

Mini-workshop in complex dynamics  
Fields Institute Nov 2008

## Lecture 2

### Super-attracting fixed points

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( w. C. Favre , RENS 2007 )

"Eigenvaluations"

## Local dynamics in $\mathbb{C}^2$

$$f: (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$$

Rough classification using  $Df(0)$ :

- attracting
- repelling
- saddle
- parabolic
- .....

We will focus on an extreme case:

Def:  $f$  is superattracting if  
 $Df(0)$  is nilpotent  
 $(\Leftrightarrow Df^2(0) = 0)$

Rem: Assume  $f$  dominant ( $Df \neq 0$ )

To simplify exposition, sometimes  
assume  $f$  finite (no contracted curves)

## Superattracting fixed points

Talk will focus on tools rather than results. Still, what can be said?

$$f = f_c + f_{c+1} + f_{c+2} + \dots$$

$$c = c(f) \geq 1 \quad \text{order of vanishing}$$

$$f \text{ superattracting} \iff c(f^2) > 1$$

$n \mapsto c(f^n)$  supermultiplicative:

$$c(f^{n+m}) \geq c(f^n)c(f^m)$$

$$\Rightarrow c_\infty := c_\infty(f) := \lim_{n \rightarrow \infty} c(f^n)^{1/n} \text{ exists.}$$

Thm A:  $\exists \delta > 0$  s.t.

$$\delta c_\infty^n \leq c(f^n) \leq C_\infty^n$$

$$\dots \quad \dots \quad \dots$$

Thm B:  $c_\infty$  is a quadratic integer:

$$c_\infty^2 = Ac_\infty + B, \quad A, B \in \mathbb{Z}$$

$$\text{Ex: } f(x,y) = (y, xy) \quad c_\infty = \frac{1}{2}(1 + \sqrt{5})$$

## Blowups I

$$f: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$$

$\pi: X \rightarrow (\mathbb{C}^2, 0)$  blowup of 0

$E_0 = \pi^{-1}(0)$  exc. div.

$$\begin{array}{ccc} X & \xrightarrow{\tilde{f}} & X \\ \pi \downarrow & & \downarrow \pi \\ (\mathbb{C}^2, 0) & \xrightarrow{f} & (\mathbb{C}^2, 0) \end{array}$$

O parabolic fixed point  $(Df|_0) = id$ )

$\Rightarrow \tilde{f}$  good tool for understanding dynamics

O superattracting fixed point  $(Df^2|_0) = 0$ )

$\Rightarrow \begin{cases} \tilde{f} \text{ usually not holomorphic} \\ \tilde{f}(E_0) = \text{point on } E_0 \end{cases}$

## Blowups II

$\pi: X_\pi \rightarrow (\mathbb{C}^2, 0)$  composition of blowups

$$\begin{array}{ccc} X_\pi & \xrightarrow{\tilde{f}} & X_\pi \\ \downarrow & & \downarrow \\ (\mathbb{C}^2, 0) & \xrightarrow{f} & (\mathbb{C}^2, 0) \end{array}$$

Can we make  $\tilde{f}$  "nice" by clever choice of  $\pi$ ?

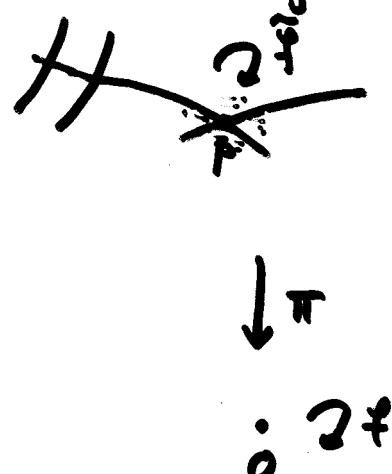
Ihm C: Can choose  $\pi: X_\pi \rightarrow (\mathbb{C}^2, 0)$  and a point  $p \in \pi^{-1}(0)$  s.t. :

- $\tilde{f}$  holomorphic at  $p$
- $\tilde{f}(p) = p$
- $\tilde{f}: (X_\pi, p) \hookrightarrow \underline{\text{rigid germ}}$

(Critical set of  $\tilde{f}$  contained in a totally invariant set with snc sing.)

Rem: Can find local normal forms for  $\tilde{f}: (X_\pi, p) \hookrightarrow$

E.g.  $\tilde{f}$  monomial map



## Blowups III

How do we pick  $\pi: X_\pi \rightarrow (\mathbb{C}^2, 0)$ ?

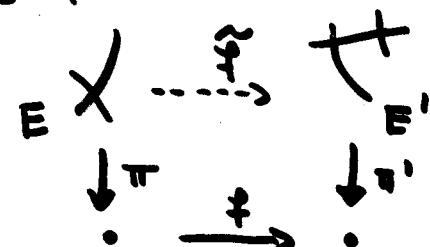
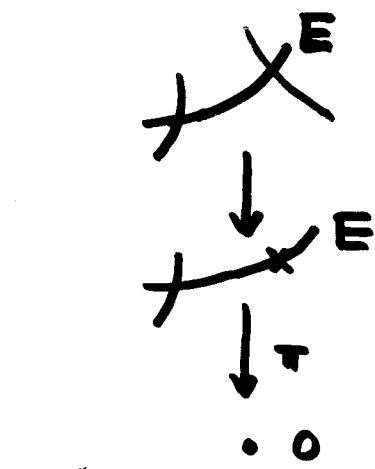
Idea: look at all possible  $\pi$  at the same time!

Def: An exceptional prime (divisor) is an irreducible component  $E \in \pi^{-1}(0)$  for some (composition of) blowup(s)  $\pi: X_\pi \rightarrow (\mathbb{C}^2, 0)$ .

Convention: identify  $E$  with its strict transform under further blowups.

Fact (Zariski):  $\tilde{\pi}$  maps exceptional primes to exc. primes: given  $E \ni E'$  s.t.  $\tilde{\pi}|_E = E'$ .

How to study induced dynamics on exc. primes?



1. Riemann-Zariski (Lecture 1)
2. Valuations (Lecture 2)
3. Combo (Lecture 3)

## Divisorial valuations

$R = \mathcal{O}_0$  ring of hol. germs at  $0 \in \mathbb{C}^2$   
 $K$  fraction field of  $R$

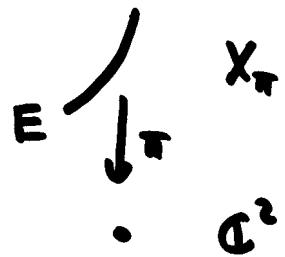
Idea: identify exc. prime  $E \subset \pi^{-1}(0)$   
 with divisorial valuation  $\text{ord}_E$

$\phi \in R \text{ (or } K) \Rightarrow$

$\text{ord}_E(\phi) = \text{order of vanishing of } \phi \text{ along } E$

(Convention: identify fcn field of  $X_\pi$  with  $K$ )

$\text{ord}_E : \begin{cases} R \rightarrow \mathbb{N} \cup \{\infty\} \\ K \rightarrow \mathbb{Z} \cup \{\infty\} \end{cases}$  satisfies



- $\text{ord}_E(\phi\psi) = \text{ord}_E(\phi) + \text{ord}_E(\psi)$
- $\text{ord}_E(\phi+\psi) \geq \min \{\text{ord}_E(\phi), \text{ord}_E(\psi)\}.$
- $\text{ord}_E(0) = \infty$
- $\text{ord}_E(c) = 0 \quad c \in \mathbb{C}^*$
- $\phi \in \mathfrak{m} \text{ i.e. } \phi(0)=0 \Rightarrow \text{ord}_E(\phi) > 0.$

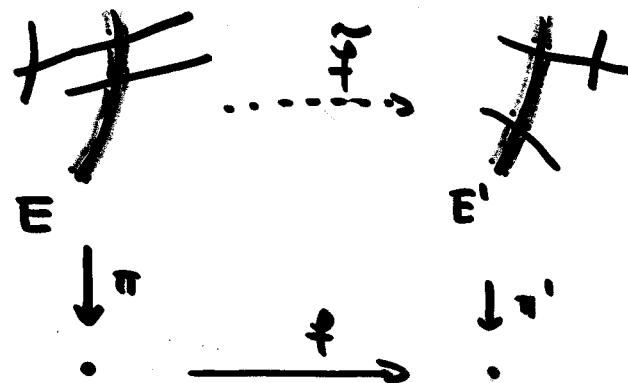
Ex:  $\text{ord}_0$  (blow up  $0$  once)

## Divisorial dynamics

Given  $\tilde{f}, E$  define new valuation  $\tilde{f}_* \text{ord}_E$ :

$$(\tilde{f}_* \text{ord}_E)(\phi) := \text{ord}_E(\tilde{f}^* \phi) = \text{ord}_E(\phi \cdot \tilde{f})$$

Fact (Zariski)  $\Rightarrow \boxed{\tilde{f}_* \text{ord}_E = k \cdot \text{ord}_{E'}}$



$k = \text{coeff of } E \text{ in } \tilde{f}^* E'$

Set  $\hat{\mathcal{V}}_{\text{div}} = \{c \cdot \text{ord}_E \mid c > 0, E \text{ exc. prime}\}$

Get induced dynamics  $\tilde{f}_*: \hat{\mathcal{V}}_{\text{div}} \rightarrow \hat{\mathcal{V}}_{\text{div}}$ .

Problem:  $\hat{\mathcal{V}}_{\text{div}}$  not "nice"

Solution: Make it nice!  $\hat{\mathcal{V}}_{\text{div}} \subseteq \hat{\mathcal{V}}$

Analogy:  $\mathbb{Q} \subseteq \mathbb{R}$

## Valuations

$R = \mathcal{O}_0$  germs at  $0 \in \mathbb{C}^2$

$m = \text{max ideal}$   $\phi \in m \Leftrightarrow \phi(0) = 0$ .

Def: A valuation on  $R$  is a function

$$\nu: R \rightarrow [0, +\infty] \text{ s.t}$$

- $\nu(\phi\psi) = \nu(\phi) + \nu(\psi)$
- $\nu(\phi+\psi) \geq \min\{\nu(\phi), \nu(\psi)\}$
- $\nu(0) = +\infty, \nu(c) = 0, c \in \mathbb{C}^*$

Def: A valuation  $\nu$  is:

- centered at 0 if  $\nu(m) := \min_{\phi \in m} \nu(\phi) > 0$
- normalized if  $\nu(m) = 1$ .

Def:  $\hat{\mathcal{V}} := \{\text{val's centered at } 0\}$   
 $\mathcal{V} := \{\text{normalized val's}\}$ .

Rem:  $\nu \in \hat{\mathcal{V}} \Leftrightarrow e^\nu$  non-archimedean seminorm  
 $\hat{\mathcal{V}} \approx \text{Berkovich space.}$

## Valuative dynamics

$$f: (\mathbb{C}^2, 0) \rightarrow \mathcal{V}$$

[Assume  $f$  finite for simplicity; need slight modification if  $f$  dominant].

Get induced map  $f_*: \hat{\mathcal{V}} \rightarrow \mathcal{V}$ .

$v \in \mathcal{V}$  normalized  $\Rightarrow f_* v$  normalized.

$$c(f, v) := v(f^* m) \quad (<\infty \text{ since } f \text{ finite})$$

$$f_* v := c(f, v)^{-1} \cdot f_* v$$

Induced map  $f_*: \mathcal{V} \rightarrow \mathcal{V}$

$$\text{Rem: } (f^n)_* = (f_*)^n \text{ and } (f^n)_* = (f_*)^n$$

$$c(f^n, v) = \prod_{j=0}^{n-1} c(f, f_*^j v) \quad \begin{matrix} \text{multiplicative} \\ \underline{\text{cocycle}} \end{matrix}$$

In particular:

$$c(f^n) = c(f^n, \text{ord}_0) = \prod_{j=0}^{n-1} c(f, f_*^j \text{ord}_0)$$

## Executive summary of approach

1)  $\mathcal{V}$  has structure of R-tree



- 2)  $f_* : \mathcal{V} \rightarrow \mathcal{V}$  preserves this structure
- 3)  $\exists$  fixed point (eigenvalue) :  $f_* v_* = v_*$
- 4) Several cases ... in one of them,  
 $v_*$  "interior" pt of  $\mathcal{V}$ , attracting :

a)  $v_*$  interior pt  $\Rightarrow$  (Skoda-Izumi)  
 $\delta v_* \leq \text{ord.} \leq v_*$

$$\text{Also: } f_* v_* = \lambda v_* \quad \lambda > 0 \Rightarrow$$

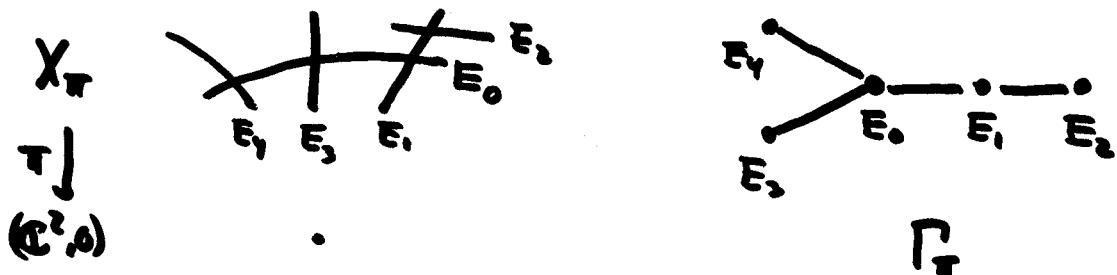
$$\delta \cdot \lambda^n \leq c(f^n) \leq \lambda^n \Rightarrow \text{Thm A}$$

b)  $v_*$  attracting  $\leadsto$   
construct  $X_{n,p}$  from suitable  
basin of attraction.

# Dual graphs I

$\pi: X_\pi \rightarrow (\mathbb{C}^2, 0)$  (composition of) blowup(s)

$\Gamma_\pi$  := dual graph of exceptional divisor



Structures on  $\Gamma_\pi$ :

- Partial ordering  $\leq$ , root =  $E_0$ .

- Metric:



$$b_E := \text{ord}_E(m)$$

$$\text{dist}(E, F) := \frac{1}{b_E b_F} \quad (E, F \text{ adjacent})$$

Compatibility:  $\pi' \geq \pi \Rightarrow \Gamma_\pi \hookrightarrow \Gamma_{\pi'}$ ,  
order-preserving isometry



## Dual graphs II

Can embed  $\Gamma_\pi$  inside  $\gamma$ :

Vertices:

$E$  exc prime  $\mapsto$  normalized div. val'n  
 $v_E := b_E^{-1} \text{ord}_E$

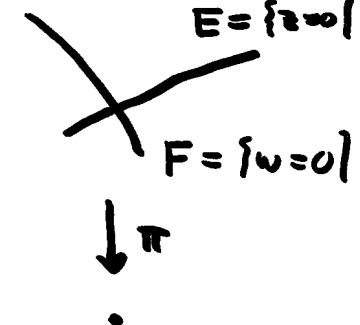
Edges:

$[E, F]$  edge in  $\Gamma_\pi$

Parametrization:  $sb_E + tb_F = 1 \quad s, t \geq 0$

Associate to  $(s, t)$  val'n  $v_{s,t}$   
 which is monomial in  $(z, w)$ :

$$v_{s,t} (\sum a_{ij} z^i w^j) = \min_{a_{ij} \neq 0} \{ si + tj \}$$



Get embedding  $\Gamma_\pi \hookrightarrow \gamma$

(Compatibility:  $\Gamma_\pi \hookrightarrow \Gamma_{\pi'} \hookrightarrow \gamma$ )

Def:  $\gamma_{qm} := \varinjlim_\pi \Gamma_\pi$       quasimonomial val'n  
 (union)

## Dual graphs III

Given blowup  $\pi: X_\pi \rightarrow (\mathbb{C}^2, 0)$   
 and valuation  $v \in \mathcal{V}$  can define  
 center of  $v$  on  $X_\pi$ :

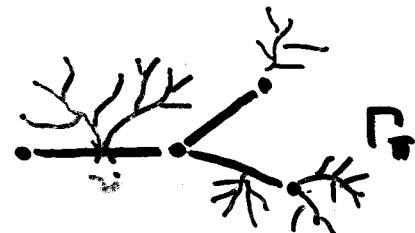
a)  $v = v_E \Rightarrow$  center = E

b) otherwise  $\exists! p \in X_\pi$  s.t.  $v > 0$  on  $M_p$   
 center = p

Refined construction  $\rightarrow$  retraction  $r_\pi: \mathcal{V} \rightarrow \Gamma_\pi$

$$\boxed{\text{Thm: } \mathcal{V} := \varprojlim \Gamma_\pi}$$

Cor:  $\mathcal{V}$  is an R-tree  
 (partial ordering + metric)



Rem:  $\mathcal{V}_{qm}$  is "interior" of  $\mathcal{V}$ .

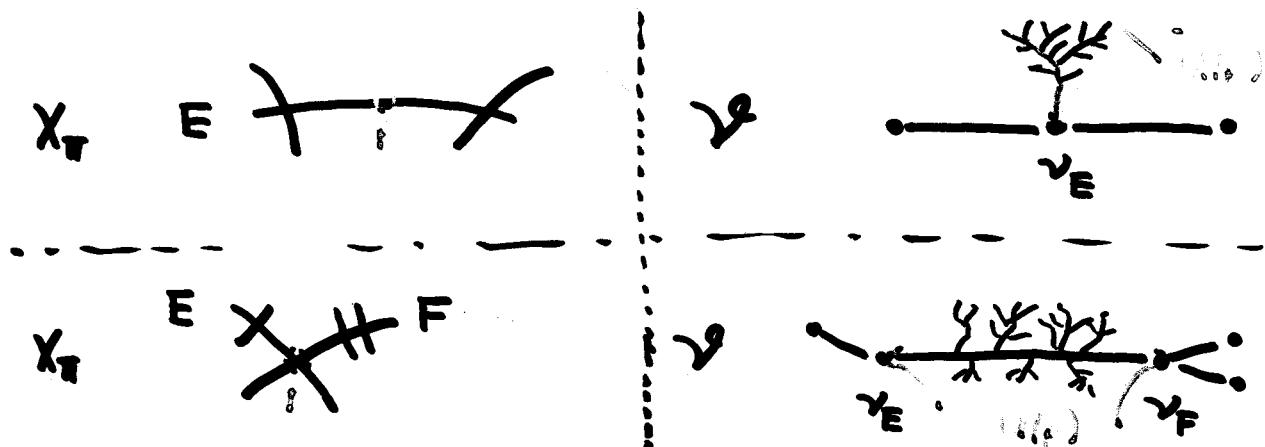
Rem: 2 types of non-quasimonomial val's:  
 • curve valuations  
 • infinitely singular valuations

## Geometric partitions of $\mathcal{V}$

Given blowup  $\pi: X_\pi \rightarrow (\mathbb{C}^2, 0)$

and a point  $p \in \pi^{-1}(0)$  "infinitely near pt"  
get open subset  $U(p) \subseteq \mathcal{V}$

$$U(p) = \{\nu \mid \text{center of } \nu \text{ on } X_\pi \text{ is } p\}$$



This gives partition of  $\mathcal{V}$  into :

- vertices  $\nu_E$  of  $\Gamma_\pi$
- open subsets  $U(p)$  ( $p \in \pi^{-1}(0)$ )

Rem: replacing  $\pi$  by  $\pi' \geq \pi$   
 $\Rightarrow$  refining partition

## Recognizing indeterminacy points

$$f: (\mathbb{C}^2, 0) \hookrightarrow$$

induced map  $f_*: \mathcal{V} \rightarrow \mathcal{V}$

$\pi, \pi'$  blowups of  $(\mathbb{C}^2, 0)$

$$\begin{array}{ccc} X_\pi & \dashrightarrow & X_{\pi'} \\ \pi \downarrow & & \downarrow \pi' \\ (\mathbb{C}^2, 0) & \xrightarrow{f_*} & (\mathbb{C}^2, 0) \end{array}$$

$$p \in \pi^{-1}(0) \quad , \quad p' \in (\pi')^{-1}(0)$$

$$\cancel{X} \dashrightarrow \cancel{X}$$

Then:  $\left\{ \begin{array}{l} \tilde{f} \text{ holo at } p \\ \tilde{f}(p) = p' \end{array} \right. \iff f_*(U(p)) \subset U(p')$

$$\cancel{X} \xrightarrow{f_*} \cancel{X}$$

## Preservation of tree structure

①  $\gamma$  (ordered) segment in  $\mathcal{Y}$   
 s.t.  $c(f, \cdot)$  constant on  $\gamma$

$$v \leq v' \Leftrightarrow \\ v(t) \leq v'(t) \forall t$$

$$\Rightarrow$$

$f_0$  maps  $\gamma$  homeom onto segment  $\gamma'$

$$\gamma \xrightarrow{f_0} \gamma'$$

"Pf":  $\begin{cases} f_0 \text{ order-preserving} \\ c(f, \cdot) \text{ constant on } \gamma \end{cases} \Rightarrow f_0 \text{ o-p on } \gamma$

②  $\exists$  finite subtree  $T_f \subset \mathcal{Y}$  s.t. :

a)  $c(f, \cdot)$  locally constant outside  $T_f$

b)  $T_f$  can be decomposed into finitely many segments on each of which  $f_0$  is a homeomorphism.

"Pf": Use monomialization  $\exists \pi, \pi'$

$$\begin{array}{ccc} X_\pi & \xrightarrow{\tilde{f}} & X_{\pi'} \\ \downarrow & & \downarrow \\ (\mathbb{C}^2, 0) & \xrightarrow{\tilde{f}} & (\mathbb{C}^2, 0) \end{array} \quad \begin{array}{l} \tilde{f} \text{ hol + locally} \\ \text{monomial} \end{array}$$

## Existence of eigenvaluation

Thm:  $\exists v_* \in \mathcal{V}$  s.t.  $f_* v_* = v_*$

Moreover, either :

- 1)  $v_*$  quasimonomial (not endpoint in  $\mathcal{V}$ )
- 2)  $v_*$  not qm , locally attracting



Pf: Arboreal  $\square$

## Growth of $c(f^n)$

Assume  $v_x$  quasimonomial

$$f \cdot v_x = v_x \Rightarrow f^* v_x = \lambda v_x \quad \lambda > 0$$

$$\Rightarrow c(f^n, v_x) = v_x(f^{n+m}) = \lambda^n$$

Izumi-Skoda inequality ( $v_x$  qm)

$$\exists \delta > 0 \text{ s.t. } \delta v_x \leq \text{ord}_0 \leq v_x$$

... etc ...

$$\Rightarrow \begin{cases} c(f^n) = \text{ord}_0(f^{n+m}) \geq \delta \cdot \lambda^n \\ c(f^n) \leq \lambda^n \end{cases}$$

$$\Rightarrow c_\infty = \lambda \text{ and } \delta c_\infty^n \leq c(f^n) \leq c_\infty^n$$

i.e Thm A

(modify argument when  $v_x$  not qm)

# Normal forms I

Assume  $\gamma_\infty$  irrational quasimonomial  
(nondegenerate)



$f_*$  must map  $I$  into  $\mathbb{F}$  for tree reasons.

Assume  $f_*$  order-preserving at  $y_\infty$   
(replace  $f$  by  $f^2$  if not)

Claim:  $f_* I \subset I$  for any sufficiently small segment  $I \ni y_\infty$ .

"Pf" In parametrization of  $I$ ,  $f_*$  given by  $t \mapsto \frac{at+b}{ct+d}$   $a,b,c,d \in \mathbb{N}$  □

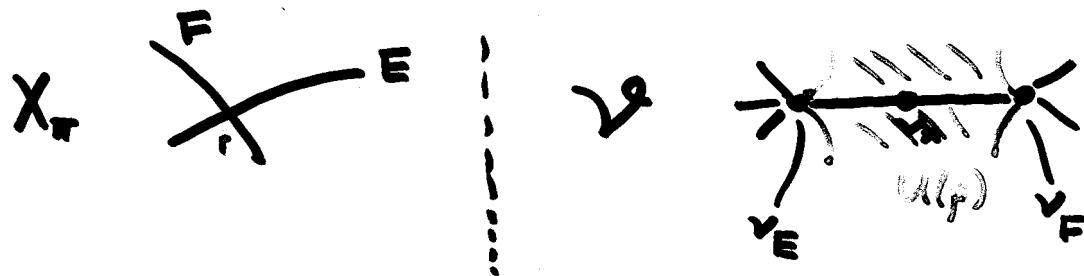
Claim: For  $I$  small enough,  $f_* U \subset U$

Pf: Preservation of tree str.



## Normal forms II

Can choose  $U$  of "geometric" form  $U = U(p)$



$$\Rightarrow \begin{cases} \tilde{f}: X_\pi \dashrightarrow X_\nu \text{ holo at } p \\ \tilde{f}(p) = p \end{cases}$$

Can make sure  $\begin{cases} C_{\tilde{f}} \subset E \cup F \\ \tilde{f}(E \cup F) \subset E \cup F \end{cases}$  locally

$\Rightarrow \tilde{f}: (X_\pi, p) \hookrightarrow$  rigid

Can make  $\tilde{f}(z, w) = (z^a w^b, z^c w^d)$

$\Rightarrow \lambda_1 = \text{spectral radius of } \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\Rightarrow$  Thm B and Thm C !