Resurgence of inner solutions for analytical perturbations of the McMillan map

P. Martin, D. Sauzin, T. M. Seara<br>Workshop on Finiteness Problems in Dynamical Systems<br>Fields Institute, Toronto, June 22-26, 2009

## The original problem: Perturbed McMillan family

$$
\binom{x}{y} \stackrel{F}{\longrightarrow}\binom{y}{-x+\mu \frac{2 y}{1+y^{2}}+\varepsilon V^{\prime}(y, h, \varepsilon)}
$$

where

- They are area preserving maps.
- $\mu=\cosh h, h>0, \varepsilon \in \mathbb{R}$,
- $V=\sum_{k \geq 2} V_{k} y^{2 k}$ is a holomorphic function in $B=\left\{(y, h, \varepsilon) \in \mathbb{C}^{3}| | y\left|<y_{0},|h|<h_{0},|\varepsilon|<\varepsilon_{0}\right\}\right.$
- $V^{\prime}$ is odd in $y$ and even in $h$,
- there exists $C>0$ such that $\left|V^{\prime}(y, 0, \varepsilon)\right| \leq C|y|^{5}$ for $|y|<y_{0}$, $|\varepsilon|<\varepsilon_{0}$.

The original problem: Dynamics around the origin

- $(0,0)$ is a fixed point for all $h, \varepsilon$.
- Spec $D F(0,0)=\left\{e^{h}, e^{-h}\right\} \Longrightarrow(0,0)$ is a weakly hyperbolic point.
- $h$ is the characteristic exponent.
- $(0,0)$ has invariant unstable and stable curves, $W^{u, s}$
- $W^{u, s}$ admit natural parametrizations

$$
z^{u, s}(t)=\left(x^{u, s}(t), y^{u, s}(t)\right)
$$

such that

$$
F\left(z^{u, s}(t)\right)=z^{u, s}(t+h)
$$

## The integrable case

When $\varepsilon=0$, the McMillan family is integrable.

$$
H^{0}(x, y)=\left(x^{2}-2 \mu x y+y^{2}+x^{2} y^{2}\right) /(2 \gamma), \quad \gamma=\sinh h
$$

is a first integral.


## Reversibility

McMillan maps of this family are reversible:

$$
F^{-1}=R \circ F \circ R, \quad R(x, y)=(y, x)
$$

that is, conjugated with its inverse by an involution $\left(R^{-1}=R\right)$.
Consequences:

- if $z^{u}(t)$ is a natural parametrization of $W^{u}, z^{s}(t)=R \circ z^{u}(-t)$ is a natural parametrization of $W^{s}$.
- the intersections of $W^{u}$ with $y=x$ are homoclinic points
- for $\varepsilon=0$, separatrix intersects transversely $y=x \Longrightarrow$ homoclinic points for $|\varepsilon|$ small.


## Perturbed case: the problem


problem: to give an asmptotic formula for the area of one of the lobes between $W^{u}$ and $W^{s}$.

## Perturbed case. Previous results

When $V$ is entire and

$$
\varepsilon=O\left(h^{6} / \ln h\right), h \rightarrow 0^{+}
$$

Delshams \& Ramírez-Ros (98) proved that the separatrix splits.
The area of one of the lobes between $W^{u}$ and $W^{s}$ is given by

$$
\begin{equation*}
A=8 \pi \varepsilon \hat{V}(2 \pi) e^{-\pi^{2} / h}\left(1+O\left(h^{2}\right)\right) \tag{1}
\end{equation*}
$$

The leading term was obtained computing a Melnikov formula. Smallness condition on $\varepsilon$ is necessary in order that the Melnikov formula gives the right prediction of the area.

The coefficient $\hat{V}(2 \pi)$ in formula (1) is the Borel transform of the perturbative potential at $2 \pi$, where

$$
\hat{V}(\zeta)=\sum_{n \geq 2} \frac{V_{n}}{(2 n-1)!} \zeta^{2 n-1}
$$

## Main Theorem

The area of the lobes between $W^{u}$ and $W^{s}$ is given by

$$
\begin{equation*}
A=B_{0}(\varepsilon) e^{-\pi^{2} / h}(1+o(1)) \tag{2}
\end{equation*}
$$

for $|\varepsilon|<1 / 2\left|V_{2}\right|, 0<h<h_{0}(\varepsilon)$,
( $o(1)$ stands for $|o(1)| \leq g(h)$, with $\left.\lim _{h \rightarrow 0^{+}} g(h)=0\right)$.

That is, the formula is valid for independent $\varepsilon$ and $h$.

Moreover,

$$
B_{0}(\varepsilon)=8 \pi \varepsilon \hat{V}(2 \pi)+O\left(\varepsilon^{2}\right)
$$

## Techniques used

We do

- use outer approximations far from the singularities of the homoclinic nonperturbed separatrix,
- use resurgence theory to study the solutions of the inner equations and their difference,
- use matching techniques to continue functions up to distance $\sim h \ln 1 / h$ of the singularities
- We do not use flow box coordinates


## Parametrizations of $W^{u, s}$

Symmetries imply that natural parametrizations of $W^{u, s}$ are

$$
z^{u, s}(t)=\left(x^{u, s}(t), y^{u, s}(t)\right)=\left(\xi^{u, s}(t-h / 2), \xi^{u, s}(t+h / 2)\right) .
$$

where $\xi(t)$ satisfies

$$
\xi(t+h)+\xi(t-h)=\mu \frac{2 \xi(t)}{1+\xi^{2}(t)}+\varepsilon V^{\prime}(\xi(t))
$$

with boundary conditions

$$
\lim _{t \rightarrow-\infty} \xi(t)=0, \quad \text { for } \xi^{u}
$$

and

$$
\lim _{t \rightarrow \infty} \xi(t)=0, \quad \text { for } \xi^{s}
$$

Moreover, we will require $\xi(-h / 2)=\xi(h / 2)$, which implies that $z^{u}(0)=z^{s}(0)$ is a homoclinic point.

## Finding $\xi^{u}$ and $\xi^{s}$ : Outer expansions (I)

$\xi^{u, s}$ are both solutions of the difference equation

$$
\xi(t+h)+\xi(t-h)=\mu \frac{2 \xi(t)}{1+\xi^{2}(t)}+\varepsilon V^{\prime}(\xi(t)), \quad \mu=\cosh h
$$

with different boundary conditions.
Expanding formally the solution in $h$,

$$
\xi(t, \varepsilon, h)=\sum_{k \geq 0} h^{2 k+1} \xi_{k}(t, \varepsilon),
$$

and imposing the boundary conditions

$$
\lim _{t \rightarrow-\infty} \xi_{k}(t, \varepsilon)=0, \quad\left(\lim _{t \rightarrow+\infty} \xi_{k}(t, \varepsilon)=0\right) \quad \xi_{k}(-h / 2, \varepsilon)=\xi_{k}(h / 2, \varepsilon)
$$

gives the same (divergent) expansion for $\xi^{u}$ and $\xi^{s}$.

## Finding $\xi^{u}$ and $\xi^{s}$ : Outer expansions (II)

Integrable case.
When $\varepsilon=0$, the map is integrable and we obtain:

$$
\xi^{u}(t, 0, h)=\xi^{s}(t, 0, h)=\xi^{0}(t, h)=\frac{\gamma}{\cosh t}, \quad \gamma=\sinh h .
$$

Its closest singularities to the real line are $\pm i \pi / 2$.
General case.
Although the series $\sum_{k \geq 0} h^{2 k} \xi_{k}$ is divergent, it satisfies (in some domain, at distance $\delta>h$ from $\pm i \pi / 2$ )

$$
\left|\xi^{u}(t, \varepsilon, h)-\sum_{k=0}^{N-1} h^{2 k+1} \xi_{k}(t, \varepsilon)\right| \leq C_{N} \frac{h^{2 N+1}}{|t \pm i \pi / 2|^{2 N+1}}
$$

It is the outer expansion. Since the outer expansion is the same for $\xi^{s}$, this implies that $\xi^{u}$ and $\xi^{s}$ coincide beyond all orders.

## The problem: Inner expansions

Study the behavior of $\xi^{u, s}$ for $t$ close to $i \pi / 2$ : change

$$
z=(t-i \pi / 2) / h, \quad \phi(z)=\xi(i \pi / 2+h z)
$$

Full Inner Equation

$$
\begin{equation*}
\phi(z+1)+\phi(z-1)=\mathcal{F}(\phi(z), h, \varepsilon) \tag{3}
\end{equation*}
$$

where $z \mapsto \phi(z)$ is the unknown scalar function and

$$
\begin{equation*}
\mathcal{F}(y, h, \varepsilon)=\frac{2(\cosh h) y}{1+y^{2}}+\varepsilon V^{\prime}(y, h, \varepsilon) \tag{4}
\end{equation*}
$$

## The problem

The $h^{2}$-expansion
We can expand

$$
\begin{equation*}
\mathcal{F}(y, h, \varepsilon)=\sum_{n \geq 0} h^{2 n} \mathcal{F}_{n}(y, \varepsilon) . \tag{5}
\end{equation*}
$$

Looking for a solution of (3) in the form

$$
\phi=\sum_{n \geq 0} h^{2 n} \phi_{n}(z, \varepsilon)
$$

and expanding in powers of $h^{2}$.

We get the "inner equation"
$\phi_{0}(z+1)+\phi_{0}(z-1)=\mathcal{F}\left(\phi_{0}(z), 0, \varepsilon\right)=\frac{2 \phi_{0}(z)}{1+\phi_{0}(z)^{2}}+\varepsilon V^{\prime}\left(\phi_{0}(z), 0, \varepsilon\right)$
and a system of "secondary inner equations"

$$
\begin{equation*}
\phi_{n}(z+1)+\phi_{n}(z-1)=F_{n}(z, \varepsilon), \quad n \geq 1 \tag{7}
\end{equation*}
$$

where the right-hand sides are determined inductively:

$$
F_{n}=\partial_{y} \mathcal{F}\left(\phi_{0}, 0, \varepsilon\right) \phi_{n}+f_{n}
$$

$f_{n}$ is the coefficient of $h^{2 n}$ in $\mathcal{F}\left(\phi_{0}+h^{2} \phi_{1}+\cdots+h^{2(n-1)} \phi_{n-1}, h, \varepsilon\right)$ (while $F_{n}$ is the coefficient of $h^{2 n}$ in $\mathcal{F}\left(\phi_{0}+h^{2} \phi_{1}+\cdots+h^{2 n} \phi_{n}, h, \varepsilon\right)$ ).

## Study of the inner equation

- Determine "formal solutions"(with respect to $z) \tilde{\Phi}_{n}(z, \varepsilon ; b)$ of equations (7) depending on a free parameter $b \in \mathbb{C}^{\mathbb{N}^{*}}$.
- $\tilde{\Phi}_{n}(z, \varepsilon ; b)$ are generically divergent
- We will study the analytic continuations of their Borel transforms $\hat{\Phi}_{n}(\zeta, \varepsilon ; b)$, which are analytic in a neighborhood of the origin.
- Borel-Laplace summation then leads to solutions $\Phi_{n}^{s}$ and $\Phi_{n}^{u}$ holomorphic in two different domains of the $z$-plane, the difference between them being related to complex singularities of the Borel transforms.
- The analysis of the singularities in the Borel plane will be performed with the help of the alien derivations introduced by J. Écalle in his resurgence theory, and will give access to the precise asymptotic behaviour of $\Phi_{n}^{s}-\Phi_{n}^{u}$.


## Formal solutions

Theorem For each $\varepsilon$, the inner equation has a unique odd formal solution $\tilde{\Phi}_{0}(z, \varepsilon)$ of the form $-\mathrm{i} z^{-1}+O\left(z^{-3}\right)$.
The solutions $\tilde{\phi} \in \mathbb{C}\left(\left(z^{-1}\right)\right)\left[\left[h^{2}\right]\right]$ of the full inner equation which are odd in $z$ and such that $[\tilde{\phi}]_{0}=\tilde{\Phi}_{0}$ are in one-to-one correspondence with the sequences $b \in \mathbb{C}^{\mathbb{N}^{*}}$ :

$$
\tilde{\Phi}(z, h, \varepsilon ; b)=\tilde{\Phi}_{0}(z, \varepsilon)+\sum_{n \geq 1} h^{2 n} \tilde{\Phi}_{n}\left(z, \varepsilon ; b_{1}, \ldots, b_{n}\right)
$$

where $\tilde{\Phi}_{n}\left(z, \varepsilon ; b_{1}, \ldots, b_{n}\right) \in z^{4 n-1} \mathbb{C}\left[\left[z^{-1}\right]\right]$ and the coefficients of the formal series $\tilde{\Phi}_{n}$ depend analytically on $\varepsilon$.

Moreover,
$b_{1}=0 \quad \Leftrightarrow \quad \forall n \geq 1, \quad \tilde{\Phi}_{n}\left(z, \varepsilon ; b_{1}, \ldots, b_{n}\right) \in z^{2 n-1} \mathbb{C}\left[\left[z^{-1}\right]\right]$
The general solution of (3) in $\mathbb{C}\left(\left(z^{-1}\right)\right)\left[\left[h^{2}\right]\right]$ is $\pm \tilde{\Phi}(z+a(h), h, \varepsilon ; b)$, with arbitrary $a(h) \in \mathbb{C}\left[\left[h^{2}\right]\right]$ and $b \in \mathbb{C}^{\mathbb{N}^{*}}$.

## The integrable case $\varepsilon=0$

For $\varepsilon=0$, we know explicitly the solution of (3) which is related to the separatrix:

$$
\begin{align*}
\Phi^{0}(z, h) & =-\mathrm{i} \frac{\sinh h}{\sinh (h z)} \\
& =-\mathrm{i} z^{-1}+\mathrm{i} \frac{h^{2}}{6}\left(z-z^{-1}\right)-\mathrm{i} \frac{h^{4}}{360}\left(7 z^{3}-10 z+3 z^{-1}\right)+\ldots \tag{8}
\end{align*}
$$

A certain choice $b^{*}(0)$ of $b$ leads to $\tilde{\Phi}\left(z, h, 0 ; b^{*}(0)\right)=\Phi^{0}(z, h)$.
In particular

$$
\begin{equation*}
\tilde{\Phi}_{0}(z, 0)=-\mathrm{i} z^{-1} \tag{9}
\end{equation*}
$$

and $\tilde{\Phi}_{1}\left(z, 0 ; b_{1}^{*}(0)\right)=\frac{i}{6}\left(z-z^{-1}\right)$,
$\tilde{\Phi}_{2}\left(z, 0 ; b_{1}^{*}(0), b_{2}^{*}(0)\right)=-\frac{\mathrm{i}}{360}\left(7 z^{3}-10 z+3 z^{-1}\right)$, etc.

## The integrable case $\varepsilon=0$

All the series $\tilde{\Phi}_{n}\left(z, 0 ; b_{1}, \ldots, b_{n}\right)$ are convergent; in fact, they are polynomials up to the factor $z^{-1}$ :
For any $b \in \mathbb{C}^{\mathbb{N}^{*}}$,

$$
\forall n \geq 1, \quad \tilde{\Phi}_{n}\left(z, 0 ; b_{1}, \ldots, b_{n}\right) \in z^{-1} \mathbb{C}[z] .
$$

When $\varepsilon=0$ the formal solution is given by a convergent series.
For nonzero $\varepsilon$, the formal solutions of (3) are deformations of $\Phi^{0}$. But they are divergent in $z$.
Wee shall deduce analytic solutions from them using resurgence theory.

## Borel transform

We define the Borel transform

$$
\begin{array}{rll}
\mathcal{B}: \mathbb{C}\left(\left(z^{-1}\right)\right) & \rightarrow & \mathbb{C}[[\zeta]] \\
\tilde{\varphi}(z) & \mapsto & \hat{\varphi}(\zeta)
\end{array}
$$

where:

$$
\tilde{\varphi}(z)=\sum_{p \geq-v} a_{p} z^{-p}, \quad v \in \mathbb{N}
$$

and

$$
\mathcal{B} \tilde{\varphi}(\zeta)=\hat{\varphi}(\zeta)=\sum_{p \geq 1} a_{p} \frac{\zeta^{p-1}}{(p-1)!}
$$

## Borel transform

- It is a linear operator which cancels out the polynomial part of $\tilde{\varphi}(z)$.
- $\hat{\varphi}(\zeta) \in \mathbb{C}\{\zeta\}$ means that $\tilde{\varphi}(z)$ is Gevrey-1, that is, there exist $C, K>0$ such that $\left|a_{p}\right| \leq C K^{p} p$ !.
- $\tilde{\varphi}(z)$ is convergent for $|z|$ large enough, then $\hat{\varphi}(\zeta)$ must define an entire function of exponential type.

In the case of the formal solutions of our equations, we shall see that the Borel transforms converge near the origin, but the holomorphic functions of $\zeta$ thus defined are generically not entire: their analytic continuations are singular at $\pm 2 \pi i$ (thus the formal solutions themselves are not convergent).

## Borel transform

We consider the cut plane $\mathcal{R}^{0}=\mathbb{C} \backslash \pm 2 \pi \mathrm{i}[1,+\infty)$, which will be the common holomorphic star of the $\mathcal{B} \tilde{\Phi}_{n}$ 's.

Definition 1 For any $\rho \in(0,2 \pi)$, we set

$$
\mathcal{R}_{\rho}^{0}=\{\zeta \in \mathbb{C} \mid \operatorname{dist}([0, \zeta], 2 \pi \mathrm{i}) \geq \rho, \operatorname{dist}([0, \zeta],-2 \pi \mathrm{i}) \geq \rho\} \subset \mathcal{R}^{0}
$$



## Borel transform

We define $\widehat{\operatorname{RES}}^{(0)}$ to be the set of all $\hat{\varphi} \in \mathbb{C}\{\zeta\}$ such that

1. $\hat{\varphi}(\zeta)$ extends analytically to $\mathcal{R}^{0}$,
2. for each $\rho \in(0,2 \pi)$, there exist $\tau, C>0$ such that $|\hat{\varphi}(\zeta)| \leq C \mathrm{e}^{\tau|\zeta|}$ for $\zeta \in \mathcal{R}_{\rho}^{0}$.
We also set $\widetilde{\mathrm{RES}^{(0)}}=\mathcal{B}^{-1} \widehat{\mathrm{RES}}^{(0)}$.

## Borel transform

Theorem Let $b \in \mathbb{C}^{\mathbb{N}^{*}}$ and $n \in \mathbb{N}$. Then the Borel transform $\hat{\Phi}_{n}\left(\zeta, \varepsilon ; b_{1}, \ldots, b_{n}\right)$ of the formal solution $\tilde{\Phi}_{n}\left(z, \varepsilon ; b_{1}, \ldots, b_{n}\right)$ is convergent for $|\zeta|<2 \pi$ and defines a holomorphic function of two variables in $\left\{(\zeta, \varepsilon) \in \mathbb{C}^{2}\left|\zeta \in \mathcal{R}^{0},|\varepsilon|<\varepsilon_{0}\right\}\right.$ which depends polynomially on $b_{1}, \ldots, b_{n}$. Moreover, for any $\varepsilon_{0}^{\prime} \in\left(0, \varepsilon_{0}\right)$ and $\rho \in(0,2 \pi)$, there exist positive constants $C_{n}, \tau_{n}$ which depend continuously on $b_{1}, \ldots, b_{n}$, such that

$$
\left|\hat{\Phi}_{n}\left(\zeta, \varepsilon ; b_{1}, \ldots, b_{n}\right)\right| \leq C_{n} \mathrm{e}^{\tau_{n}|\zeta|}, \quad \zeta \in \mathcal{R}_{\rho} \beta 0,|\varepsilon| \leq \varepsilon_{0}^{\prime}
$$

In particular $\tilde{\Phi}_{n}\left(z, \varepsilon ; b_{1}, \ldots, b_{n}\right) \in \widetilde{\operatorname{RES}}^{(0)}$ for each $\varepsilon$.

## Borel-Laplace summation

Take $\tilde{\varphi}(z)=\sum_{p \geq-v} a_{p} z^{-p}$ belonging to $\widetilde{\operatorname{RES}}^{(0)}$,
And the angle $\theta$ is such that the half-line of integration $\mathrm{e}^{\mathrm{i} \theta} \mathbb{R}^{+}$be contained in $\mathcal{R}_{\rho}^{0}$.
The formula

$$
\begin{equation*}
\left(\mathcal{S}^{\theta} \tilde{\varphi}\right)(z)=\sum_{p=0}^{v} a_{-p} z^{p}+\int_{0}^{\mathrm{e}^{\mathrm{i} \theta} \infty} \mathrm{e}^{-z \zeta} \hat{\varphi}(\zeta) \mathrm{d} \zeta \tag{10}
\end{equation*}
$$

defines a function $\mathcal{S}^{\theta} \tilde{\varphi}$ which is holomorphic in the half-plane $\Pi_{\theta, \tau}=\left\{z \in \mathbb{C} \mid \operatorname{Re} e\left(z \mathrm{e}^{\mathrm{i} \theta}\right)>\tau\right\}$,
where $\tau=\tau(\rho), \rho \in(0,2 \pi)$.

## Borel-Laplace summation

Such angles correspond to two intervals:

$$
\theta \in I_{\rho}^{+}=\left[-\frac{\pi}{2}+\delta, \frac{\pi}{2}-\delta\right] \text { or } \theta \in I_{\rho}^{-}=\left[\frac{\pi}{2}+\delta, \frac{3 \pi}{2}-\delta\right]
$$

with $\delta=\arcsin \frac{\rho}{2 \pi}$


## Borel-Laplace summation

Cauchy theorem: functions $\mathcal{S}^{\theta} \tilde{\varphi}$ corresponding to angles $\theta$ from the same interval mutually extend.

We get two holomorphic functions:
$\mathcal{S}^{+} \tilde{\varphi}(z)=\left(\mathcal{S}^{\theta} \tilde{\varphi}\right)(z)$ for any $\theta \in I_{\rho}^{+}$holomorphic in $\mathcal{D}_{\rho, \tau}^{+}=\bigcup_{\theta \in I_{\rho}^{+}} \Pi_{\theta, \tau}$,
and
$\mathcal{S}^{-} \tilde{\varphi}(z)=\left(\mathcal{S}^{\theta} \tilde{\varphi}\right)(z)$ for any $\theta \in I_{\rho}^{-}$holomorphic in $\mathcal{D}_{\rho, \tau}^{-}=\bigcup_{\theta \in I_{\rho}^{-}} \Pi_{\theta, \tau}$,

## Borel-Laplace summation: properties

- The domains $\mathcal{D}_{\rho, \tau}^{+}$and $\mathcal{D}_{\rho, \tau}^{-}$can be considered as sectorial neighbourhoods of infinity of opening $2 \pi-2 \delta$ centred respectively on $\mathbb{R}^{+}$and $\mathbb{R}^{-}$
- $\tilde{\varphi}$ is the asymptotic expansion of $\mathcal{S}^{ \pm} \tilde{\varphi}$ in the Gevrey-1 sense uniformly in $\mathcal{D}_{\rho, \tau}^{ \pm}$:

$$
\mathcal{S}^{ \pm} \tilde{\varphi}(z) \sim_{1} \tilde{\varphi}(z), \quad z \in \mathcal{D}_{\rho, \tau}^{ \pm}
$$

- The intersection of $\mathcal{D}_{\rho, \tau}^{+}$and $\mathcal{D}_{\rho, \tau}^{-}$has two connected components, in which $\mathcal{S}^{+} \tilde{\varphi}$ and $\mathcal{S}^{-} \tilde{\varphi}$ generically do not coincide;

In fact, $\mathcal{S}^{+} \tilde{\varphi}$ and $\mathcal{S}^{-} \tilde{\varphi}$ mutually extend if and only if the original series $\tilde{\varphi}$ has positive radius of convergence (then the union $\mathcal{D}_{\rho, \tau}^{+} \cup \mathcal{D}_{\rho, \tau}^{-}$contains a full neighbourhood of infinity, $\{|z|>R\}$, in which $\tilde{\varphi}(z)$ converges to $\mathcal{S}^{ \pm} \tilde{\varphi}(z)$ ).

## Borel-Laplace summation: properties

- By letting $\rho$ vary in $(0,2 \pi)$, we see that $\mathcal{S}^{+} \tilde{\varphi}$ and $\mathcal{S}^{-} \tilde{\varphi}$ admit an analytic continuation to $\mathcal{D}^{s}=\bigcup \mathcal{D}_{\rho, \tau(\rho)}^{+}$and $\mathcal{D}^{u}=\bigcup \mathcal{D}_{\rho, \tau(\rho)}^{-}$.
- $\widetilde{\mathrm{RES}}^{(0)}$ is a differential subalgebra of $\mathbb{C}\left(\left(z^{-1}\right)\right)$ (it is stable by multiplication and differentiation), the operators $\mathcal{S}^{ \pm}$are differential algebra morphisms (they map the product of formal series on the product of analytic functions and they commute with $\partial_{z}$ ) and they commute with the shift operator $\tilde{\varphi}(z) \mapsto \tilde{\varphi}(z+1)$.

Consequently, when $\mathcal{S}^{+}$and $\mathcal{S}^{-}$can be applied to a formal solution of a (possibly non-linear) difference equation, it yields an analytic solution of this equation (Ecalle's theory).

## Borel-Laplace summation: results

## corollary

Let $b \in \mathbb{C}^{\mathbb{N}^{*}}$. Then there exist two decreasing sequences of domains $\mathcal{D}_{n}^{s}$ and $\mathcal{D}_{n}^{u}$, each of which contains sectorial neighborhoods of infinity with opening arbitrarily close to $2 \pi$ centered respectively on $\mathbb{R}^{+}$and $\mathbb{R}^{-}$, such that for any $n \in \mathbb{N}$, the functions

$$
\Phi_{n}^{s}\left(z, \varepsilon ; b_{1}, \ldots, b_{n}\right):=\mathcal{S}^{+} \tilde{\Phi}_{n}, \quad \Phi_{n}^{u}\left(z, \varepsilon ; b_{1}, \ldots, b_{n}\right):=\mathcal{S}^{-} \tilde{\Phi}_{n}
$$

are holomorphic for $z \in \mathcal{D}_{n}^{s}$, respectively $z \in \mathcal{D}_{n}^{u}$, and $|\varepsilon|<\varepsilon_{0}$, and solve the inner equations

Moreover, for each $\rho \in(0,2 \pi)$, there exists $\tau_{n}>0$ such that

$$
\Phi_{n}^{s, u}\left(z, \varepsilon ; b_{1}, \ldots, b_{n}\right) \sim_{1} \tilde{\Phi}_{n}\left(z, \varepsilon ; b_{1}, \ldots, b_{n}\right), \quad z \in \mathcal{D}_{\rho, \tau_{n}}^{s} \text { or } \mathcal{D}_{\rho, \tau_{n}}^{u}
$$

and $\Phi_{n}^{s}$ and $\Phi_{n}^{u}$ coincide for $\varepsilon=0$.

## Borel-Laplace summation

The solutions $\Phi_{n}^{s}$ and $\Phi_{n}^{u}$ are characterized by the beginning of their asymptotic expansion: If, for $\tilde{\varphi}(z)=\sum_{p \geq-v} a_{p} z^{-p}$, we denote

$$
[\tilde{\varphi}]_{\leq 2}=\sum_{p=-v}^{2} a_{p} z^{-p}
$$

(for instance $\left[\tilde{\Phi}_{0}(z, \varepsilon)\right]_{\leq 2}=-\mathrm{i} z^{-1}$, we indeed have

## Proposition

Let $b_{1}, \ldots, b_{n_{0}} \in \mathbb{C}, \sigma \in(2,3], z_{0} \in \mathcal{D}_{n_{0}}^{u}$ and $\varepsilon \in \mathbb{C}$ such that $|\varepsilon|<\left|\varepsilon_{0}\right|$. The functions $\left(\phi_{n}\right)_{0 \leq n \leq n_{0}}$ defined by $\phi_{n}(z)=\Phi_{n}^{u}\left(z, \varepsilon ; b_{1}, \ldots, b_{n}\right)$ are the only solutions of the inner equations, $0 \leq n \leq n_{0}$, such that each $\phi_{n}$ is defined on the half-line $z_{0}+\mathbb{R}^{-}$and satisfies

$$
\phi_{n}(z)=\left[\tilde{\Phi}_{n}\left(z, \varepsilon ; b_{1}, \ldots, b_{n}\right)\right]_{\leq 2}+O\left(|z|^{-\sigma}\right)
$$

Similarly for $\Phi_{n}^{s}\left(z, \varepsilon ; b_{1}, \ldots, b_{n}\right)$, with $z_{0}+\mathbb{R}^{+}$

## The alien derivatives of the formal solution

## Definition

Let $\tilde{\varphi} \in \widetilde{\operatorname{RES}^{(0)}}$. We say that $\hat{\varphi}=\mathcal{B} \tilde{\varphi}$ has a simply ramified singularity at $\omega= \pm 2 \pi \mathrm{i}$ if there exist reg $(\zeta) \in \mathbb{C}\{\zeta\}$ and
$\tilde{\psi}(z)=\sum_{p \geq-v} b_{p} z^{-p} \in \mathbb{C}\left(\left(z^{-1}\right)\right)$ (with $v \in \mathbb{N}$ ), such that $\hat{\psi}=\mathcal{B} \tilde{\psi} \in \mathbb{C}\{\zeta\}$ and

$$
\begin{equation*}
\hat{\varphi}(\zeta)=\sum_{p=0}^{v} b_{-p} \frac{(-1)^{p} p!}{2 \pi \mathrm{i}(\zeta-\omega)^{p+1}}+\hat{\psi}(\zeta-\omega) \frac{\log (\zeta-\omega)}{2 \pi \mathrm{i}}+\operatorname{reg}(\zeta-\omega) \tag{11}
\end{equation*}
$$

for $\zeta \in \mathcal{R}^{0}$ with $|\zeta-\omega|$ small enough.
In this situation, we use the notation

$$
\begin{equation*}
\Delta_{\omega} \tilde{\varphi}=\tilde{\psi} \tag{12}
\end{equation*}
$$

## The alien derivatives of the formal solution

- The Gevrey- 1 formal series $\tilde{\psi}$ is indeed determined by $\tilde{\varphi}$ (by $\mathcal{B} \tilde{\varphi}$ in fact).
- $\hat{\varphi}$ extends holomorphically to the universal cover of a punctured disc centered at $\omega$ and $\hat{\psi}(\xi)$ is the variation (or monodromy) of $\hat{\varphi}$ at $\omega+\xi$ around $\omega$, that is, the difference between two consecutive branches

$$
\hat{\psi}(\xi)=\hat{\varphi}(\omega+\xi)-\hat{\varphi}\left(\omega+\xi \mathrm{e}^{-2 \pi \mathrm{i}}\right)
$$

- The polynomial part of $\tilde{\psi}(z)$ is determined by the polar part of the Laurent expansion at the origin of

$$
\stackrel{\vee}{P}(\xi)=\hat{\varphi}(\omega+\xi)-\hat{\psi}(\xi) \frac{\log \xi}{2 \pi \mathrm{i}}
$$

(which is meromorphic in a small disc centred at the origin);

- The regular function $\operatorname{reg}(\xi)$ depends on the branch of the logarithm which is chosen in (11).


## The alien derivatives of the formal solution

- We have two linear operators $\Delta_{2 \pi \mathrm{i}}$ and $\Delta_{-2 \pi \mathrm{i}}$ defined on the subspace of $\widetilde{R E S}(0)$ consisting of the formal series whose Borel transforms have simply ramified singularities at $\pm 2 \pi \mathrm{i}$, with values in the space of Gevrey- 1 formal series $\mathbb{C}\left(\left(z^{-1}\right)\right)_{\text {Gev }}$.
- These operators are particular instances of Ecalle's alien derivations.
- They are indeed derivations: it can be proved that

$$
\Delta_{\omega}\left(\tilde{\varphi}_{1} \tilde{\varphi}_{2}\right)=\left(\Delta_{\omega} \tilde{\varphi}_{1}\right) \tilde{\varphi}_{2}+\tilde{\varphi}_{1}\left(\Delta_{\omega} \tilde{\varphi}_{2}\right)
$$

## The alien derivatives of the formal solution

- It will turn out that the $\hat{\Phi}_{n}$ 's have simply ramified singularities at $\pm 2 \pi \mathrm{i}$.
- We will describe these singularities through the action of the alien derivations $\Delta_{ \pm 2 \pi \mathrm{i}}$ on $\tilde{\Phi}_{n}$.
- We will compute the alien derivations in term of some auxiliary formal series $\tilde{\Psi}_{1, n}, \tilde{\Psi}_{2, n}$ which we now introduce.


## The variational equation

"formal" variational equation associated with the formal solution $\tilde{\Phi}(z, h, \varepsilon ; b)=\tilde{\Phi}_{0}(z, \varepsilon)+\sum_{n \geq 1} h^{2 n} \tilde{\Phi}_{n}\left(z, \varepsilon ; b_{1}, \ldots, b_{n}\right):$

$$
\Psi(z+1)+\Psi(z-1)=\partial_{y} \mathcal{F}(\tilde{\Phi}(z, h, \varepsilon ; b), h, \varepsilon) \Psi(z)
$$

for an unknown $\Psi=\sum_{n \geq 0} h^{2 n} \Psi_{n}(z) \in \mathbb{C}\left(\left(z^{-1}\right)\right)\left[\left[h^{2}\right]\right]$.
variational equation associated with the solution
$\Phi^{u}(z, h, \varepsilon ; b)=\Phi_{0}^{u}(z, \varepsilon)+\sum_{n \geq 1} h^{2 n} \Phi_{n}^{u}\left(z, \varepsilon ; b_{1}, \ldots, b_{n}\right)$ (formal in $h$, analytic in $z$ ):

$$
\Psi(z+1)+\Psi(z-1)=\partial_{y} \mathcal{F}\left(\Phi^{u}(z, h, \varepsilon ; b), h, \varepsilon\right) \Psi(z)
$$

for an unknown $\Psi=\sum_{n \geq 0} h^{2 n} \Psi_{n}(z)$ with coefficients analytic in $z$.

## The "formal"variational equation

We call normalized fundamental system of solutions a pair of solutions $\left(\Psi_{1}, \Psi_{2}\right)$ such that

$$
\Psi_{1}(z) \Psi_{2}(z+1)-\Psi_{1}(z+1) \Psi_{2}(z) \equiv 1
$$

There exists a normalized fundamental system of solutions $\left(\tilde{\Psi}_{1}, \tilde{\Psi}_{2}\right)$ for the "formal" variational equation, of the form
$\tilde{\Psi}_{j}(z, h, \varepsilon ; b)=\tilde{\Psi}_{j, 0}(z, \varepsilon)+\sum_{n \geq 1} h^{2 n} \tilde{\Psi}_{j, n}\left(z, \varepsilon ; b_{1}, \ldots, b_{n}\right), \quad j=1,2$,
with all $\tilde{\Psi}_{j, n} \in \widetilde{\operatorname{RES}^{(0)}}$.

## The "formal"variational equation

We obtain $\tilde{\Psi}_{1}, \tilde{\Psi}_{2}$ by using "formally"the theory of linear difference equations.

- $\tilde{\Psi}_{1}=\partial_{z} \tilde{\Phi}$ even in $z$, and $\tilde{\Psi}_{2}$ odd in $z$.
- $\tilde{\Psi}_{1,0}(z, \varepsilon)=\mathrm{i} z^{-2}+O\left(z^{-4}\right), \tilde{\Psi}_{2,0}(z, \varepsilon)=-\frac{\mathrm{i}}{5} z^{3}+O(z)$
- $\tilde{\Psi}_{1, n} \in z^{4 n-2} \mathbb{C}\left[\left[z^{-1}\right]\right]$ and $\tilde{\Psi}_{2, n} \in z^{4 n+3} \mathbb{C}\left[\left[z^{-1}\right]\right]$ in general
- If we choose $b_{1}=0$ then

$$
\tilde{\Psi}_{1, n} \in z^{2 n-2} \mathbb{C}\left[\left[z^{-1}\right]\right], \quad \tilde{\Psi}_{2, n} \in z^{2 n+3} \mathbb{C}\left[\left[z^{-1}\right]\right]
$$

## The variational equation

Applying the Borel resumation process we obtain

- The formulas

$$
\Psi_{j}^{u}=\sum_{n \geq 0} h^{2 n} \Psi_{j, n}^{u}, \quad \Psi_{j, n}^{u}=\mathcal{S}^{-} \tilde{\Psi}_{j, n},
$$

define a normalized fundamental system of solutions $\left(\Psi_{1}^{u}, \Psi_{2}^{u}\right)$ for the variational equation
We thus have at our disposal formal series $\tilde{\Psi}_{1, n}, \tilde{\Psi}_{2, n}$, and analytic functions which admit them as Gevrey- 1 asymptotic expansions.

The coefficients of these formal series can be determined inductively, as was the case for the formal series $\tilde{\Phi}_{n}$.

## Computation of the alien derivatives of the formal solution

## Theorem

Let $b \in \mathbb{C}^{\mathbb{N}^{*}}$. Then the Borel transforms $\hat{\Phi}_{n}(\zeta, \varepsilon ; b)$ have simply ramified singularities at $\pm 2 \pi \mathrm{i}$ whose alien derivatives can be computed as:

$$
\Delta_{ \pm 2 \pi \mathrm{i}} \tilde{\Phi}_{n}=\sum_{n_{1}+n_{2}=n}\left(A_{n_{1}}^{ \pm} \tilde{\Psi}_{1, n_{2}}+\mathrm{i} B_{n_{1}}^{ \pm} \tilde{\Psi}_{2, n_{2}}\right), \quad n \in \mathbb{N}
$$

$A^{ \pm}(h, \varepsilon ; b)=\sum_{n \geq 0} A_{n}^{ \pm}\left(\varepsilon ; b_{1}, \ldots, b_{n}\right) h^{2 n}$
$B^{ \pm}(h, \varepsilon ; b)=\sum_{n \geq 0} B_{n}^{ \pm}\left(\varepsilon ; b_{1}, \ldots, b_{n}\right) h^{2 n}$,
and $A^{ \pm}(h, \varepsilon ; b)$ and $B^{ \pm}(h, \varepsilon ; b)$ are formal series in $h^{2}$, the coefficients of which are complex polynomials in $b_{1}, b_{2} \ldots$ that depend analytically on $\varepsilon$ for $|\varepsilon|<\varepsilon_{0}$ and vanish at $\varepsilon=0$.

Computation of the alien derivatives of the formal solution
The analytic functions $A_{0}^{ \pm}(\varepsilon)$ and $B_{0}^{ \pm}(\varepsilon)$ do not depend on $b$. One has

$$
\begin{align*}
A_{0}^{ \pm}(\varepsilon)=\varepsilon A_{0,1}^{ \pm}+O\left(\varepsilon^{2}\right), & A_{0,1}^{ \pm}=2 \pi D \hat{V}_{0}( \pm 2 \pi)  \tag{13}\\
B_{0}^{ \pm}(\varepsilon)=\varepsilon B_{0,1}^{ \pm}+O\left(\varepsilon^{2}\right), & B_{0,1}^{ \pm}= \pm 4 \pi^{2} \hat{V}_{0}( \pm 2 \pi) \tag{14}
\end{align*}
$$

where $\hat{V}_{0}$ is the entire function obtained as Borel transform with respect to $1 / y$ of a primitive of $V^{\prime}(y, 0,0)$ :
$V^{\prime}(y, 0,0)=\sum_{p \geq 5} v_{p} y^{p}, \quad V_{0}(y)=\sum_{p \geq 5} v_{p} \frac{y^{p+1}}{p+1}, \quad \hat{V}_{0}(\xi)=\sum_{p \geq 5} v_{p} \frac{\xi^{p}}{(p+1)!}$,
and $D=\frac{1}{5} \xi \partial_{\xi}^{5}+\partial_{\xi}^{4}+\frac{1}{3} \xi \partial_{\xi}^{3}+\partial_{\xi}^{2}+\frac{2}{15} \xi \partial_{\xi}+\frac{2}{15}$ Id.

Computation of the alien derivatives of the formal solution
We can extend the action of the linear operators $\Delta_{\omega}$ to formal series in $h^{2}$ by the formula

$$
\Delta_{\omega}\left(\sum h^{2 n} \tilde{\varphi}_{n}\right)=\sum h^{2 n} \Delta_{\omega} \tilde{\varphi}_{n}
$$

then we obtain:

$$
\Delta_{ \pm 2 \pi \mathrm{i}} \tilde{\Phi}=A^{ \pm} \tilde{\Psi}_{1}+\mathrm{i} B^{ \pm} \tilde{\Psi}_{2}
$$

This equation is an example of what is called the bridge equation in Écalle's terminology.

## Consequences for the splitting of separatrices

Let $n \in \mathbb{N}$. $\hat{\Phi}_{n}$ has a simply ramified singularity at $\omega=2 \pi \mathrm{i}$, the variation of which is $\hat{\psi}=\sum_{n_{1}+n_{2}=n}\left(A_{n_{1}}^{+} \hat{\Psi}_{1, n_{2}}+\mathrm{i} B_{n_{1}}^{+} \hat{\Psi}_{2, n_{2}}\right) \in \widehat{\operatorname{RES}}^{(0)}$.
This implies that $\hat{\Phi}_{n}$ admits a multivalued analytic continuation through the cut between $2 \pi \mathrm{i}$ and $4 \pi \mathrm{i}$ : if $\zeta=\omega+\xi \in \mathcal{R}^{0}$ with $\xi \in \mathcal{R}^{0}$, we can consider $\hat{\Phi}_{n}\left(\omega+\xi \mathrm{e}^{2 \pi \mathrm{i}}\right)=\hat{\Phi}_{n}(\omega+\xi)+\hat{\psi}(\xi)$ as defining the branch of the analytic continuation of $\hat{\Phi}_{n}$ which is obtained from the principal one (the branch holomorphic in $\mathcal{R}^{0}$ ) by turning anticlockwise around $2 \pi \mathrm{i}$.

## Consequences for the splitting of separatrices

Let $\lambda \in(0,1), \beta \in(0, \pi / 2)$. Consider the path $\Gamma_{\lambda, \beta}$ consisting of two half-lines with vertex at $2 \pi(1+\lambda) \mathrm{i}$ and angle $\beta$ with respect to the horizontal, oriented from left to right, as on


## Consequences for the splitting of separatrices

Let $\varepsilon_{0}^{\prime} \in\left(0, \varepsilon_{0}\right)$. There exist constants $C_{n}^{*}, \tau_{n}^{*}>0$ which depend only on $\lambda, \beta, \varepsilon_{0}^{\prime}, b_{1}, \ldots, b_{n}$ such that

$$
\left|\hat{\Phi}_{n}\left(\zeta, \varepsilon ; b_{1}, \ldots, b_{n}\right)\right| \leq C_{n}^{*} \mathrm{e}^{\tau_{n}^{*}|\zeta-2 \pi(1+\lambda) \mathrm{i}|}, \quad \zeta \in \Gamma_{\lambda, \beta},|\varepsilon| \leq \varepsilon_{0}^{\prime}
$$

where the branch of $\hat{\Phi}_{n}$ considered is determined by the convention that the right part of $\Gamma_{\lambda, \beta}$ lies in $\mathcal{R}^{0}$, while on its left part one should use the branch of $\hat{\Phi}_{n}$ obtained by crossing the cut from right to left.

## Consequences for the splitting of separatrices

We now estimate the differences $\Phi_{n}^{s}-\Phi_{n}^{u}$ for $z$ belonging to the intersection of half-planes

$$
\mathcal{D}_{n}=\left\{z \in \mathbb{C} \mid \operatorname{Re}\left(z e^{\mathrm{i} \beta}\right) \geq 2 \tau_{n}^{*} \text { and } \operatorname{Re}\left(z e^{-\mathrm{i} \beta}\right) \geq 2 \tau_{n}^{*}\right\}
$$

Taking $\tau_{n}^{*}$ large enough, we can assume that $\mathcal{D}_{n}$ is contained in the lower component of the intersection $\mathcal{D}_{n}^{s} \cap \mathcal{D}_{n}^{u}$.


## Consequences for the splitting of separatrices

## Theorem

Let $n \geq 0$. For any $\varepsilon \in \mathbb{C}$ such that $|\varepsilon| \leq \varepsilon_{0}^{\prime}$ and any $z \in \mathcal{D}_{n}$,

$$
\begin{align*}
& \Phi_{n}^{s}-\Phi_{n}^{u}=\sum_{n_{1}+n_{2}=n}\left(A_{n_{1}}^{+} \Psi_{1, n_{2}}^{u}+\mathrm{i} B_{n_{1}}^{+} \Psi_{2, n_{2}}^{u}\right) \mathrm{e}^{-2 \pi \mathrm{i} z}+R, \\
& \text { with }|R| \leq K_{n}|\varepsilon| \mathrm{e}^{-2 \pi(1+\lambda)|\operatorname{Im} z|}, \tag{16}
\end{align*}
$$

where $K_{n}=\frac{2 C_{n}^{*}}{\varepsilon_{0}^{\prime} \tau_{n}^{*}}$.

## Consequences for the splitting of separatrices

Idea of the proof
For such $\varepsilon$ and $z$, we can write

$$
\left(\Phi_{n}^{s}-\Phi_{n}^{u}\right)\left(z, \varepsilon ; b_{1}, \ldots, b_{n}\right)=\int_{\mathrm{e}^{\mathrm{i}(\pi-\beta) \infty}}^{\mathrm{e}^{\mathrm{i} \beta} \infty} \mathrm{e}^{-z \zeta} \hat{\Phi}_{n}\left(\zeta, \varepsilon ; b_{1}, \ldots, b_{n}\right) \mathrm{d} \zeta
$$

By the Cauchy theorem, we can deform the contour: $\Phi_{n}^{s}-\Phi_{n}^{u}=D+R$ with

$$
D=\int_{\gamma_{\beta}} \mathrm{e}^{-z \zeta} \hat{\Phi}_{n} \mathrm{~d} \zeta, \quad R=\int_{\Gamma_{\lambda, \beta}} \mathrm{e}^{-z \zeta} \hat{\Phi}_{n} \mathrm{~d} \zeta
$$

where the path $\Gamma_{\lambda, \beta}$ was already defined, while $\gamma_{\beta}$ comes from $\mathrm{e}^{\mathrm{i}(\pi-\beta)} \infty$ in $\mathcal{R}^{0}$, encircles the point $2 \pi \mathrm{i}$ anticlockwise and goes back to $\mathrm{e}^{\mathrm{i}(\pi-\beta)} \infty$ (thus on another sheet of the Riemann surface of $\hat{\Phi}_{n}$.

Consequences for the splitting of separatrices


## Consequences for the splitting of separatrices

We can express $\hat{\Phi}_{n}$ along $\gamma_{\beta}$ by a formula of the form:

$$
\hat{\Phi}_{n}(\zeta)=\sum_{p=0}^{v} b_{-p} \frac{(-1)^{p} p!}{2 \pi \mathrm{i}(\zeta-\omega)^{p+1}}+\hat{\psi}(\zeta-\omega) \frac{\log (\zeta-\omega)}{2 \pi \mathrm{i}}+\operatorname{reg}(\zeta-\omega)
$$

with $\omega=2 \pi \mathrm{i}$; the change of variable $\zeta=2 \pi \mathrm{i}+\xi$ then yields

$$
\begin{aligned}
\int_{\gamma_{\beta}} \mathrm{e}^{-z \zeta} \frac{(-1)^{p} p!}{2 \pi \mathrm{i}(\zeta-\omega)^{p+1}} \mathrm{~d} \zeta & =\mathrm{e}^{-2 \pi \mathrm{i} z} z^{p} \\
\int_{\gamma_{\beta}} \mathrm{e}^{-z \zeta} \hat{\psi}(\zeta-\omega) \frac{\log (\zeta-\omega)}{2 \pi \mathrm{i}} \mathrm{~d} \zeta & =\mathrm{e}^{-2 \pi \mathrm{i} z} \int_{0}^{\mathrm{e}^{\mathrm{i}(\pi-\beta)} \infty} \mathrm{e}^{-z \xi} \hat{\psi}(\xi) \mathrm{d} \xi
\end{aligned}
$$

thus the contribution of the singularity at $2 \pi \mathrm{i}$ is given by the operator $\mathcal{S}^{-}$ applied to the alien derivative $\Delta_{2 \pi \mathrm{i}} \tilde{\Phi}_{n}$ :

## Consequences for the splitting of separatrices

$$
D=\mathrm{e}^{-2 \pi \mathrm{i} z} \mathcal{S}^{-} \Delta_{2 \pi \mathrm{i}} \tilde{\mathrm{I}}_{n}=\sum_{n_{1}+n_{2}=n}\left(A_{n_{1}}^{+} \Psi_{1, n_{2}}^{u}+\mathrm{i} B_{n_{1}}^{+} \Psi_{2, n_{2}}^{u}\right) \mathrm{e}^{-2 \pi \mathrm{i} z} .
$$

As for the remainder $R$, we use the change of variable $\zeta=2 \pi(1+\lambda) \mathrm{i}+\xi$ and get

$$
|R(z, \varepsilon)| \leq \leq \frac{2 C_{n}^{*}}{\tau_{n}^{*}} \mathrm{e}^{-2 \pi(1+\lambda)|\operatorname{Im} z|}
$$

- $\left|\mathrm{e}^{-2 \pi \mathrm{i} z}\right|=\mathrm{e}^{-2 \pi|\operatorname{Im} m z|}$ is exponentially small
- We know the asymptotics of the functions $\Psi_{j, n}^{u}$ 's
- $\mathrm{e}^{-2 \pi(1+\lambda)|\operatorname{Im} m z|}$ is exponentially smaller.

The singularity analysis in the Borel plane gave us access to the precise measure of the exponentially small splitting phenomenon.

