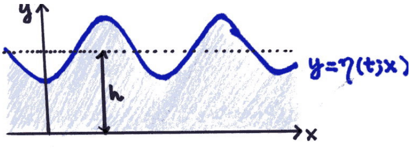
Local smoothing effect for the water-wave problem with surface tension

G. Staffilani (MIT)
joint with H. Christianson (MIT) and V. Hur (MIT)

Mathematical formulation

In the simplest setting, in 2D fluid region
$\{(x, y) : -\infty < y < \eta(t; x)\}$



the velocity $(u(t; x, y), v(t; x, y))$ & the pressure $P(t; x, y)$ satisfy
the Euler equations

$$\boxed{u_x + v_y = 0, \quad \begin{aligned} u_t + uu_x + vu_y &= -P_x, \\ v_t + uv_x + vv_y &= -P_y - g; \end{aligned}}$$

$g > 0$ is the gravity constant.

The **vorticity** $\boxed{\omega = v_x - u_y}$ measures the local swirl or eddy.

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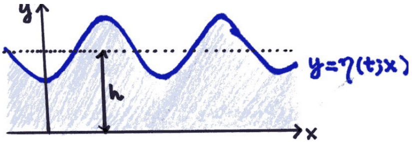

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The wave profile is a priori unknown! **Free-boundary problem**.

The **top** boundary conditions on $y = \eta(t;x)$ are

$$\boxed{\begin{array}{rcll} v &=& \eta_t + u\eta_x & \text{(kinematic)}, \\ P &=& P_{atm} + S\dfrac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} & \text{(dynamic)}; \end{array}}$$

where $S > 0$ is the coefficient of surface tension.

The **bottom** and **horizontal** boundary conditions are

$$\boxed{\begin{array}{ll} (u,v) \to (0,0) & \text{as } y \to -\infty \\ \eta \to 0 & \text{as } |x| \to \infty. \end{array}}$$

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Boundary evolution

In case $\omega \equiv 0$ (irrotational setting), with **velocity potential** $\boxed{\phi_x = u, \quad \phi_y = v}$ the problem reduces to

$$\boxed{\begin{array}{lll} \Delta\phi = 0 & \text{in} & -d < y < \eta(t;x), \\ \eta_t = \phi_y - \phi_x\eta_x & \text{on} & y = \eta(t;x), \\ \phi_t = -g\eta + S\dfrac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} - \frac{1}{2}|\nabla\phi|^2 & \text{on} & y = \eta(t;x), \\ \nabla\phi \to 0 & \text{as} & |x| \to \infty. \end{array}}$$

- The evolutionary nature is on the **free surface**.
- In case of general vorticities (rotational setting), **additional nonlinearity** in the field equation.
- Why vorticity? (i) Real flows possess vorticities. (ii) Stagnation (Ko-Strauss), Stability (Lin-Hur.)

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Time evolution of the wave profile and velocity distribution pose as an **initial value problem** provided with the initial data. For this initial value problem one studies **well-posedness**. Well-posedness usually summarizes existence, uniqueness and continuity properties in time and with respect to initial data.

Chief difficulties are:
(i) possible **Rayleigh-Taylor instability** of the linear part and
(ii) severe nonlinearity (**quasilinear**).

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Overview on the Cauchy problem

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Brief history, in the Sobolev setting

gravity waves, $S = 0$

- Nalimov (1974), Yosihara (1982): local well-posedness when the free surface is small localized perturbation from being flat.

  The smallness implies the **Taylor-Young inequality**
  $$\boxed{\nabla P \cdot \hat{\mathbf{n}} < 0}$$
  (connected to the Rayleigh-Taylor instability).

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Capillary waves, $S > 0$

- Yosihara (1983), Iguchi & Tani (2002): local well-posedness for small data

- Ambrose & Masmoudi (2005) for general data, Shatah & Zeng (06, 08) with geometric approach.

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Question

Due to its nonlinearity, all existing well-posedness results relative to the water-wave initial value problem rely on **energy estimate** only. Further analysis would help to better understand the behavior of solutions.

Surface tension ($S > 0$) has certain **regularizing** effects in the linear analysis.
Is it possible to use them in in order to prove that the profile of the wave gains some smoothness in later time (Smoothing Effect)?

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FURTHER FORMULATION WHEN $\omega \equiv 0$

Let $\psi(t,x) = \phi(t,x,\eta(t,x))$. With **Dirichlet-Neumann operator**

$$G(\eta)\psi := \sqrt{1+(\partial_x\eta)^2}\,\frac{\partial\phi}{\partial n}\Big|_{y=\eta(t,x)},$$

the system may be written: for $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$

$$\begin{aligned}
\partial_t\eta &= G(\eta)\psi,\\
\partial_t\psi &= -g\eta + S\frac{\partial_x^2\eta}{(1+(\partial_x\eta)^2)^{3/2}}\\
&\quad -\frac{1}{2}(\partial_x\psi)^2 - \frac{1}{2}\frac{1}{(1+(\partial_x\eta)^2)}(G(\eta)\psi + \partial_x\eta\,\partial_x\psi)^2.
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Further formulation when $\omega \equiv 0$

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Dispersion relation

Linearize around $\eta \equiv 0$ & $\psi \equiv 0$ (flat surface, no motion) to obtain

$$\partial_t \eta = H \partial_x \psi, \qquad \partial_t \psi = -g\eta + S\partial_x^2 \eta.$$

H is the Hilbert transform, $\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi)\hat{f}(\xi)$. Equivalently,

$$\boxed{\partial_t^2 \psi = -gH\partial_x \psi + SH\partial_x^3 \psi.}$$

Compare to the wave equation $\partial_t^2 \psi - \partial_x^2 \psi = 0$.

Dispersion relation:

without surface tension	with surface tension
$c(\xi) = \sqrt{g/\lvert\xi\rvert}$	$c(\xi) = \sqrt{S\lvert\xi\rvert}$
	$\downarrow$
	smoothing effects!

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Finally, our formulation (inspired by Ambrose & Masmoudi)

(The interface is a general parametrized curve.)

$$\boxed{\partial_t^2 u - H\partial_\alpha^3 u = -2u\partial_\alpha\partial_t u - u^2\partial_\alpha^2 u + R(t,\alpha),}$$

(weakly) coupled with

$$\partial_t\theta = -u\partial_\alpha\theta + H\partial_\alpha u + r(t,\alpha).$$

$\alpha \in \mathbb{R}$: arclength parametrization,
u: modified tangential velocity,
θ: angle between tangent & x-axis.

Remainders satisfy

$$\|r(t,\alpha)\|_{H^s} \le C(\|u\|_{H^s}, \|u_t\|_{H^{s-1}}),$$
$$\|R(t,\alpha)\|_{H^s} \le C(\|u_t\|_{H^s}, \|u_\alpha\|_{H^s}),$$

that is, they are lower order terms.

Kato's smoothing for linear equation

Solution of

$$\begin{cases} (\partial_t^2 - H\partial_\alpha^3)u = 0, \\ u(0,\alpha) = u_0(\alpha) \quad \text{and} \quad \partial_t u(0,\alpha) = u_1(\alpha) \end{cases}$$

is given via Frourier transform

$$\hat{u}(t,\xi) = \hat{u}_0(\xi)\cos|\xi|^{3/2}t + \hat{u}_1(\xi)\frac{\sin|\xi|^{3/2}t}{|\xi|^{3/2}}$$
$$:= \hat{W}_0(t,\xi)\hat{u}_0(\xi) + \hat{W}_1(t,\xi)\hat{u}_1(\xi).$$

Smoothing effects for W_0:

(i) $\|\partial_\alpha^{1/4} W_0(t)u_0\|_{L^\infty_\alpha L^2_t} \le C\|u_0\|_{L^2_\alpha}$,

(ii) $\|\partial_\alpha^{1/4} \int_0^t W_0(t-t')f(\cdot,t')dt'\|_{L^\infty_t L^2_\alpha} \le C\|f\|_{L^1_\alpha L^2_t}$,

(iii) $\|\partial_\alpha^{1/2} \int_0^t W_0(t-t')f(\cdot,t')dt'\|_{L^\infty_\alpha L^2_t} \le C\|f\|_{L^1_\alpha L^2_t}$.

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Proof of (i). Let $\hat{\mathcal{U}}(t) = e^{i\xi^{3/2}t}$. Then,

$$\begin{aligned}\partial_\alpha^{1/4}\mathcal{U}(t)u_0(\alpha) &= c\int e^{i(\xi^{3/2}t+\alpha\xi)}\xi^{1/4}\hat{u}_0(\xi)d\xi \\ &= c\int e^{i(\eta t+\alpha\eta^{2/3})}\eta^{-1/6}\hat{u}_0(\eta^{2/3})d\eta.\end{aligned}$$

By Plancherel,

$$\begin{aligned}\int |\partial_\alpha^{1/4}\mathcal{U}(t)u_0|^2 dt &= \int |\eta|^{-1/6}|\hat{u}_0(\eta^{2/3})|^2 d\eta \\ &= c\int |\hat{u}_0(\xi)|^2 d\xi = c\|u_0\|_{L^2}^2.\end{aligned}$$

SMOOTHING EFFECTS FOR INHOMOGENEOUS EQUATION

Solution to

$$\begin{cases} (\partial_t^2 - H\partial_\alpha^3)u = f(t,\alpha), \\ u(0,\alpha) = 0 \quad \text{and} \quad \partial_t u(0,\alpha) = 0 \end{cases}$$

is given via Duhamel's principle as

$$u(t,\alpha) = \int_0^t W_1(t-t')f(t',\alpha)dt', \qquad \hat{W}_1(t,\xi) = \frac{\sin|\xi|^{3/2}t}{|\xi|^{3/2}}.$$

This has the smoothing property

$$\boxed{\|\partial_\alpha^2 u\|_{L_\alpha^\infty L_t^2} \leq C\|f\|_{L_\alpha^1 L_t^2}.}$$

The smoothing effects make up for **two** derivatives in nonlinearity. HOWEVER, $u\partial_\alpha\partial_t u$ has $2+1/2$ derivatives in α. More than what smoothing effects can treat.

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- Surface tension effects produce strong nonlinear effects as well as dispersion property. Indeed, $\partial_t \sim \partial_\alpha^{3/2}$.

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Parametrix to linear problem

View the equation as

$$\boxed{(\partial_t^2 + 2u\partial_\alpha\partial_t + u^2\partial_\alpha^2 - H\partial_\alpha^3)u = R(t,\alpha).}$$

That is to say, view $u\partial_\alpha\partial_t u + u^2\partial_\alpha^2 u$ as a "linear" component of the equation too, but with a **variable coefficient** which happens to depend on the solution itself.

Establish analogous Kato's smoothing effects for the equation

$$\boxed{(\partial_t^2 + 2c(t,\alpha)\partial_\alpha\partial_t + c^2(t,\alpha)\partial_\alpha^2 - H\partial_\alpha^3)u = 0,}$$

where $c(t,\alpha)$ is a (variable) coefficient function, via techniques in **microlocal analysis**.

Why is there a chance that this may work? If one goes back to the linear equation

$$(\partial_t^2 - H\partial_\alpha^3)u = 0$$

one can see that the space time Fourier transform of the solution u lives on the surface

$$\Omega = \{(\tau, \xi)/\tau^2 - |\xi|^3 = 0\},$$

and the "curvature" of this surface is responsible for the smoothing effect presented above. If one assumes that $c(t, \alpha) = c$, then the space time Fourier transform of the solution u of

$$(\partial_t^2 + 2c\partial_\alpha\partial_t + c^2\partial_\alpha^2 - H\partial_\alpha^3)u = 0$$

lives on the surface

$$\tilde{\Omega} = \{(\tau, \xi)/\tau^2 + c\xi\tau + c^2\xi^2 - |\xi|^3 = 0\},$$

that geometrically looks very much like Ω.

What one can prove in fact is the following: the solution of

$$\begin{cases} (D_t^2 + 2c(t,\alpha)D_\alpha D_t + c^2(t,\alpha)D_\alpha^2 + iHD_\alpha^3)u = 0, \\ u(0,\alpha) = u_0(\alpha), \qquad \partial_t u(0,\alpha) = u_1(\alpha) \end{cases}$$

($D_t = -i\partial_t$ and $D_\alpha = -i\partial_\alpha$) satisfies

$$\|D_\alpha^{1/4} u\|_{L^\infty_\alpha L^2_t} \leq C(\|u_0\|_{L^2} + \|u_1\|_{H^{-3/2}}),$$

and the solution to

$$\begin{cases} (D_t^2 + 2c(t,\alpha)D_\alpha D_t + c^2(t,\alpha)D_\alpha^2 + iHD_\alpha^3)u = f(t,\alpha), \\ u(0,\alpha) = 0, \qquad \partial_t u(0,\alpha) = 0 \end{cases}$$

satisfies

$$\boxed{\|D_\alpha^{2-\epsilon} u\|_{L^\infty_\alpha L^2_t} \leq C t^{4\epsilon/3} \|f\|_{L^2_\alpha L^2_t},}$$

where C is a polynomial in $\|c\|_{W^{s_\alpha,\infty}_\alpha W^{s_t,\infty}_t}$.

Putting thing together

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energy method $\Rightarrow$ local well-posedness of solution to

$$\begin{cases} (\partial_t + u\partial_\alpha)^2 u - H\partial_\alpha^3 u = R(t,\alpha) \\ u(0,\alpha) = u_0(\alpha), \qquad \partial_t u(0,\alpha) = u_1(\alpha) \end{cases}$$

in $(u, \partial_t u) \in C([0,T]; H^s(\mathbb{R})) \times C([0,T]; H^{s-3/2}(\mathbb{R}))$ for $s > s_0$ for some s_0, satisfying

$$\| < \alpha >^{-s-1} \partial_\alpha^{s+1/4} u\|_{L^2_\alpha L^2_T} < \infty$$

(smoothing property of solutions).

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