

Mini-workshop in complex dynamics
Fields Institute Nov 2008

Lecture 3

Polynomial dynamics at infinity

Mattias Jonsson

(w C.Favre , www.arxiv.org)
"Dynamical compactifications of \mathbb{C}^2 "

Polynomial Mappings

$f: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ poly, dominant

Study behavior at ∞ of f^n

Similar to $g: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ but

- f not proper in general so can't say $f(\infty) = \infty$
- situation less local than $(\mathbb{C}^2, 0)$: "holomorphic objects" defined near ∞ in \mathbb{C}^2 extend to all of \mathbb{C}^2 (Hartogs)

Strategy : combine methods from

- Lecture 1 (global merom selfmaps)
- Lecture 2 (local dynamics)

Dynamical degrees

$\lambda_2 = \text{top. deg of } f$

$$\lambda_1 = \lim_{n \rightarrow \infty} (\deg f^n)^{1/n}$$

(Assume $\lambda_1 > 1$ for the most part).

Thm A: 2 possibilities:

a) $\deg f^n \sim \lambda_1^n$

b) $\deg f^n \sim n \cdot \lambda_1^n$. In this case,
 f conjugate to a skew product

$$(x, y) \mapsto (P(x), Q(x, y)) \quad (+ \text{cond'n on } P, Q)$$

Thm B: λ_1 quadratic integer : $\lambda_1^2 = A\lambda_1 + B$ $A, B \in \mathbb{Z}$

Thm C: $\deg f^n$ satisfies recursion formula:

$$\deg f^n = \sum_{j=1}^N a_j \deg f^{n-j} \quad a_j \in \mathbb{Z}$$

Can approach these results by studying
dynamics at ∞ .

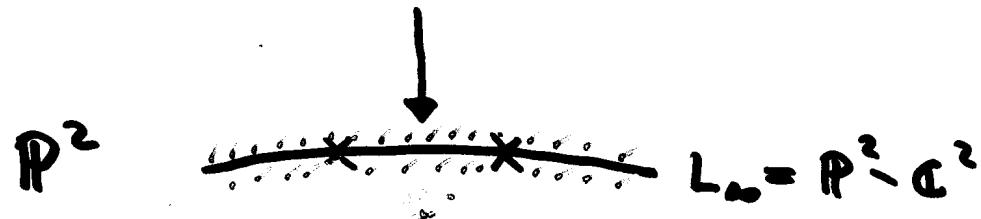
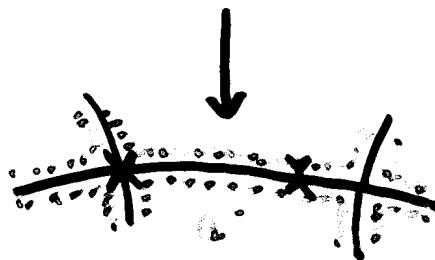
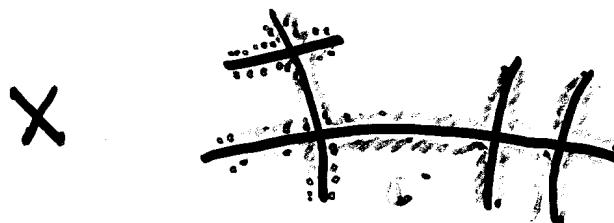
Compactifications

Fix embedding $\mathbb{C}^2 \hookrightarrow \mathbb{P}^2$

(\Rightarrow can talk about affine fcns on \mathbb{C}^2)

Def: An admissible compactification $X \supseteq \mathbb{C}^2$ is obtained from \mathbb{P}^2 by finitely many blowups at ∞

Def: Primes of X = irr. comp's of $X \setminus \mathbb{C}^2$
(e.g. $L_\infty = \mathbb{P}^2 \setminus \mathbb{C}^2$)



Dynamical compactifications

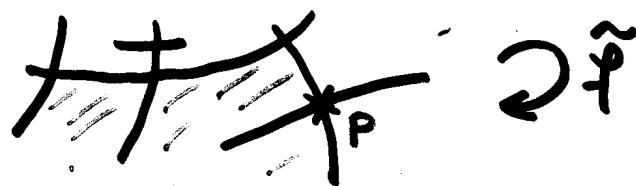
Thm D: Assume $\lambda_2 < \lambda_1$, "low top deg."

Then there exists:

- An admissible comp'n $X \supset \mathbb{C}^2$
- A point $p \in X \setminus \mathbb{C}^2$
- An integer $N \geq 1$

s.t.:

- the lift $\tilde{f}: X \rightarrow X$ is halo at p , $\tilde{f}(p) = p$
- the germ $\tilde{f}: (X, p) \rightarrow (X, p)$ is superattracting and admits "simple" normal form
- $\tilde{f}^N E = \{p\}$ for all primes E of X (with at most one exception)



Can deduce Thms B, C (when $\lambda_2 < \lambda_1$) from this.

Can also deduce that Green fn

$$G := \lim_{n \rightarrow \infty} \lambda_1^{-n} \log^+ \|f^n\|$$

is well behaved

Focus on proving Thm A, D

The Riemann-Zariski approach

$$\mathbb{X} := \varprojlim X \quad X \text{ admissible compn}$$

Riemann-Zariski space at ∞ (don't touch \mathbb{C}^2)

$$W(\mathbb{X}) := \varprojlim H_R^{(1)}(X) \quad \text{Weil classes}$$

$$C(\mathbb{X}) := \varinjlim H_R^{(1)}(X) \quad \text{Cartier —}$$

$$L^2(\mathbb{X})$$

$$Nef(\mathbb{X})$$

f_*, f^* act on W, C, L^2, Nef

[Use f holo on \mathbb{C}^2 and $f(\mathbb{C}^2) \subset \mathbb{C}^2$]

Thm: Assume $\lambda_1^2 > \lambda_2$. Then:

a) $\exists \Theta_x, \Theta^* \in Nef(\mathbb{X})$, unique up to scaling,
s.t. $f_* \Theta_x = \lambda_1 \Theta_x$, $f^* \Theta^* = \lambda_2 \Theta^*$

b) $\alpha \in L^2(\mathbb{X}) \Rightarrow$

$$\lambda_1^{-n} f_*^n \alpha \rightarrow \frac{(\Theta^*. \alpha)}{(\Theta^*. \Theta_x)} \Theta_x \quad (\text{fast})$$

(+ same for f^*)

Will interpret Θ_x

Valuations

$R = \mathbb{C}[x,y]$ coordinate ring of \mathbb{C}^2

$\hat{\mathcal{V}}_0 = \{ \text{valuations } v : R \rightarrow (-\infty, +\infty] \text{ centered at } \infty :$

$v(P) < 0 \text{ for some polynomial } P \}$.

$\mathcal{V}_0 = \{ \text{normalized valuations in } \hat{\mathcal{V}}_0 \}$

Normalization: $\min \{v(L) \mid L \text{ affine}\} = -1$
 $\Leftrightarrow v(L) = -1 \text{ for generic } L$.

Prime $E \subset X \rightsquigarrow \text{divisorial valuations}$

$$\text{ord}_E \in \hat{\mathcal{V}}_0$$

$$v_E \in \mathcal{V}_0$$

$$v_E = b_E^{-1} \text{ord}_E$$

$$b_E = -\text{ord}_E(L)$$



$$\text{Ex: } E = L_\infty = \mathbb{P}^2 - \mathbb{C}^2 \Rightarrow$$

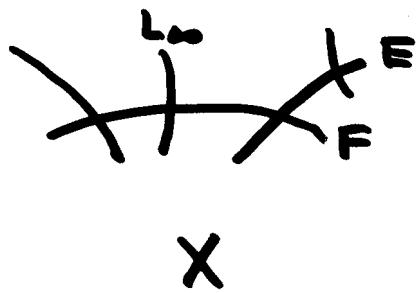
$$v_E = \text{ord}_E = -\deg \quad (b_E = 1)$$

Dual graphs

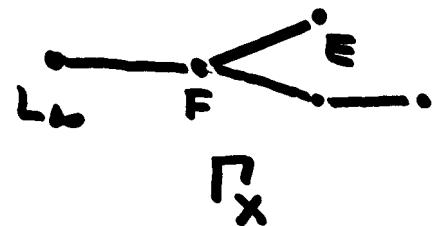
X admissible comp'n of \mathbb{C}^2

Γ_X dual graph:

- partial ordering (root = L_∞)
- metric



$$\text{dist}(E, F) = \frac{1}{\pi}$$



$X' \geq X$ (can get from X to X' by blowing up)

$\Rightarrow \Gamma_X \hookrightarrow \Gamma_{X'}$ order-preserving isometry

Also have:

- Embedding: $\Gamma_X \hookrightarrow \mathcal{V}_0$
- Retraction: $\mathcal{V}_0 \twoheadrightarrow \Gamma_X$

Thm: $\mathcal{V}_0 \cong \varprojlim \Gamma_X$

Cor: \mathcal{V}_0 is an R-tree



Valuative dynamics

$f: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ poly, dominant

Want to define: $\begin{cases} f_x: \hat{\mathcal{V}}_0 \rightarrow \hat{\mathcal{V}}_0 \\ f_0: \mathcal{V}_0 \rightarrow \mathcal{V}_0 \end{cases}$

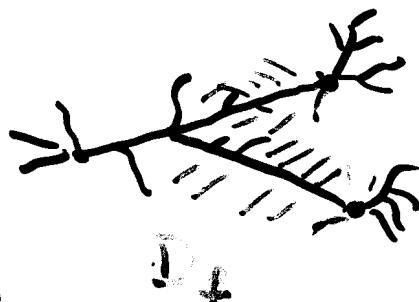
as in local case.

Problem: $f_x v$ may not be centered at ∞ .

Get subtree $D_f \subseteq \mathcal{V}_0$ on which f_0 is defined.

$$f_0: D_f \rightarrow \mathcal{V}_0$$

respects tree structure,
has dense image.



Problem: D_f depends on f .

Is $D_f \cap D_{f_2} \cap \dots \cap D_{f_n} \dots \neq \emptyset$?

Answer: YES!

The subtree $\gamma_1 = I$

The IR-tree γ_0 admits

2 natural parametrizations:

$$\alpha : \gamma_0 \longrightarrow [-\infty, 1] \quad \text{"skewness"}$$

$$R : \gamma_0 \longrightarrow [-2, \infty] \quad \text{"thinness"}$$

$$1) \alpha(v) = 1 - \text{dist}(v, \text{root})$$

2) Thinness R defined as "log-discrepancy"

$$R(\text{ord}_E) = 1 + \text{ord}_E(\omega) \quad \omega = dx dy \text{ on } E^*$$

$$R(v_E) = b_E^{-1} R(\text{ord}_E)$$

~~E~~ E

Def: $v \in \gamma_1$, iff $\begin{cases} \alpha(v) \geq 0 \\ R(v) \leq 0 \end{cases}$

" v close enough to the root of γ_0 "



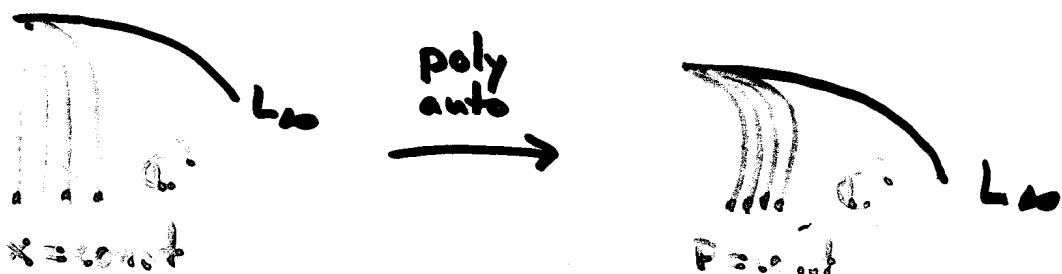
The subtree \mathcal{V}_i : II

Valuations in \mathcal{V}_i have good properties

Thm: If $\alpha(v) > 0$ (e.g. v in "interior" of \mathcal{V}_i) \Rightarrow

- 1) $v(P) < 0$ for every nonconst poly P
- 2) $v(P) \leq -\alpha(v) \cdot \deg P$

Thm: If $\alpha(v) = 0$, $R(v) < 0$ and
 v is divisorial ($\Rightarrow v$ endpt in \mathcal{V}_i)
then v is a rational pencil valuation:



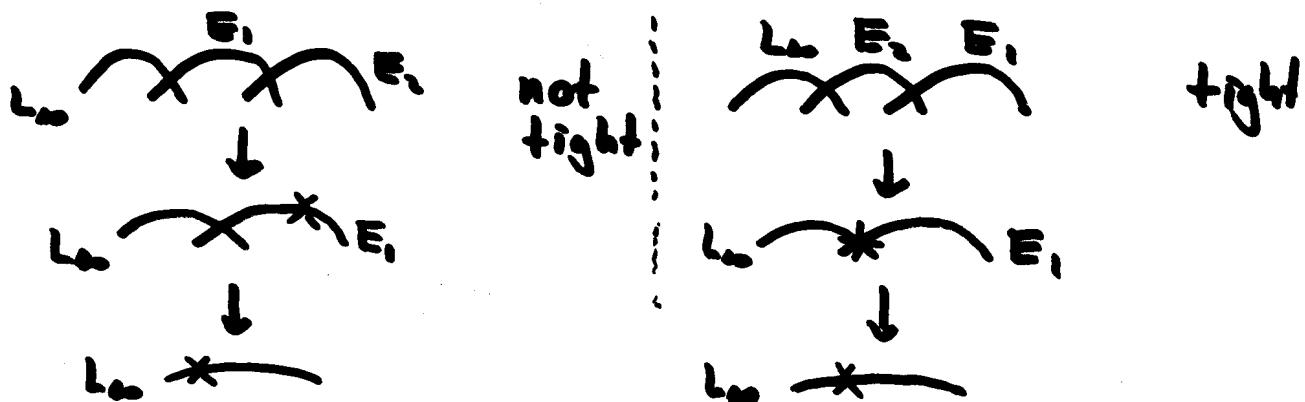
$$v(Q) = \text{const} \cdot \text{ord}_{\infty}(Q|_{P=\text{const}})$$

Tight compactifications

$X \supset \mathbb{C}^2$ admissible compactification

Def: X is tight if V prime $\in E$ of X ,
the valuation v_E lies in \mathcal{V}_1 ,

[Restriction on blowups from \mathbb{P}^2 to X].



Tight compactifications $X \supset \mathbb{C}^2$
have nice geometric properties:

- $\text{Nef}(X)$ is a simplicial cone
- Every nef line bundle on X is generated by global sections

Dynamics on \mathcal{X}_1 , I

$$f: \mathbb{C}^2 \rightarrow \mathbb{C}^2$$

Thm: f_* well defined on \mathcal{X}_1 ($D_f \supset \mathcal{X}_1$)
and $f_* \mathcal{X}_1 \subset \mathcal{X}_1$

Cor: \exists eigenvaluation $v_* \in \mathcal{V}_1$:

$$f_* v_* = v_* \quad f_* v_* = \lambda v_*$$

Pf of Thm A [Behavior of $\deg f^n$].

(i) v_* rat'l pencil valuation
 $\Rightarrow f \sim$ skew product

(ii) $\alpha(v_n) > 0 \Rightarrow \lambda_1^n \leq \deg f^n \leq D \lambda_1^n$
 where $D = \alpha(v_n)^{-1}$.

[(iii) one more case ...]

Rem: $\lambda_2 < \lambda_1 \Rightarrow v_*$ not divisorial

Rem: $A(v_n) = 0 < \alpha(v_*) \Rightarrow$
 f counterexample to FC!

Dynamics of γ , II

Thm: Assume $\lambda_1^2 > \lambda_2$. Then:

$$f_n v \rightarrow v_* \text{ as } n \rightarrow \infty$$

for all $v \in \gamma$, with at most one exception v^* , for which $f_n v^* = v^*$.

Pf: Associate Weil class $Z_v \in W(\mathbb{X})$ to any valuation $v \in \gamma$.

$$v \in \gamma \Rightarrow Z_v \text{ nef}$$

$$f_* Z_v = Z_{f_v v}$$

$$\Rightarrow \dots \Rightarrow$$

$$f_n v \rightarrow v_* \text{ unless}$$

$$(Z_v, \Theta^*) = 0 \Rightarrow Z_v = \Theta^* \dots \square$$

Construction of Z_v when $v = \text{ord}_E$

E prime of X . $Z_{\text{ord}_E} = Z_E$ defined by:

$$(Z_E, F) = \begin{cases} 1 & F = E \\ 0 & F \neq E \end{cases}$$

[General v by homog. + approx.]

Proof of Thm D

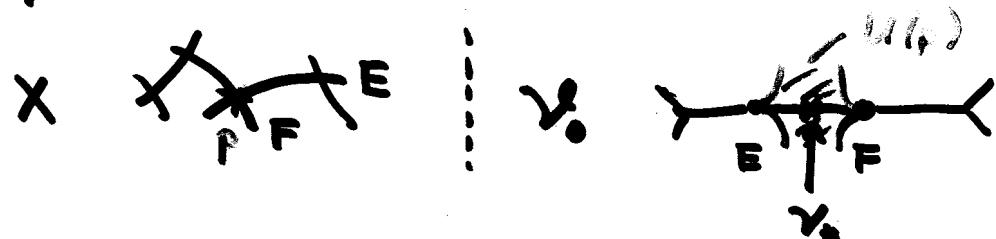
Thm D: Assume $\lambda_2 < \lambda_1$. Then $\exists X, p, N \dots$



$\tilde{f}: (X, p) \supset$ superattr. fixed pt germs
 $\tilde{f}^N E = p \quad \forall E.$

PF: γ_x cannot be divisible ($\lambda_2 < \lambda_1$)

- Assume irrational quasimonomial
- Successively blow up center of γ_x many times
- Get tight compactification X ,
 $p =$ center of γ_x on X .



- γ_x locally attracting \Rightarrow
 $f_* U(p) \subset U(p) \Rightarrow \tilde{f}$ hole at p etc
- $f_* \gamma_E \rightarrow \gamma_x \quad \forall E \Rightarrow \tilde{f}^N E = f_p ?.$