

Definable semi-germs

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1. The ring $\mathcal{F}^{(n)}$

${}^*{\mathbb{R}}$ denotes an extension of \mathbb{R} suitable for Non-Standard Analysis (NSA).

- For definiteness, ${}^*{\mathbb{R}} = {\mathbb{R}}^{\mathbb{I}}/\mu$: a suitably saturated ultrapower of \mathbb{R} .
- Also, ${}^*\mathbb{C} := \mathbb{C}^{\mathbb{I}}/\mu$ ($= {}^*{\mathbb{R}} \times {}^*{\mathbb{R}}$
 $= {}^*{\mathbb{R}}[i]$).
- Indeed

... all functions $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$, and all sets $S \subseteq \mathbb{R}^n$ have canonical extensions, denoted *f , *S to ${}^*\mathbb{R}$.

- Lö's Theorem states that ${}^*\mathbb{R}$ is an elementary extension of \mathbb{R} for any 1st order structure on \mathbb{R} .
- Let us fix an o-minimal structure on \mathbb{R} expanding the field structure.
- Then, in particular, $\mathbb{R} \leq {}^*\mathbb{R}$ for this structure, and ${}^*\mathbb{R}$ is o-minimal.

• "Definable" means definable in ${}^*\mathbb{R}$ (or in \mathbb{R}) with parameters, unless otherwise stated. This is also applied in ${}^*\mathbb{C}$ (and \mathbb{C}) via the identification ${}^*\mathbb{C} = {}^*\mathbb{R} \times {}^*\mathbb{R}$ (or $\mathbb{C} = \mathbb{R} \times \mathbb{R}$).

• It is convenient to assume that every $t \in \mathbb{R}$ is a constant of our language.

Some notation

For $\bar{t} \in \mathbb{R}_+^n$, $\Delta^{(n)}(\bar{t}) = \{\bar{z} \in \mathbb{C}^n : |z_i| < t_i, i=1,\dots,n\}$
 (where $\bar{t} = \langle t_1, \dots, t_n \rangle$).

So ${}^*\Delta^{(n)}(\bar{t})$ denotes its extension to ${}^*\mathbb{C}$.



Definition

- (1) $\mathfrak{F}^{(\infty)}(\bar{\tau}) := \{ f : {}^*\Delta^{(\infty)}(\bar{\tau}) \rightarrow {}^*\mathbb{C} \mid$
 f is holomorphic and definable}
- (2) Clearly $\bar{\tau} < \bar{s}$ implies $\mathfrak{F}^{(\infty)}(\bar{s}) \subseteq \mathfrak{F}^{(\infty)}(\bar{\tau})$
and $\mathfrak{F}^{(\infty)}$ denotes the direct limit
of the directed set $\{ \mathfrak{F}^{(n)}(\bar{\tau}) : \bar{\tau} \in {}^*\mathbb{R}_+^n \}$
- NOT ${}^*\mathbb{R}_+^\infty$

Then $\mathfrak{F}^{(\infty)}$ is called the differential ring of definable, holomorphic semi-germs ($\text{in } {}^*\mathbb{C}, \bar{o}$).

(5)

- Let $\mu^* :=$ set of infinitessimals of ${}^*\mathbb{C}$.
 $= \{\bar{z} \in {}^*\mathbb{C}^n : \forall r \in \mathbb{R}_+, |z_i| < r \text{ for } i=1, \dots, n\}$

Then each $f \in {}^*\mathcal{F}$ may be regarded as a function $f : \mu^* \rightarrow {}^*\mathbb{C}$ (having the property that it extends to a definable, holomorphic $f : {}^*\Delta^{(n)}(\bar{r}) \rightarrow {}^*\mathbb{C}$, for some $\bar{r} \in \mathbb{R}_+^*$).

- For any $\bar{\alpha} \in \mu^*$, we have a homomorphism $H_{\bar{\alpha}} : {}^*\mathcal{F} \rightarrow {}^*\mathbb{C}[[z_1, \dots, z_n]] : f \mapsto \sum_{\sigma \in \mathbb{N}^n} \frac{f^{(\sigma)}(\bar{\alpha})}{\bar{\alpha}!} \bar{z}^\sigma$.

Proposition (Peterzil - Starchenko)

This homomorphism $H_{\bar{\alpha}}$ is injective.

[Remark: Does not need poly.-boundedness.]

- Situation for ${}^*\mathbb{R}^*$ is obscure.

I shall show here the following:

Theorem

Assume that the given o-minimal structure is polynomially bounded.

Then $\mathcal{F}^{(n)}$ is Noetherian.

Remarks (algebraically)

- 1) $\mathcal{F}^{(n)}$ is not flat in ${}^*\mathbb{C}[[z_1, \dots, z_n]]$,
e.g. if $\alpha \in \mu \setminus \{0\}$, then $z_1 - \alpha$ is
invertible in ${}^*\mathbb{C}[z_1]$ (via H_0), but not
in $\mathcal{F}^{(n)}$.
- 2) I don't know if the theorem holds
without poly-boundedness.

Motivation

2. Zilber's Conjecture

$\mathbb{C}_{\exp} (= \langle \mathbb{C}; +, \cdot, \exp \rangle)$ is quasi-minimal, i.e. every \mathbb{C}_{\exp} -definable subset of \mathbb{C} is either countable or no-countable.

This clearly involves studying sets of the form

$$V = \{ \bar{z} \in \mathbb{C}^n : f_1(\bar{z}) = \dots = f_m(\bar{z}) = 0 \}$$

where $f_j(\bar{z}) \in \mathbb{C}[z_1, \dots, z_n, e^{z_1}, \dots, e^{z_n}]$.

Crucial Case: $m = n - 1$ and V is non-singular w.r.t. $\bar{z}' (= z_1, \dots, z_n)$.

Crucial Question - - - -

What is the nature of the set of asymptotes of V , i.e. of the set of those $z \in \mathbb{C}$ for which

$\exists w_i \rightarrow z$, and $\langle w_i, \bar{z}'_i \rangle \in V$ with
 $\|\langle w_i, \bar{z}'_i \rangle\| \rightarrow \infty$ as $i \rightarrow \infty$.

Idea: Consider ${}^*V \subseteq {}^*\mathbb{C}^n$ and pick $i_0 \in {}^*N - N$. Let $\bar{x} = \langle w_{i_0}, \bar{z}'_{i_0} \rangle$.

Then $\bar{x} \in {}^*V$ and

$${}^*V \cap (\bar{x} + {}^*\Delta^{(n)}(\bar{T}))$$

is definable (for the o-minimal, poly-bdcl. structure ${}^*\mathbb{R} \gtrsim \langle \mathbb{R}; +, ;, \exp \restriction [0,1], \sin \restriction [0,1], \{\bar{T}\}_{\bar{T} \in {}^*\mathbb{R}} \rangle$) with parameters from ${}^*\mathbb{R}$ (actually ${}^*\mathbb{C}$).

3. Robinson's fundamental theorem of nonstandard complex analysis.

Let $\tilde{\tau} \in \mathbb{R}_+^n$, $f \in \mathcal{F}^{(\infty)}(\tilde{\tau})$ and assume that $\exists R \in \mathbb{R}_+$ such that $|f(z)| < R$ for all $\bar{z} \in {}^*\Delta^{(\infty)}(\tilde{\tau})$.

Then we may define a function

$$\tilde{f} : \Delta^{(\infty)}(\tilde{\tau}) \rightarrow \mathbb{C} \text{ by } \tilde{f}(\bar{z}) = \text{st.pt.}(f(\bar{z})).$$

Then \tilde{f} is holomorphic and $\frac{\partial \tilde{f}}{\partial z_i} = \frac{\partial f}{\partial z_i}$ on $\Delta^{(\infty)}(\tilde{\tau})$.

Remarks

(1) False in real case, e.g. consider

$$f : [-1, 1] \rightarrow [0, 1] : x \mapsto \frac{\epsilon}{x^2 + \epsilon} \quad (\epsilon \text{ a positive infmt}). \quad \tilde{f} \text{ not even continuous.}$$

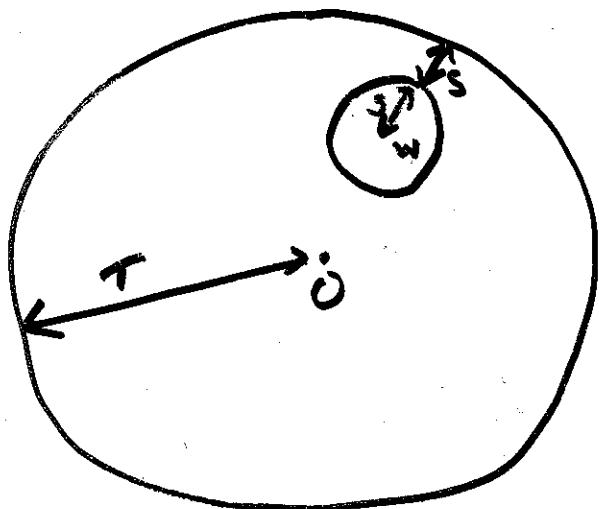
(2) The Marker-Steinhorn theorem implies that \tilde{f} is also definable (in the o-minimal structure on \mathbb{R}), and hence ${}^*\tilde{f} \in {}^*\mathcal{F}^{(n)}(\mathbb{F})$, and notice that ${}^*\tilde{f}$ is definable without parameters. More on this later.

Proof of Robinson's Theorem.

- One easily reduces to case $n=1$.
- Let $f: {}^*\Delta^{(n)}(\tau) \rightarrow {}^*\Delta^{(n)}(R)$, $f \in \mathcal{F}^{(n)}(\tau)$.
Let $w \in \Delta^{(n)}(\tau)$ and set $s = \frac{1}{2}(\tau - |w|)$.
Then f is $\frac{2R}{\delta}$ -Lipschitz on ${}^*\Delta_w^{(n)}(s)$

.... just apply the Maximum Modulus theorem (in ${}^*\mathbb{C}$ - this is NSA) to

$$F(z) := \frac{f(z) - f(w)}{z - w} \quad \text{for } |z - w| < \delta : -$$



Since $\frac{2R}{\delta} \in \mathbb{R}_+$, this easily gives continuity, and same Lipschitz property for \tilde{f} . For analyticity, can we use Morera's theorem, or same argument applied

$$\text{to } g(z) := \frac{f(z) - f(w) - (z-w)f'(w)}{(z-w)^2}$$

□

4. Using the Marker-Steinhorn Theorem.

- We have $*\mathbb{C} \subseteq \mathcal{F}^{(n)}$ (as constant functions).
- If I is an ideal of $\mathcal{F}^{(n)}$ then
 $I \cap *\mathbb{C} = \{0\}$
- I prefer to work with the rings
 $\mathcal{A}^{(n)}(\bar{\tau}) := \{f \in \mathcal{F}^{(n)} : \exists R \in \mathbb{R}, |f(z)| < R,$
 $\forall z \in *D^{(n)}(\bar{\tau})\}$.
- and the corresponding direct limit,
denoted $\mathcal{A}^{(n)}$.
- Sufficient to show $I \cap \mathcal{A}^{(n)}$ is a fin. gen. ideal of $\mathcal{A}^{(n)}$, since $\forall f \in \mathcal{F}^{(n)}$
 $\exists b \in *\mathbb{C}$ s.t. $b \cdot f \in \mathcal{A}^{(n)}$.
- For $n=0$, $\mathcal{A}^{(0)} = \text{Fin}(*\mathbb{C})$ and $I \cap \mathcal{A}^{(0)} = (0)$.

Let $n > 0$.

Let $A^{(n)}(\bar{r})$ be the (subring) of $A^{(\infty)}(\bar{r})$ consisting of only the parameter-free definable $f \in A^{(\infty)}(\bar{r})$. Let $A_0^{(n)}$ be the corresponding direct limit of the $A^{(n)}(\bar{r})$'s.

By Robinson's Theorem and the Marker - Steinhorn Theorem we have a differential ring homomorphism

$$\phi_n : A^{(n)} \rightarrow A_0^{(n)} : f \mapsto {}^*\tilde{f}$$

(which splits the inclusion $A_0^{(n)} \subseteq A^{(n)}$).

NB: For $f \in A^{(\infty)}(\bar{r})$, $|f(\bar{z}) - {}^*\tilde{f}(\bar{z})| \in \mu$
 $\forall \bar{z} \in {}^*\Delta^{(\infty)}(\bar{r})$.

So $f \in A^{(n)}$ lies in $\ker(\phi_n)$

iff $\exists \bar{r} \in \mathbb{R}_+$ s.t. $f \in A^{(n)}(\bar{r})$ and
 $|f(\bar{z})| \leq \mu \quad \forall \bar{z} \in {}^*A^{(n)}(\bar{r}).$

We now obtain (by transfer):

5. The WPT for semi-germs

Suppose $f \in A^{(n)} - \ker(\phi_n)$ and $f(\bar{o}) = 0$. Then after a linear, homogeneous change of coordinates,

$\exists d \geq 1$, there exists $\bar{r} \in \mathbb{R}_+$ and a representative $f \in A^{(n)}(\bar{r})$ with a representation

$$f(\bar{x}', x_n) = u(\bar{x}', x_n)(x_n^d + a_1(\bar{x}') \cdot x_n^{d-1} + \dots)$$

with $u(\bar{o}', 0) \notin \mu$, $u \in A^{(n)}(\bar{r})$, $a_i \in A_1^{(n-1)}(\bar{r}')$.

This only uses o-minimality. We can now complete the proof that I is finitely generated in the usual way, except for the hypothesis in WPT that $f \notin \text{ker}(\phi_a)$.

• May assume $\sup_{\bar{z} \in {}^*\Delta^{(n)}(\bar{\tau})} |f(\bar{z})| = 1$.

However, this is not enough to guarantee that $f \notin \text{ker}(\phi_a)$.

E.g. for $\varepsilon > 0$, $\varepsilon \in \mu$, consider $\frac{\varepsilon}{2 - (1 + \varepsilon)} = g(z)$

Then $\sup_{z \in {}^*\Delta^{(n)}(1)} |g(z)| = 1$, $\sup_{z \in {}^*\Delta^{(n)}(\frac{1}{2})} |g(z)| < 4\varepsilon$

So $g \in \text{ker}(\phi_1)$.

Theorem

Assume that our o-minimal structure is polynomially bounded.

Let $f \in A^{(n)}(\bar{\tau})$. Then there exists $\bar{s} \in \mathbb{R}_+^n$ with $\bar{s} < \bar{\tau}$, and $c \in {}^*\mathbb{C}$ such that

$$\sup_{\bar{z} \in {}^*\Delta^{(n)}(\bar{s})} |kf(\bar{z})| = \sup_{\bar{z} \in {}^*\tilde{f}^{(n)}(\bar{s})} |{}^*\tilde{c}\tilde{f}(\bar{z})| = i,$$

and hence $c \cdot f \in A^{(n)} \setminus \ker(\phi_n)$.

... and of course $c \cdot f \in I \Leftrightarrow f \in I$.