Arithmetic Dynamics

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What Is Dynamics?

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$$\boxed{\phi : S \longrightarrow S}$$

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We write

$$\phi^n = \underbrace{\phi \circ \phi \circ \phi \cdots \phi}_{n \text{ iterations}}$$

for the n^{th} iterate of ϕ and

$$\mathcal{O}_\phi(\alpha) = \{\alpha, \phi(\alpha), \phi^2(\alpha), \phi^3(\alpha), \ldots\}$$

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for the **(forward) orbit of** $\alpha \in S$.

A primary goal in the study of dynamics is to classify the points of S according to the behavior of their orbits.

Polynomials and Rational Maps

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A rational function is the same as a rational map (morphism)

$$\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

from the projective line to itself.

Some Dynamical Terminology

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An Example: The Map $\phi(z) = z^2$

- 2 and $\frac{1}{2}$ are *wandering points.*
- 0 and 1 are *fixed points.*
- -1 is a *preperiodic* point that is not periodic.
- $\frac{-1+\sqrt{-3}}{2}$ is a periodic point of period 2.

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Diophantine Equations		**Dynamical Systems**
rational and integral points on varieties	$\longleftrightarrow$	rational and integral points in orbits
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In the rest of this talk I'll survey some current areas of research in arithmetic dynamics.

Periodic Points from an Arithmetic Perspective

Periodic Points and Number Theory

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The answer is obviously

Yes.

We've seen several examples. This leads to the. . .

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The answer is given by a fundamental theorem:

Theorem. (Northcott 1949) A rational function $\phi(z) \in \mathbb{Q}(z)$ has only finitely many periodic points that are rational numbers. More generally, a morphism $\phi : \mathbb{P}^N \to \mathbb{P}^N$ defined over a number field K has only finitely many K-rational preperiodic points.

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Notice that for any constant B, there are only finitely many rational numbers $\alpha \in \mathbb{Q}$ with height $H(\alpha) \leq B$.

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Lemma. If $\phi(z)$ has degree d, then there is a constant $C = C_\phi > 0$ so that for all rational numbers $\beta \in \mathbb{Q}$,

$$H\bigl(\phi(\beta)\bigr) \ge C \cdot H(\beta)^d.$$

This is intuitively reasonable if you write out $\phi(z)$ as a ratio of polynomials. The tricky part is making sure there's not too much cancellation.

Proof (Sketch) of Northcott's Theorem

Suppose that α is periodic, say $\phi^n(\alpha) = \alpha$. We apply the lemma repeatedly:

$$\begin{aligned}
H\bigl(\phi(\alpha)\bigr) & & &\geq C \cdot H(\alpha)^d \\
H\bigl(\phi^2(\alpha)\bigr) &\geq C \cdot H\bigl(\phi(\alpha)\bigr)^d &&\geq C^{1+d} \cdot H(\alpha)^{d^2} \\
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But $\phi^n(\alpha) = \alpha$, so we get

$$H(\alpha) = H\bigl(\phi^n(\alpha)\bigr) \ge C^{(d^n-1)/(d-1)} H(\alpha)^{d^n}.$$

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Then a little bit of algebra yields

$$H(\alpha) \leq C^{-1/(d-1)}.$$

This proves that the rational periodic points have bounded height, hence there are only finitely many of them. QED

Rational Periodic Points

We now know that $\phi(z)$ has only finitely many rational periodic points. This raises the question:

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If we don't restrict the degree of ϕ, then we can get as many as we want. Simply take ϕ to have large degree and set

$$\phi(0)=1, \quad \phi(1)=2, \quad \phi(2)=3, \quad \dots, \quad \phi(n-1)=0.$$

This leads to a system of n linear equations for the coefficients of ϕ in the coefficients of ϕ, so if $\deg(\phi)>n$, we can solve for the coefficients of ϕ.

A Uniformity Conjecture

Hence in order to pose an interesting question, we should restrict attention to rational functions of a fixed degree.

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Uniform Boundedness Conjecture for Rational Periodic Points. (Morton–Silverman) Fix an integer $d \geq 2$. Then there is a constant $C(d)$ so that every rational function $\phi(z) \in \mathbb{Q}(z)$ of degree d has at most $C(d)$ rational periodic points.

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More generally,

Uniform Boundedness Conjecture (D, N, d). There is a constant $C(D, N, d)$ so that for every number field $K/\mathbb{Q}$ of degree D and for every morphism $\phi : \mathbb{P}^N \to \mathbb{P}^N$ of degree d defined over K, we have

$$\#\bigl(\operatorname{PrePer}(\phi) \cap \mathbb{P}^N(K)\bigr) \le C(D, N, d).$$

Applications of the Uniformity Conjecture

- The special case $(D, N, d) = (1, 1, 4)$ implies uniform boundedness of torsion points on elliptic curves over $\mathbb{Q}$ (Mazur's theorem), and $(D, 1, 4)$ implies the same for number fields of degree D (Merel's theorem).

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- Fakhruddin has shown that the general conjecture implies uniform boundedness of torsion points on abelian varieties of dimension N.

Rational Periodic Points of $\phi_c(z) = z^2 + c$

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We can write down some examples:

$\phi(z) = z^2$	has 1 as a point of period 1,
$\phi(z) = z^2 - 1$	has -1 as a point of period 2,
$\phi(z) = z^2 - \frac{29}{16}$	has $-\frac{1}{4}$ as a point of period 3,

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Can $\phi(z) = z^2 + c$ have a rational point of period 4?

Rational Periodic Points of $\phi_c(z) = z^2 + c$

Theorem.
(a) There are many values of c such that $\phi_c(z)$ has a rational periodic point of period 1, 2, or 3.

(b) (Morton) The polynomial $\phi_c(z)$ cannot have a rational periodic point of period 4.

(c) (Flynn, Poonen, Schaefer) The polynomial $\phi_c(z)$ cannot have a rational periodic point of period 5.

(d) (Stoll) The polynomial $\phi_c(z)$ cannot have a rational periodic point of period 6 (conditional on the Birch–Swinnerton-Dyer conjecture).

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(d) (Stoll) The polynomial $\phi_c(z)$ cannot have a rational periodic point of period 6 (conditional on the Birch–Swinnerton-Dyer conjecture).

And that is the current state of our knowledge! No one knows if $\phi_c(z)$ can have rational periodic points of period 7 or greater. (Poonen has conjectured that it cannot.)

Integer Points in Orbits

Integers and Wandering Points

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Question. Is it possible for an orbit $\mathcal{O}_\phi(\alpha)$ to contain infinitely many integers?

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The obvious answer is **Yes**, of course it can. For example, take $\phi(z) = z^2 + 1$ and $\alpha = 1$.

More generally, if $\phi(z)$ is any polynomial with integer coefficients and if we start with an integer point, then the entire orbit consists of integers.

Are there any other possibilities?

Rational Functions with Polynomial Iterate

Here is an example of a nonpolynomial with an orbit containing infinitely many integer points. Let

$$\phi(z) = \frac{1}{z^d} \quad \text{and let} \quad \alpha \in \mathbb{Z}.$$

Then

$$\mathcal{O}_\phi(\alpha) = \left\{ \alpha, \frac{1}{\alpha^d}, \alpha^{d^2}, \frac{1}{\alpha^{d^3}}, \alpha^{d^4}, \frac{1}{\alpha^{d^5}}, \alpha^{d^6}, \ldots \right\}.$$

Thus half the points in the orbit are integers.

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This is not a surprising phenomenon, since $\phi^2(z) = z^{d^2}$ is a polynomial. And in principle, the same thing would happen if some higher iterate of ϕ were a polynomial. Somewhat surprisingly, this does not occur.

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Theorem. If some iterate $\phi^n(z)$ is a polynomial, then already $\phi^2(z)$ is a polynomial.

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Theorem. (Silverman) Let $\alpha \in \mathbb{Q}$ be a wandering point for ϕ, and assume that $\phi^2(z)$ is not a polynomial. Then the orbit $\mathcal{O}_\phi(\alpha)$ contain only finitely many integers.

Integer-Like Points in Wandering Orbits

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Theorem. (Silverman) Assume that neither $\phi^2(z)$ nor $1/\phi^2(z^{-1})$ are polynomials and that $\alpha \in \mathbb{Q}$ has infinite orbit. Then

$$\lim_{n\to\infty} \frac{\log|A_n|}{\log|B_n|} = 1.$$

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- However, the proof in the dynamical setting is more complicated because the map ϕ is always ramified, while Siegel was able to use unramified covering maps of curves.
- Ultimately the proof reduces to a Diophantine approximation problem.
- For particular functions and orbits it may be possible to give an elementary proof of finiteness, but I don't know a general proof that does not ultimately rely on Roth's theorem or one of its variants.

Additional Topics
in
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Canonical Heights

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For convenience, we introduce the **logarithmic height**

$$h\left(\frac{a}{b}\right) = \log H\left(\frac{a}{b}\right) = \log\max\bigl\{|a|,|b|\bigr\},$$

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The **canonical height associated to ϕ** is

$$\hat{h}_\phi(\alpha) = \lim_{n\to\infty}\frac{1}{d^n}h\bigl(\phi^n(\alpha)\bigr).$$

One proves that the limit exists by using a telescoping sum argument to show that the sequence is Cauchy. (This idea is due to Tate.)

Properties of Canonical Heights

The canonical height has many nice properties:

Theorem.

(a) $\hat{h}_\phi(\alpha) = h(\alpha) + O(1)$.

(b) $\hat{h}_\phi\bigl(\phi(\alpha)\bigr) = d\hat{h}_\phi(\alpha)$.

(c) $\hat{h}_\phi(\alpha) \ge 0$.

(d) $\hat{h}_\phi(\alpha) = 0$ if and only if α is preperiodic.

Properties of Canonical Heights

The canonical height has many nice properties:

Theorem.
(a) $\hat{h}_\phi(\alpha) = h(\alpha) + O(1)$.
(b) $\hat{h}_\phi\bigl(\phi(\alpha)\bigr) = d\hat{h}_\phi(\alpha)$.
(c) $\hat{h}_\phi(\alpha) \ge 0$.
(d) $\hat{h}_\phi(\alpha) = 0$ if and only if α is preperiodic.

There are many conjectures about classical (canonical) heights that have analogs for dynamical heights, e.g.,

Dynamical Lehmer Conjecture. There is a constant $C = C(\phi) > 0$ so that for all algebraic numbers $\alpha \in \bar{\mathbb{Q}}$ that are not preperiodic for ϕ,

$$\hat{h}_\phi(\alpha) \ge \frac{C}{[\mathbb{Q}(\alpha) : \mathbb{Q}]}.$$

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Using the analogy

$$\boxed{\text{Rational Points on Varieties}} \longleftrightarrow \boxed{\text{Points in Orbits}}$$

Dynamical Mordell Conjecture.
Let $\phi : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism, let $\alpha \in \mathbb{P}^N$ be a wandering point for ϕ, and let $V \subset \mathbb{P}^N$ be a closed subvariety. Suppose

$$\mathcal{O}_\phi(\alpha) \cap V \text{ is infinite.}$$

Then there is a subvariety $W \subset V$ of dimension ≥ 1 that is preperiodic for ϕ, i.e, $\phi^{i+j}(W) \subset \phi^i(W)$.

A Dynamical Manin–Mumford Conjecture

The Manin–Mumford conjecture (Raynaud's theorem) says that if A is an abelian variety and if $V \subset A$ is a subvariety that contains infinitely many torsion points of A, then V contains a translate of an abelian subvariety of A.

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Example. Let $E/\mathbb{Q}$ be an elliptic curve and let p be a prime for which E has *good reduction.* (This means that $\tilde{E}$ mod p is nonsingular.) Then

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A map $\phi : \mathbb{P}^1_{\mathbb{Q}} \to \mathbb{P}^1_{\mathbb{Q}}$ of degree d has **good reduction** if the degree of the reduced map $\tilde{\phi} : \mathbb{P}^1_{\mathbb{F}_p} \to \mathbb{P}^1_{\mathbb{F}_p}$ is also d.

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I.e., $\phi = F(z)/G(z)$ has good reduction if $\tilde{F}(z)$ mod p and $\tilde{G}(z)$ mod p have no common roots in $\bar{\mathbb{F}}_p$.

Equivalently, ϕ has good reduction if it extends to a scheme-theoretic morphism $\phi : \mathbb{P}^1_{\mathbb{Z}} \to \mathbb{P}^1_{\mathbb{Z}}$.

Reduction of Periodic Points Modulo p

Let $\phi : \mathbb{P}^1_{\mathbb{Q}} \to \mathbb{P}^1_{\mathbb{Q}}$ be a rational map of degree d that has good reduction at p. Let $\alpha \in \mathbb{P}^1(\mathbb{Q})$ be a periodic point of period n. The **multiplier of ϕ at α** is

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Theorem. Let

$$n = \text{period of } \alpha \text{ in } \mathbb{P}^1(\mathbb{Q}).$$
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$$r = \text{order of } \lambda_\alpha(\tilde{\phi} \bmod p) \text{ in } \mathbb{F}_p^*.$$

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$$n = m \quad \text{or} \quad mr \quad \text{or} \quad mrp^e \quad \text{for some } e \geq 1.$$

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There is also a bound for e due to Zieve.

Application to Rational Periodic Points

Since

$$\#\mathbb{P}^1(\mathbb{F}_p) = p+1 \qquad \text{and} \qquad \#\mathbb{F}_p^* = p-1,$$

we clearly have

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Corollary. Let p and q be the two smallest primes of good reduction for ϕ. Let $\alpha \in \mathbb{P}^1(\mathbb{Q})$ be a periodic point. Then

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With more work, one can significantly improve this estimate, but the bound will depend on ϕ.

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One can create **cyclotomic units** by taking expressions of the form

$$\frac{\zeta^i - \zeta^j}{\zeta^k - \zeta^\ell}.$$

The set of all cyclotomic units has finite index in the full unit group.

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However, the dynatomic units generate only a portion of the full group of units.

Arithmetic Dynamics

Joseph H. Silverman

Brown University

Workshop on p-adic Dynamics
Fields Institute, Toronto

Monday, October 27, 2008

Moduli Spaces in Arithmetic Dynamics

Joseph H. Silverman

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The "additional structure" might be as a manifold or algebraic variety or scheme.

Moduli Spaces for Dynamical Systems

The Space of Rational Functions

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To see this, we identify a rational map

$$\phi(z) = \frac{F_{\mathbf{a}}(z)}{F_{\mathbf{b}}(z)} = \frac{a_d z^d + \cdots + a_1 z + a_0}{b_d z^d + \cdots + b_1 z + b_0} \in \mathrm{Rat}_d$$

with a point in projective space,

$$[F_{\mathbf{a}}, F_{\mathbf{b}}] = [a_d, a_{d-1}, \ldots, a_0, b_d, b_{d-1}, \ldots, b_0] \in \mathbb{P}^{2d+1}.$$

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The requirement that $\deg \phi = d$ is equivalent to

$$\mathrm{Resultant}(F_{\mathbf{a}}, F_{\mathbf{b}}) \neq 0,$$

so $\mathrm{Rat}_d \subset \mathbb{P}^{2d+1}$ is the complement of a hypersurface.

The Conjugation Action of $\mathrm{Aut}(\mathbb{P}^1)$

The dynamics of a rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ does not depend on a particular choice of coordinates. Let

$$f \in \mathrm{PGL}_2 = \mathrm{Aut}(\mathbb{P}^1)$$

be an automorphism of $\mathbb{P}^1$. The associated change of coordinates for ϕ is given by conjugation

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Thus we have a commutative diagram

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{\phi^f} & \mathbb{P}^1 \\ \Big\downarrow f & & \Big\downarrow f \\ \mathbb{P}^1 & \xrightarrow{\phi} & \mathbb{P}^1 \end{array}$$

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Notice that iteration of ϕ^f is given by

$$(\phi^f)^n = (f^{-1}\phi f)\cdots(f^{-1}\phi f) = f^{-1}\phi^n f = (\phi^n)^f.$$

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More generally, we can look at

$$\mathcal{M}_d^N = \mathrm{Rat}_d^N \,/\, \mathrm{PGL}_{N+1},$$

where

$$\mathrm{Rat}_d^N = \{\text{degree } d \text{ rational maps } \mathbb{P}^N \to \mathbb{P}^N\} \subset \mathbb{P}^L$$

and $\mathrm{PGL}_{N+1} = \mathrm{Aut}(\mathbb{P}^N)$ acts via conjugation.

Dynamical Moduli Spaces: Existence

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Theorem. (a) (Milnor) *The quotient*

$$\mathcal{M}_d(\mathbb{C}) = \mathrm{Rat}_d(\mathbb{C}) / \mathrm{PGL}_2(\mathbb{C})$$

has a natural structure as a complex orbifold.

(b) (Silverman) *The quotient*

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The construction of $\mathcal{M}_d$ as a variety, or as a scheme over $\mathbb{Z}$, uses Mumford's geometric invariant theory.

The Construction of the Moduli Space $\mathcal{M}_d$

Mumford's GIT provides larger sets of "stable" and "semistable" points and associated quotient spaces:

$$\mathrm{Rat}_d \subset \mathrm{Rat}_d^s \subset \mathrm{Rat}_d^{ss} \subset \mathbb{P}^{2d+1},$$
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These quotients have various nice properties, such as:

$$\mathcal{M}_d^s(\mathbb{C}) = \frac{\mathrm{Rat}_d^s(\mathbb{C})}{\mathrm{PGL}_2(\mathbb{C})} \quad \text{and} \quad \mathcal{M}_d^{ss}(\mathbb{C}) \text{ is compact.}$$

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Proposition. *If d is even, then*

$$\mathcal{M}_d^s = \mathcal{M}_d^{ss},$$

so in this case there is a geometric quotient space that is a natural compactification $\mathcal{M}_d$.

For even d, we write $\overline{\mathcal{M}}_d$ for $\mathcal{M}_d^s = \mathcal{M}_d^{ss}$.

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This structure carries over in the scheme-theoretic setting, even in characteristic 2.

Theorem. (Silverman)

$$\mathcal{M}_{2/\mathbb{Z}} \cong \mathbb{A}^2_{\mathbb{Z}} \quad \text{and} \quad \overline{\mathcal{M}}_{2/\mathbb{Z}} \cong \mathbb{P}^2_{\mathbb{Z}}.$$

An Explicit Isomorphism $\mathcal{M}_2 \cong \mathbb{A}^2$

The isomorphism

$$(\sigma_1, \sigma_2) : \mathcal{M}_2 \longrightarrow \mathbb{A}^2$$

may be given quite explicitly, although it is somewhat complicated. The image of a rational map

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$$\sigma_1 = \frac{\begin{array}{r} a_1^3 b_0 - 4a_0a_1a_2b_0 - 6a_2^2b_0^2 - a_0a_1^2b_1 + 4a_0^2a_2b_1 + 4a_1a_2b_0b_1 - 2a_0a_2b_1^2 \\ + a_2b_1^3 - 2a_1^2b_0b_2 + 4a_0a_2b_0b_2 - 4a_2b_0b_1b_2 - a_1b_1^2b_2 + 2a_0^2b_2^2 + 4a_1b_0b_2^2, \end{array}}{a_2^2b_0^2 - a_1a_2b_0b_1 + a_0a_2b_1^2 + a_1^2b_0b_2 - 2a_0a_2b_0b_2 - a_0a_1b_1b_2 + a_0^2b_2^2}$$

$$\sigma_2 = \frac{\begin{array}{r} -a_0^2a_1^2 + 4a_0^3a_2 - 2a_1^3b_0 + 10a_0a_1a_2b_0 + 12a_2^2b_0^2 - 4a_0^2a_2b_1 - 7a_1a_2b_0b_1 \\ - a_1^2b_1^2 + 5a_0a_2b_1^2 - 2a_2b_1^3 + 2a_0^2a_1b_2 + 5a_1^2b_0b_2 - 4a_0a_2b_0b_2 \\ - a_0a_1b_1b_2 + 10a_2b_0b_1b_2 - 4a_1b_0b_2^2 + 2a_0b_1b_2^2 - b_1^2b_2^2 + 4b_0b_2^3. \end{array}}{a_2^2b_0^2 - a_1a_2b_0b_1 + a_0a_2b_1^2 + a_1^2b_0b_2 - 2a_0a_2b_0b_2 - a_0a_1b_1b_2 + a_0^2b_2^2}$$

Multipliers of a Rational Map

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Let $\alpha \in \mathbb{P}^1$ be a periodic point of period n for ϕ. An easy chain rule calculation shows that the **multiplier**

$$\lambda_\phi(\alpha) = (\phi^n)'(\alpha)$$

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(If $\alpha = \infty$, the fomula for $\lambda_\phi(\alpha)$ is slightly different.)

We form combinations of $\lambda_\phi(\alpha)$ as α varies over the points of period n, a set we denote by

$$\mathrm{Per}_n(\phi) = \{\alpha \in \mathbb{P}^1 : \phi^n(\alpha) = \alpha\}.$$

Multiplier Systems

Let

$$\mathrm{Per}_n(\phi) = \{\alpha_1, \alpha_2, \ldots, \alpha_\rho\}, \qquad (\rho = d^n + 1),$$

where the α_i appear with multiplicities. We define

$$\sigma_n^{(i)} = \begin{pmatrix} i^{\text{th}} \text{ symmetric polynomial} \\ \text{in } \lambda_\phi(\alpha_1), \ldots, \lambda_\phi(\alpha_\rho) \end{pmatrix}.$$

Then each $\sigma_n^{(i)}(\phi)$ is a rational function in the coefficients $a_d, \ldots, a_0, b_d, \ldots, b_0$ of ϕ.

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Each $\sigma_n^{(i)}(\phi)$ is in the affine coordinate ring $\mathbb{Z}[\mathcal{M}_d]$, so they can be used to define a morphism
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$$(\sigma_n^{(1)}, \sigma_n^{(2)}, \ldots, \sigma_n^{(\rho)}) : \mathcal{M}_d \longrightarrow \mathbb{A}^\rho.$$

More generally, we can use periodic points of different periods to define a map

$$\boldsymbol{\sigma}_N = \left(\sigma_n^{(i)} : 1 \le n \le N,\ 1 \le i \le \rho_n\right) : \mathcal{M}_d \longrightarrow \mathbb{A}^L.$$

An Example: Rational Maps of Degree 2

The functions $\sigma_n^{(i)}$ are not independent. For example:

$$d = 2 \qquad \text{and} \qquad n = 1.$$

A rational map ϕ of degree two has three fixed points,

$$\operatorname{Per}_1(\phi) = \operatorname{FixedPoints}(\phi) = \{\alpha, \beta, \gamma\}.$$

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$$\begin{aligned}
\sigma_1^{(1)} &= \lambda(\alpha) + \lambda(\beta) + \lambda(\gamma), \\
\sigma_1^{(2)} &= \lambda(\alpha)\lambda(\beta) + \lambda(\alpha)\lambda(\gamma) + \lambda(\beta)\lambda(\gamma), \\
\sigma_1^{(3)} &= \lambda(\alpha)\lambda(\beta)\lambda(\gamma).
\end{aligned}$$

An elementary calculation shows that

$$\sigma_1^{(3)} = \sigma_1^{(1)} - 2,$$

and $\sigma_1^{(1)}, \sigma_1^{(2)}$ give the isomorphism described earlier:

$$(\sigma_1^{(1)}, \sigma_1^{(2)}) : \mathcal{M}_2 \xrightarrow{\ \sim\ } \mathbb{A}^2.$$

McMullen's Theorem

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Theorem. (McMullen) If d is not a square, then for all sufficiently large N, the map

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As far as I know, this is still an open problem in characteristic p.

The Exceptional Set in McMullen's Theorem

The exceptional line in McMullen's Theorem is the set of Lattès maps. These are rational maps $\phi_{E,m}$ constructed from elliptic curves via diagrams

$$\begin{array}{ccc} E & \xrightarrow{[m]} & E \\ \big\downarrow x & & \big\downarrow x \\ \mathbb{P}^1 & \xrightarrow{\phi_{E,m}} & \mathbb{P}^1 \end{array}$$

For fixed m, the maps $\phi_{E,m}$ all have the same multipliers, but are not PGL_2-conjugate for non-isomorphic E.

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For fixed m, the maps $\phi_{E,m}$ all have the same multipliers, but are not PGL_2-conjugate for non-isomorphic E.

Using elliptic curves with complex mulitplication and the fact that class numbers of imaginary quadratic fields go to infinity, it is not hard to prove that the degree of

$$\boldsymbol{\sigma}_N : \mathcal{M}_d(\mathbb{C}) \longrightarrow \mathbb{C}^L$$

is unbounded as $d \to \infty$.

Questions Concerning the Geometry of $\mathcal{M}_d$

The space Rat_d is **rational**, which means that there is a generically 1-to-1 rational map

$$\mathbb{P}^{2d+1} \longrightarrow \mathrm{Rat}_d\,.$$

This is clear, since Rat_d is an open subset of $\mathbb{P}^{2d+1}$.

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Questions.
1. $\mathcal{M}_2 \cong \mathbb{A}_2$ is rational. Is $\mathcal{M}_3$ rational?
2. What do the singularities of $\mathcal{M}_d$ look like?
3. Let $\mathcal{M}_d(n)$ classify rational maps of degree d with a marked periodic point of period n. For fixed d, is $\mathcal{M}_d(n)$ of general type for sufficiently large n?
4. Same questions for the moduli spaces $\mathcal{M}_d^N$.

Lang's Height Conjecture

Conjecture. (Lang) Let $E/\mathbb{Q}$ be an elliptic curve and let $P \in E(\mathbb{Q})$ a nontorsion point. Then

$$\hat{h}(P) \ge c \log \bigl|\mathrm{Disc}(E/\mathbb{Q})\bigr|,$$

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Lang's conjecture has applications to the distribution of integer points on elliptic curves. Various weaker versions are known, and the conjecture has been proved conditional on the ABC-conjecture.

A Dynamical Analogue of Lang's Height Conjecture

We fix an embedding

$$\boxed{i_{\mathcal{M}} : \mathcal{M}_d \hookrightarrow \mathbb{P}^L}$$

and use it to define a height function

$$\boxed{h_{\mathcal{M}} : \mathcal{M}_d(\bar{\mathbb{Q}}) \longrightarrow \mathbb{R}, \qquad h_{\mathcal{M}}(\phi) = h\bigl(i_{\mathcal{M}}(\phi)\bigr).}$$

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The height $h_{\mathcal{M}}(\phi)$ is a measure of the arithmetic complexity of the rational map ϕ.

Conjecture. Fix a number field $K/\mathbb{Q}$. There is a constant $c > 0$, depending only on K and the embedding $\mathcal{M}_d \hookrightarrow \mathbb{P}^L$, such that for all maps $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d and all wandering points $\alpha \in \mathbb{P}^1(K)$,

$$\hat{h}_\phi(\alpha) \geq c h_{\mathcal{M}}(\phi).$$

Dynamical Modular Curves

Dynatomic Polynomials

The $n^{\mathbf{th}}$ **cyclotomic polynomial**

$$\prod_{k|n}(z^k - 1)^{\mu(n/k)}$$

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Similarly, the $n^{\mathbf{th}}$ **dynatomic polynomial** associated to a polynomial $\phi(z)$ is

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The roots α of $\Phi_\phi(z)$ are said to have **formal period n,** because although they satisfy

$$\phi^n(\alpha) = \alpha,$$

they occasionally have period strictly smaller than n. (This is different from the cyclotomic case.)

Dynatomic Polynomials (continued)

The polynomial $\phi^n(z) - z$ may have multiple roots. E.g.,

$$\phi(z) = z^2 - \frac{3}{4}, \quad \Phi_2 = \left(z + \frac{1}{2}\right)^2, \quad \phi\left(-\frac{1}{2}\right) = -\frac{1}{2}.$$

It is not clear that $\Phi_{\phi,n}(z)$ is a polynomial, but we have:

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Theorem. The dynatomic "polynomial" is indeed a polynomial. Equivalently, for every point α,

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More generally, every rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ has an associated dynatomic polynomial $\Phi_{\phi,n}(X,Y)$. The roots of $\Phi_{\phi,n}$ define a **dynatomic divisor** in $\operatorname{Div}(\mathbb{P}^1)$, and the theorem says that this divisor is effective (positive).

The Dynatomic Polynomial of $z^2 + c$

We now restrict attention to the family of quadratic polynomials

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$$\begin{aligned}
\Phi_1(c, z) &= z^2 - z + c \\
\Phi_2(c, z) &= z^2 + z + (c + 1) \\
\Phi_3(c, z) &= z^6 + z^5 + (3c + 1)z^4 + (2c + 1)z^3 + (3c^2 + 3c + 1)z^2 \\
&\quad + (c^2 + 2c + 1)z + (c^3 + 2c^2 + c + 1) \\
\Phi_6(c, z) &= z^{54} - z^{53} + 27cz^{52} + (-26c + 1)z^{51} + (351c^2 + 13c - 1)z^{50} + \cdots \\
&\quad + (c^{27} + 13c^{26} + 78c^{25} + 293c^{24} + 792c^{23} + \cdots + 3c^3 + c^2 - c + 1)
\end{aligned}$$

The Dynatomic Curve $Y_1^{\mathsf{dyn}}(n)$

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consists of a pair such that α is a periodic point of formal period n for the polynomial $\phi_c(z) = z^2 + c$.

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Notice the close analogy with the classical modular curve $Y_1(n)$ whose points classify pairs

$$(E, P)$$

consisting of an elliptic curve E and a point $P \in E$ of order n.

Rational Points on $Y_1^{\mathsf{dyn}}(n)$

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In particular, the set of rational points

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classifies all quadratic polynomials with $\mathbb{Q}$-coefficients having a $\mathbb{Q}$-rational periodic point of (formal) period n.

Thus the uniformity conjecture for quadratic polynomials is equivalent to the statement:

$$Y_1^{\text{dyn}}(n)(\mathbb{Q}) = \emptyset \quad \text{for sufficiently large } n,$$

and the strong uniformity conjecture asserts that

$$Y_1^{\text{dyn}}(n)(\mathbb{Q}) = \emptyset \qquad \text{for all } n \geq 4.$$

The Dynatomic Curve $X_1^{\mathsf{dyn}}(n)$

The curve $Y_1^{\mathrm{dyn}}(n)$ has a unique nonsingular completion, which we denote by $X_1^{\mathrm{dyn}}(n)$.

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The finitely many points in the complement

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Theorem. (Bousch, Morton) The curve $X_1^{\mathrm{dyn}}(n)$ is geometrically irreducible.

The proofs are far from trivial and ultimately depend on dynamical arguments. (Remark: There are families for which the analogous dynatomic curves are reducible.)

The Dynatomic Curves $Y_0^{\mathsf{dyn}}(n)$ and $X_0^{\mathsf{dyn}}(n)$

There is a natural automorphism of $Y_1^{\mathrm{dyn}}(n)$ defined by

$$\begin{aligned} \hat{\phi} : Y_1^{\mathrm{dyn}}(n) &\longrightarrow & Y_1^{\mathrm{dyn}}(n), \\ (y,z) &\longmapsto & (y, z^2+y) = \bigl(y, \phi_y(z)\bigr). \end{aligned}$$

This automorphism extends to $X_1^{\mathrm{dyn}}(n)$.

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Note that $\hat{\phi}^n = 1$. The quotients of $Y_1^{\mathrm{dyn}}(n)$ and $X_1^{\mathrm{dyn}}(n)$ by the action of $\hat{\phi}$ are denoted

$$\begin{aligned} Y_0^{\mathrm{dyn}}(n) &= Y_1^{\mathrm{dyn}}(n)/\hat{\phi}, \\ X_0^{\mathrm{dyn}}(n) &= X_1^{\mathrm{dyn}}(n)/\hat{\phi}. \end{aligned}$$

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The points of $Y_0^{\text{dyn}}(n)$ classify pairs $(c, \mathcal{O})$, where $\mathcal{O}$ is the orbit of a point of formal period n for $\phi_c(z) = z^2 + c$.

The Genera of $X_1^{\mathsf{dyn}}(n)$ and $X_0^{\mathsf{dyn}}(n)$

Bousch and Morton found explicit, but rather messy, formulas for the genera of $X_1^{\mathrm{dyn}}(n)$ and $X_0^{\mathrm{dyn}}(n)$. The genera grow quite rapidly:

n	1	2	3	4	5	6	7	8	9	10
genus $X_1(n)$	0	0	0	2	14	34	124	285	745	1690
genus $X_0(n)$	0	0	0	0	2	4	16	32	79	162

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genus $X_1(n)$	0	0	0	2	14	34	124	285	745	1690
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Theorem. For $c \in \mathbb{Q}$, the polynomial $z^2 + c$ has no $\mathbb{Q}$-rational periodic point of exact period 6.

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The proof relies on the fact that the genus of $X_0^{\mathrm{dyn}}(6)$ is (barely) small enough to allow arithmetic computations.

Fields of Definition
and
Field of Moduli

The Map from Rat_d to $\mathcal{M}_d$

We let $\langle \quad \rangle$ denote the natural map

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Question: Does this imply that there is some change of variables $f \in \mathrm{PGL}_2(\bar{\mathbb{Q}})$ such that

$$\phi^f \in \mathrm{Rat}_d(\mathbb{Q}).$$

Two Examples

Example 1. The polynomial

$$\phi(z) = z^2 + 2\sqrt{2}z + 1 - \sqrt{2}$$

is not defined over $\mathbb{Q}$, but the change of variables

$$f(z) = z - \sqrt{2}$$

transforms it to

$$\phi^f(z) = f^{-1} \circ \phi \circ f(z) = \phi(z - \sqrt{2}) + \sqrt{2} = z^2 - 1.$$

Thus not only is

$$\langle \phi \rangle \in \mathcal{M}_2(\mathbb{Q}),$$

but we can find a change variables $f \in \mathrm{PGL}_2(\bar{\mathbb{Q}})$ such that

$$\phi^f \in \mathrm{Rat}_2(\mathbb{Q}).$$

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Example 2. The rational map

$$\phi(z) = \sqrt{-1}\left(\frac{z-1}{z+1}\right)^3$$

has the property that

$$\overline{\phi}(z) = \phi^g(z) \qquad \text{with} \quad g(z) = -1/z,$$

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Thus although ϕ seems as if it should be defined over $\mathbb{Q}$, it is not possible to change variables and make it actually defined over $\mathbb{Q}$.

Fields of Definition and Field of Moduli

Let $\phi \in \mathrm{Rat}_d(\bar{\mathbb{Q}})$.

Definition. A **Field of Definition** for ϕ is any field K for which there is an $f \in \mathrm{PGL}_2(\bar{\mathbb{Q}})$ such that

$$\phi^f \in \mathrm{Rat}_d(K).$$

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Equivalently, let

$$G_\phi = \left\{\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) : \exists g \in \mathrm{PGL}_2(\bar{\mathbb{Q}}) \text{ with } \sigma(\phi) = \phi^g\right\}.$$

Then

$$K_\phi = \text{Field of Moduli of } \phi = \text{Fixed field of } G_\phi.$$

The Automorphism Group of a Rational Map

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Example.

$$\phi = \frac{az}{bz^2 + c}, \qquad \mathrm{Aut}(\phi) = \{z, -z\}.$$

A Cohomological Criterion

We make the simplifying assumption that

$$\mathrm{Aut}(\phi) = 1.$$

Then for every $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/K_\phi)$ there is a *unique* element $g_\sigma \in \mathrm{PGL}_2(\bar{\mathbb{Q}})$ satisfying

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Proposition. (a) The map

$$\mathrm{Gal}(\bar{\mathbb{Q}}/K_\phi) \longrightarrow \mathrm{PGL}_2(\bar{\mathbb{Q}}), \qquad \sigma \longrightarrow g_\sigma,$$

is a 1-cocycle.

(b) The field of moduli K_ϕ is a field of definition for ϕ if and only if $\sigma \to g_\sigma$ is a coboundary, i.e.,

$$\mathrm{FOM} = \mathrm{FOD} \quad \Longleftrightarrow \quad [g_\sigma] = 0 \text{ in } H^1\bigl(\mathrm{Gal}, \mathrm{PGL}_2\bigr).$$

Some Cases of $\mathrm{FOM} = \mathrm{FOD}$

Using the cohomological criterion, one can prove the following.

Theorem. Let $\phi \in \mathrm{Rat}_d(\bar{\mathbb{Q}})$. Then

$$\boxed{\mathrm{FOM} = \mathrm{FOD} \;\text{ for } \phi}$$

in the following two situations:

(a) $\phi(z)$ has even degree.

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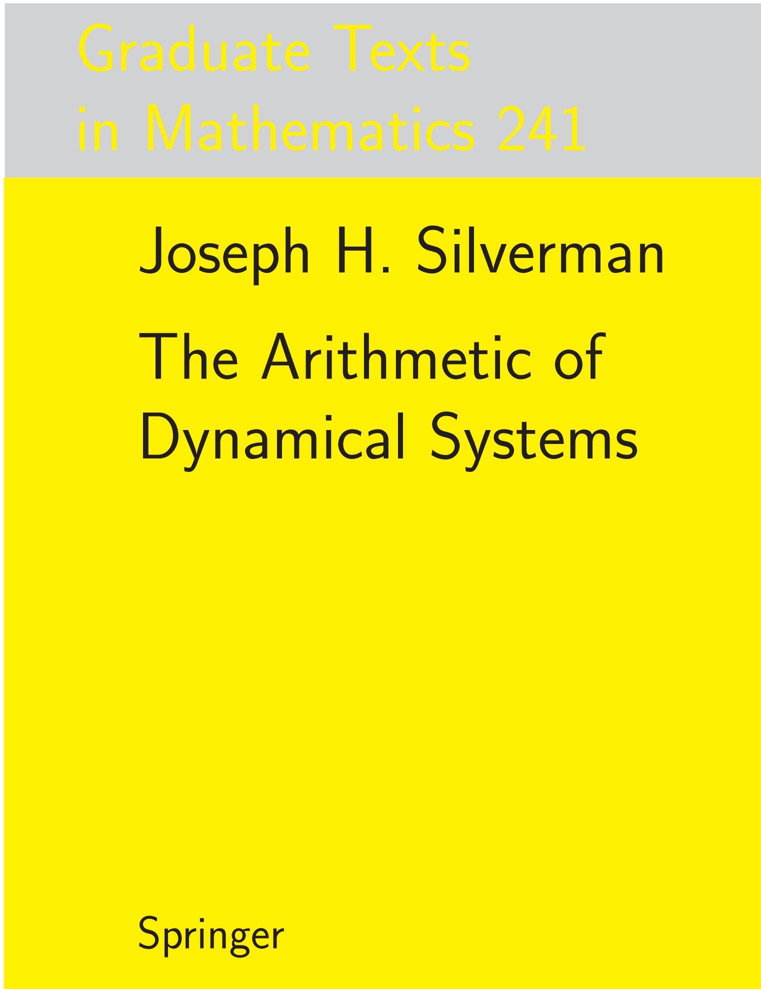
Proof Ideas. Interpret the elements of $H^1(\mathrm{Gal}, \mathrm{PGL}_2)$ as twists of $\mathbb{P}^1$. Then use the given properties of ϕ to prove that the twists have a rational point, hence are trivial.

And In Conclusion, . . . A Blatant Advertisement

For those who are interested in learning more about arithmetic dynamics, there is now an introductory graduate textbook on the subject.



www.math.brown.edu/~jhs/ADSHome.html

Moduli Spaces in Arithmetic Dynamics

Joseph H. Silverman

Brown University

Workshop on p-adic Dynamics
Fields Institute, Toronto
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