then

- $K \supseteq \frac{1}{n} E_{min}$
- $K \supseteq \frac{1}{\sqrt{n}} E_{min}$ *if K is centrally symmetric ($x \in K$ iff $-x \in K$)*

Algorithmically??



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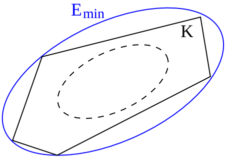
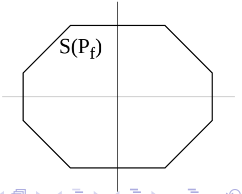


Make P_f centrally symmetric:
$K = S(P_f) = \{x : (|x_1|, |x_2|, \cdots, |x_n|) \in P_f\}$



Ellipsoids Basics

- Ellipsoids centered at 0
- For $A \succ 0$ (positive definite), let

$$E(A) = \{x \in \mathbb{R}^n : x^T A x \leq 1\} = \{x : \|x\|_A \leq 1\}$$

  where $\|x\|_A = \sqrt{x^T A x}$
- Linear image of unit ball: $E(A) = A^{-1/2}(B_n)$
- $\mathrm{vol}(E(A)) = \mathrm{vol}(B_n)/\det(A^{1/2})$
- $\max_{x \in E(A)} \langle c, x \rangle = \|c\|_{A^{-1}}$

Definition

E is a λ-ellipsoidal approximation to centrally symmetric K if

$$\lambda \geq \frac{\inf\{\alpha : \alpha E \supseteq K\}}{\sup\{\alpha : \alpha E \subseteq K\}}$$

Min Volume Circumscribed Ellipsoids
as convex semi-infinite programs

Restrict to centrally symmetric convex bodies

Formulation for $E_{min} = E(A)$

$$\begin{array}{ll} \min & -\log\det(A) \\ \text{s.t.} & x^T A x \leq 1 \qquad x \in K \\ & A \succ 0 \end{array}$$

- Convex: $\det(A)$ log-concave over pos. def. matrices [Fan '50]
- Strict log-concavity: E_{min} is unique [Löwner]
- John's theorem from optimality conditions

Maximum Volume Inscribed Ellipsoid E_{max}

aka. John ellipsoid or Löwner-John ellipsoid

By polarity ($K^* = \{y : \langle x, y \rangle \leq 1 \ \ \forall x \in K\}$):

Formulation for $E_{max} = E(A)$ (John ellipsoid or Löwner-John ellipsoid)

$$\begin{array}{ll} \max & \log\det(A^{-1}) \\ \text{s.t.} & c^T A^{-1} c \leq 1 \quad c \in K^* \\ & A^{-1} \succ 0 \end{array}$$

- Can restrict c to vertices of K^* (maximal faces of K)

Algorithms for Ellipsoidal Approximations

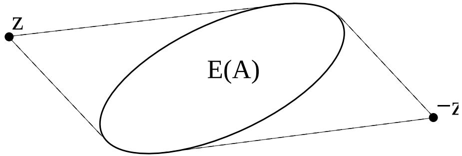
- Nestervov and Nemirovski '89 and Khachiyan and Todd '93:
  - Can find E_{min} in P-time (up to ϵ) if explicitly given as $K = \mathrm{conv}(a_1, \cdots, a_m)$
  - Can find E_{max} in P-time (up to ϵ) if explicitly given as $K = \{x : Ax \leq b\}$
- Grötschel, Lovász and Schrijver:
  - $\sqrt{n+1}$-ellipsoidal approximation in P-time for explicitly given polytopes $K = \{x : -b \leq Ax \leq b\}$
  - only $n + 1$-ellipsoidal approximation for convex bodies given by weak separation oracle
- No (even randomized) $o(n/\log n)$-ellipsoidal approximation for convex bodies given by a separation oracle [J. Soto]
  Same idea as in [BriedenGKKLS'99] for approximating diameter

Finding Larger and Larger Inscribed Ellipsoids

Theorem

If $A \succ 0$ and $z \in \mathbb{R}^n$ with $l = \|z\|_A^2 \geq n$ then $E(A')$ is max volume ellipsoid inscribed in $\mathrm{conv}\{E(A), \{z, -z\}\}$ where

$$A' = \frac{n}{l}\frac{l-1}{n-1}A + \frac{n}{l^2}\left(1 - \frac{l-1}{n-1}\right)Azz^TA$$



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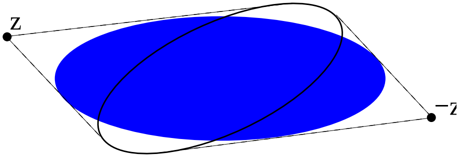


$\mathrm{vol}(E(A')) = k_n(l)\mathrm{vol}(E(A))$ where

$$k_n(l) = \sqrt{\left(\frac{l}{n}\right)^n \left(\frac{n-1}{l-1}\right)^{n-1}}$$

Remarks

- $k_n(l) > 1$ for $l > n$ proves (polar to) John's theorem
- Significant volume increase for $l \geq n+1$:
  $k_n(n+1) = 1 + \Theta(1/n^2)$

Ellipsoidal Norm Maximization
for Ellipsoidal Approximations

- Ellipsoidal Norm Maximization

  Given $A \succ 0$, given well-bounded ($B(r) \subseteq K \subseteq B(R)$) convex body K by separation oracle, find

  $$\max_{x \in K} \|x\|_A$$

  - P-time α-approximation algorithm for Ellipsoidal Norm Maximization gives P-time $\alpha\sqrt{n+1}$-ellipsoidal approximation for K (in $O(n^3 \log(R/r))$ iterations)

- Complexity

  Ellipsoidal Norm Maximization NP-complete for $S(P_f)$ (same for P_f) even if f is a matroid rank function

Definition

$\mathrm{Aut}(K) = \{T(x) = Cx : T(K) = K\}$

- Uniqueness of $E_{max} \implies \mathrm{Aut}(K) \subseteq \mathrm{Aut}(E_{max})$
- Same for E_{min}

- $S(P_f)$ is axis-aligned $(\mathrm{Aut}(\cdot) \supseteq \{\mathrm{Diag}(\{\pm 1\}^n)\})$
  $\Rightarrow E_{max} = E(A^*)$ is axis-aligned, i.e. A^* is diagonal

Keeping Ellipsoids Axis-Aligned
when K is axis-aligned

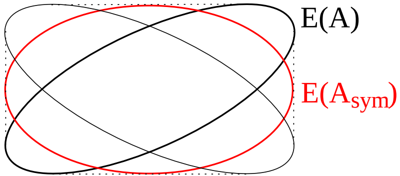
Lemma

Given $A \succ 0$ with $E(A) \subseteq K$, let

$$A_{sym} = \left(\mathrm{Diag}\left(\mathrm{diag}\left(A^{-1}\right)\right)\right)^{-1}$$

(zero out all non-diagonal entries of A^{-1}). Then

1. $\mathrm{vol}(E(A_{sym})) > \mathrm{vol}(E(A))$ *(Hadamard's ineq)*
2. $E(A_{sym}) \subseteq \mathrm{conv}(\bigcup_{C=Diag(\{\pm 1\}^n)} C(E(A))) \subseteq K$



Maximum Ellipsoidal Norm for Diagonal A

Only need $A \succ 0$ diagonal

Squared Norm formulation

$$\begin{array}{ll} \max & \sum_i d_i x_i^2 \\ \text{s.t.} & x \in P_f \end{array}$$

- Maximizing concave function over convex set
  $\Rightarrow$ max attained at vertex

If f is matroid rank function

- P_f: matroid polytope [Edmonds]. Vertices are $0-1$: $x_i^2 = x_i$. [Recall: [LS] matrix cuts]
- Squared norm maximization equivalent to

$$\begin{array}{ll} \max & \sum_i d_i x_i \\ \text{s.t.} & x \in P_f \end{array}$$

  $\rightarrow$ max weight matroid base, solved exactly by greedy alg
- In $O(n^3 \log n)$ iterations, can find axis-aligned ellipsoid $E = E(\mathrm{Diag}(1/c))$ ($\sim$ Löwner ellipsoid E_{max}) with $E \subseteq S(P_f) \subseteq \sqrt{n+1}E$

$$\rightarrow g(S) = \max_{x \in E} \langle 1_S, x \rangle = \|1_S\|_{\mathrm{Diag}(c)} = \sqrt{\sum_{i \in S} c_i}$$

Extension

Theorem

Can find $\sqrt{n+1}$ ellipsoidal approximation for any axis-symmetric $\{-1,0,1\}$ given by a separation oracle.

Ellipsoidal Norm Maximization
for general monotone submodular functions

Steps:

- $(1-\frac{1}{e})$-approximation algorithm for Euclidean Norm Maximization, i.e. $A = I$
- General case reduces to Euclidean Norm Maximization over scaled polymatroid
- Scaled polymatroid approximated by polymatroid at a loss of $O(\log n)$

Euclidean Norm Maximization ($A = I$)
for general monotone submodular functions

Problem

$\max_{x \in P_f} \|x\|$

Algorithm

Define permutation σ by

$$\sigma(i+1) = \arg\max_j f(\{\sigma(1), \sigma(2), \cdots, \sigma(i), j\})$$

Output vertex x^σ

Theorem

Algorithm is a $1 - 1/e$*-approximation algorithm for* $\max_{x \in P_f} \|x\|$

Uses Nemhauser et al. '78 result for maximizing submodular function over cardinality constraint

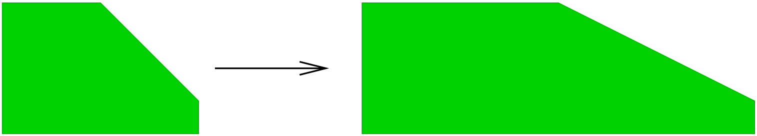
Rescaling

$$\max_{x \in P_f} \sqrt{\sum_i c_i^2 x_i^2} \longrightarrow \max_{y \in T(P_f)} \|y\|$$

where

$$T : \mathbb{R}^n \to \mathbb{R}^n : x \to y = (c_1 x_1, \cdots, c_n x_n)$$

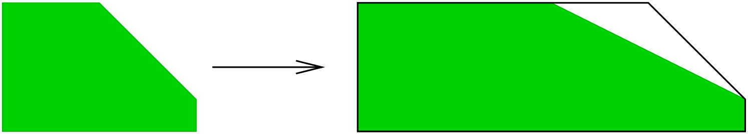
- $T(P_f)$ not a polymatroid



Approximating Scaled Polymatroid by Polymatroid

$T(P_f) \subseteq P_g$ where

$$g(S) = \max\{\langle 1_S, y\rangle : y \in T(P_f)\} = \max\{\langle c, x\rangle : x \in P_f\}.$$



Theorem

g is submodular

Proof.

$f \longrightarrow$ Lovász extension $\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{R} : c \rightarrow \max\{\langle c, x\rangle : x \in P_f\}$.

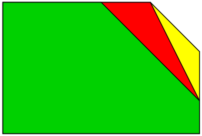
$\tilde{f}$ is L-convex: $\tilde{f}(w_1) + \tilde{f}(w_2) \geq \tilde{f}(w_1 \vee w_2) + \tilde{f}(w_1 \wedge w_2)$
[$\vee$ (resp. $\wedge$): component-wise max (resp. min)]

Sandwiching $T^{-1}(P_g)$

- Can approx. $\max\{\|y\| : y \in P_g\}$, or $\max\{\sqrt{\sum_i c_i^2 x_i^2} : x \in T^{-1}P_g\}$
- How close is P_f and $T^{-1}(P_g)$?
- Polymatroid approximation again: $P_f \subseteq T^{-1}(P_g) \subseteq P_h$

$$h(S) = \max\{\langle c, x\rangle : x \in T^{-1}(P_g)\} = \max\{\langle 1/c, y\rangle : y \in P_g\}$$

submodular (by Lovász extension)



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Explicit Expressions for g and h

For $S = \{1, \cdots, k\}$ with $c_1 \leq c_2 \leq \cdots \leq c_k$, we have

- $g(S) = \sum_{i=1}^{k} c_i \left[f(i,k) - f(i+1,k)\right]$
- $h(S) = \sum_{l,m:1\leq l\leq m\leq k} (c_l - c_{l-1}) \left(\frac{1}{c_m} - \frac{1}{c_{m+1}}\right) f(l,m)$

Theorem

For every S, $h(S) \leq O(\log n) f(S)$

Summarizing

Theorem

Construct in deterministic P-time a (submodular) function $g(S) = \sqrt{\sum_{i \in S} c_i}$ *with*

- $\alpha(n) = \sqrt{n+1}$ *for matroid rank functions* f*, or*
- $\alpha(n) = O(\sqrt{n} \log n)$ *for general monotone submodular* f

$\Omega(\sqrt{n}/\log n)$ Lower Bound
for matroid rank functions

With polynomially many oracle calls, cannot distinguish between

- Uniform matroid U with rank $\alpha = \lceil \sqrt{n} \rceil$:
  $r_U(S) = \min\{|S|, \alpha\}$
- Matroid M_R parameterized by a fixed set R of cardinality α.
  Independent sets: $\mathcal{I} = \{I \subseteq [n] : |I| \leq \alpha \text{ and } |I \cap R| \leq \log n\}$

Theorem

With polynomially many oracle calls, $\alpha(n) = \Omega(\sqrt{n}/\log n)$ (even for randomized algs)

Back to Submodular Max-Min Fair Allocation Problem

Goal:

$$\max_{\text{partitions } (P_1, P_2, \cdots, P_m) \text{ of } [n]} \min_j f_j(P_j).$$

- Can replace f_j by
$$g_j(S) = \sqrt{\sum_{i \in S} c_{ij}}$$
- Reduces to Santa Claus problem (at a loss of $O(\sqrt{n} \log n)$)
$\longrightarrow O(n^{\frac{1}{2}} m^{\frac{1}{4}} \log n \log^{\frac{3}{2}} m)$-approximation
- Not great, but better than previous results...

Submodular Load-Balacing

[Svitkina and Fleischer '08]:

Problem

Given submodular $f : 2^{[n]} \to \mathbb{R}$ *and integer* k*, partition* $[n]$ *into* $P_1, \cdots, P_k$ *to*

$$\min \max_k f(P_i)$$

For $g(S) = \sqrt{\sum_{i \in S} c_i}$, equivalent to scheduling on unrelated parallel machines: 2-approx alg. [LenstraST90]

Can immediately get $O(\sqrt{n} \log n)$-approximation for submodular load-balancing

Degree-Bounded MST

Definition

Minimum Degree-Bounded Spanning Tree (MST) problem:

- Given $G = (V, E)$ with costs $c : E \longrightarrow \mathbb{R}$, integer k
- find Spanning Tree T of maximum degree $\leq k$ and of minimum total cost $\sum_{e \in T} c(e)$

Even feasibility is hard ($k = 2$: Hamiltonian path).

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Results:

Let $OPT(k)$ be the cost of the optimum tree of max degree $\leq k$.

- [G. 2006]: Find a tree of cost $\leq OPT(k)$ and of max degree $\leq k + 2$
  (or prove that no tree of max degree $\leq k$ exists)
- [Singh and Lau 2007]: same result with $k + 1$

Our Algorithm

- Solve natural LP relaxation (spanning tree polytope intersected with degree constraints) and get an extreme point x^* of cost LP

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- Hence, E^* can be oriented so that indegree of every vertex is at most 2.

  Define matroid $M_2(x^*)$ on ground set E^* such that independent sets have at most k outgoing edges from any vertex

  $\Rightarrow$ any independent set has degree at most $k + 2$

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- Find a minimum cost spanning tree of E^* which is also independent in $M_2(x^*)$ (by matroid intersection)

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- Argue (polyhedrally) that cost of solution obtained $\leq LP$

D.R. Fulkerson Prize
2009 Call for Nominations

- For outstanding papers in the area of discrete mathematics published between 1/2003 and 12/2008
- Prize Committee: Bill Cook (chair), Michel Goemans, Danny Kleitman
- To be awarded in Aug 2009
- Send nominations to Bill Cook (bico@isye.gatech.edu) by 1/15/2009