# Space Time Resonances

Pierre Germain
Nader Masmoudi
Jalal Shatah
Courant Institute
New York University

## How to guess which nonlinear dispersive equation behaves as a linear one for small data?

- $\partial_t^2 u - \Delta u = -u^3 \quad x \in \mathbb{R}^3$. All solutions exist globally and asymptotically behave like linear solutions. Same result holds for the Klein-Gordon equation (C. Morawetz and W. Strauss 1972).
- $\partial_t^2 u - \Delta u = u^2 \quad x \in \mathbb{R}^3$. Fritz John 1979 proved that small solutions blow up. (Generalization of this result to higher dimension was given by Walter Strauss)
- $\partial_t^2 u - \Delta u = |\partial_t u|^2 - |\nabla u|^2 \quad x \in \mathbb{R}^3$. Small solution exist globally and behave like linear solutions $v = e^u - 1$. More generally, if the nonlinearity satisfies the null condition we have global existence of small solutions, D. Christodoulou , S. Klainerman 1984.

- $i\partial_t u - \Delta u = u^2 \quad (\text{or} \ \ \bar{u}^2) \quad x \in \mathbb{R}^3$. Small solutions behave like linear solutions. Hayashi, Mizumachi, and Naumkin 2002
- $i\partial_t u - \Delta u = |u|^2 \quad x \in \mathbb{R}^3$. Small solutions do not behave like a linear solutions.
- $i\partial_t u - \Delta u = |u|^2 u \quad \text{vs} \ \ u^3 \quad x \in \mathbb{R}$. T. Ozawa 1991

**The Aim of the Talk**

To derive and explain all of these results and to prove new results from a simple computation involving *space time* resonant sets.

## Time resonance

*Time resonance* is an ODE phenomena.

$$\begin{cases} \partial_t u = 2iu + v^2 \\ \partial_t v = iv \end{cases}$$

The solution $u$ behaves very differently from the linear equation.

$$u = u_0 e^{2it} + v_0^2 t e^{2it}$$

## Poincaré - Dulac Normal Forms

Given an analytic ODE

$$\dot{u} = Au + f(u) = Au + M_2(u,u) + \dots \quad u \in \mathbb{R}^d$$

Find an analytic transformation

$$v = G(u) = u + H_2(u) + H_3(u) + \dots$$

where $H_k$ is $k$ multilinear map of $u$, that transform solutions of the above equation to solutions of the linear system

$$\dot{v} = Av$$

## Poincaré - Dulac Normal Forms (continued)

Plug $v = G(u) = u + H_2 + \dots$ into the equation to get

$$G'(u)(Au + f(u)) = AG(u)$$

and expand in a power series to find the quadratic term of $G$

$$H_2(Au, u) + H_2(u, Au) + M_2(u, u) = AH_2(u, u),$$

which can be solved provided $\lambda_{k_1} + \lambda_{k_2} \neq \lambda_k$. Repeating this process for the higher order terms of $G$ leads to the condition on the eigenvalues $\lambda_k$ of the matrix $A$

$$\lambda_{k_1} + \lambda_{k_2} \cdots + \lambda_{k_\ell} \neq \lambda_k$$

## Poincaré - Dulac Normal Forms (continued)

Our way of computing $G(u)$ or more precisely $G^{-1}(v)$ is to look at the integral equation

$$u = e^{At}u_0 + e^{At}\int_0^t e^{-As}f(u(s))ds$$

and write it in terms of the profile of $u$, i.e., $w = e^{-At}u$

$$w = u_0 + \int_0^t e^{-As}f(e^{As}w(s))ds = u_0 + \int_0^t e^{-As}M_2(e^{As}w, e^{As}w) + \ldots ds$$

Each $i$-multilinear term consists of matrices whose eigenvalues are given by

$$e^{(\lambda_{k_1}+\lambda_{k_2}\cdots+\lambda_{k_i}-\lambda_k)s}$$

## Poincaré - Dulac Normal Forms (continued)

Since $\dot{w} = e^{-At} M_2(e^{At}w, e^{At}w) + \dots$ which is at least quadratic in $w$, then integrating by parts on $w$ in

$$\int_0^t e^{-As} M_2(e^{As}w, e^{As}w) + \dots ds$$

eliminates $M_2$ in favor of a cubic term. If

$$\lambda_{k_1} + \lambda_{k_2} \cdots + \lambda_{k_i} - \lambda_k \neq 0$$

then $w$ is a function of $e^{-At}\tilde{M}_2(e^{At}w, e^{At}w) + \dots$. Otherwise we get power of $t$ which makes the behavior of $u$ and $e^{At}v_0$ quit different.

## Time resonance for PDEs

The same analysis can be applied to PDEs which are translation invariant with eigenvalues and eigenvectors substituted by $\xi$ and $e^{-i\xi\cdot x}$ (Shatah 1985).

EXAMPLE:

$$i\partial_t u - \Delta u = u^2$$

Writing $\hat{u}$ for the Fourier transform, we have the equation for $\hat{w} = e^{-i|\xi|^2 t}\hat{u}$

$$\hat{w}(\xi, s) = \hat{u}_0(\xi) - i \int\limits_{\xi_1+\xi_2=\xi} \Big( \int\limits_0^t e^{i(-\xi^2+\xi_1^2+\xi_2^2)s}\hat{w}(\xi_1, s)\hat{w}(\xi_2, s)ds\Big)$$

## Time resonance for PDEs

(Example continued)

Thus if we let $\phi = -\xi^2 + \xi_1^2 + \xi_2^2$ then if $\phi \neq 0$ we can integrate by parts in $s$ and eliminate the quadratic term in favor of a cubic term. Thus quadratic time resonances correspond to

$$\mathcal{T} = \{(\xi_1, \xi_2); \quad \phi = 0 \ \text{ whenever } \ \xi_1 + \xi_2 = \xi\}.$$

Unfortunately the set $\mathcal{T}$ is too big, it corresponds to $\xi_1 \cdot \xi_2 = 0$, which means that the quadratic term is relevant and can not be eliminated via a transform.

## Time resonance for PDEs

(Example continued)

However we *know* that small solutions to

$$i\partial_t u - \Delta u = u^2$$

behave like linear solutions, Hayashi, Mizumachi, and Naumkin. So the normal forms method fails for this PDE.

## Time resonance for PDEs

EXAMPLE $\quad \Box u + u = Q(u,u) \qquad x \in \mathbb{R}^3.$

$\phi \geq \sqrt{|\xi_1|^2 + |\xi_2|^2 + 1}$, which implies no time resonance. Consequently we can eliminate the quadratic term in favor of the cubic.

EXAMPLE $\quad \Box u =$ Null form $\qquad x \in \mathbb{R}^3.$

Null form $\approx |\partial_t u|^2 - |\nabla u|^2 \Rightarrow \phi = 1.$

All other null forms give $\phi \approx |\xi_1 \wedge \xi_2|$.

## Time resonance for PDEs

EXAMPLE

$$\begin{cases} \partial_t u - \partial_x u = uv \\ \partial_t v - c\partial_x v = u^2 \end{cases}$$

$\phi_u = (1-c)\xi_2$ which always vanishes. However L. Tartar (1981) proved that if $c \neq 1$ we have global existence for small data.

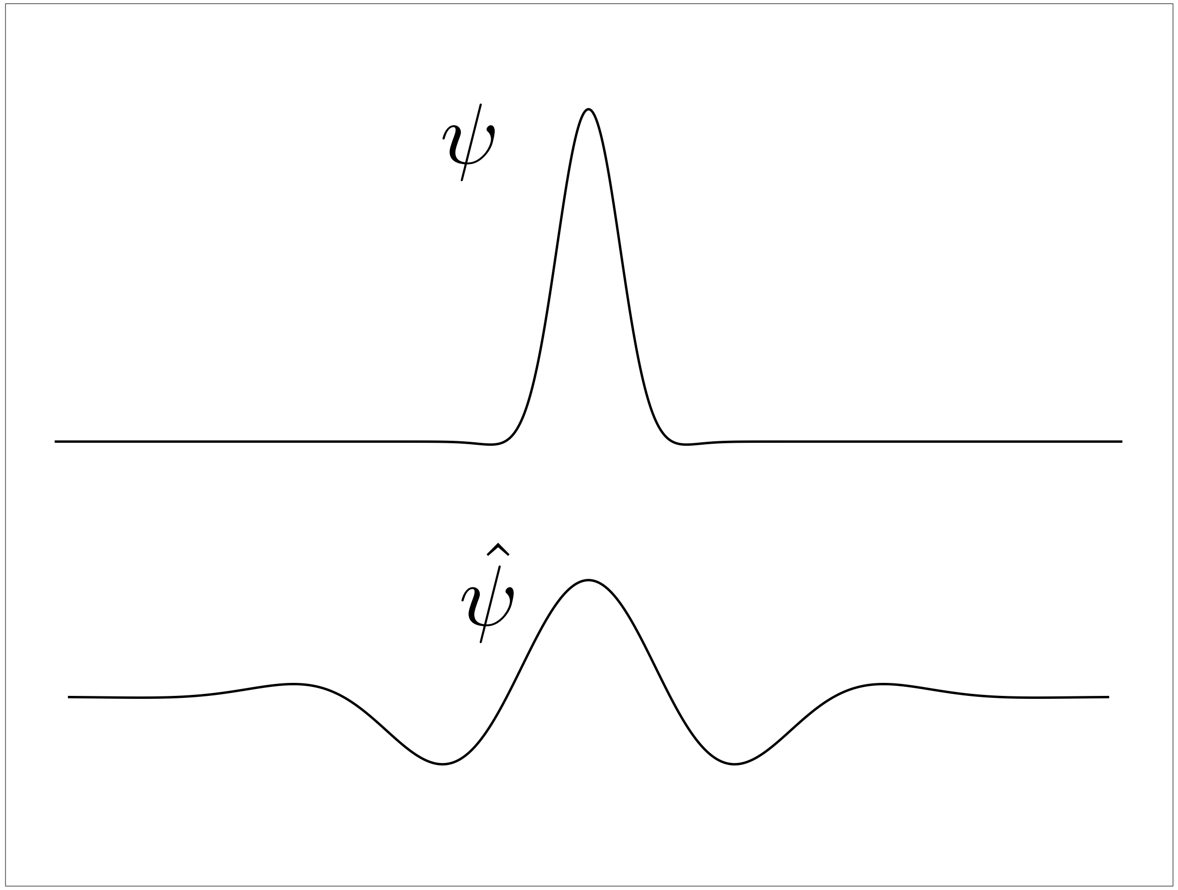
Too many failures and too few successes. Are we missing something? *Yes.*

## Plane waves vs wave packets

The problem lies in the way we computed resonances. We considered eigenvectors $e^{-i\xi\cdot x}$ which are not in the space of functions where nonlinear solutions behave like linear ones. For that to happen we need to consider functions that decay at spatial infinity.

Thus instead of $e^{-i\xi\cdot x}$ we need to consider a wave packet $\psi$, i.e., a smooth function on $\mathbb{R}^d$ which is strongly localized in space and in frequency.

# Wave Packets




Instead of compact support we can take Gaussians

$$\psi = c2^{kd}e^{ix\cdot\xi_0}e^{-\frac{|x-x_0|^2 2^{2k}}{4}} \qquad \hat{\psi} = e^{ix_0\cdot(\xi-\xi_0)}e^{-\frac{|\xi-\xi_0|^2}{2^{2k}}}$$

## Wave packets in dispersive equations

Thus if we propagate a wave packet, adapted to the origin with frequency $\xi_1$, by an equation

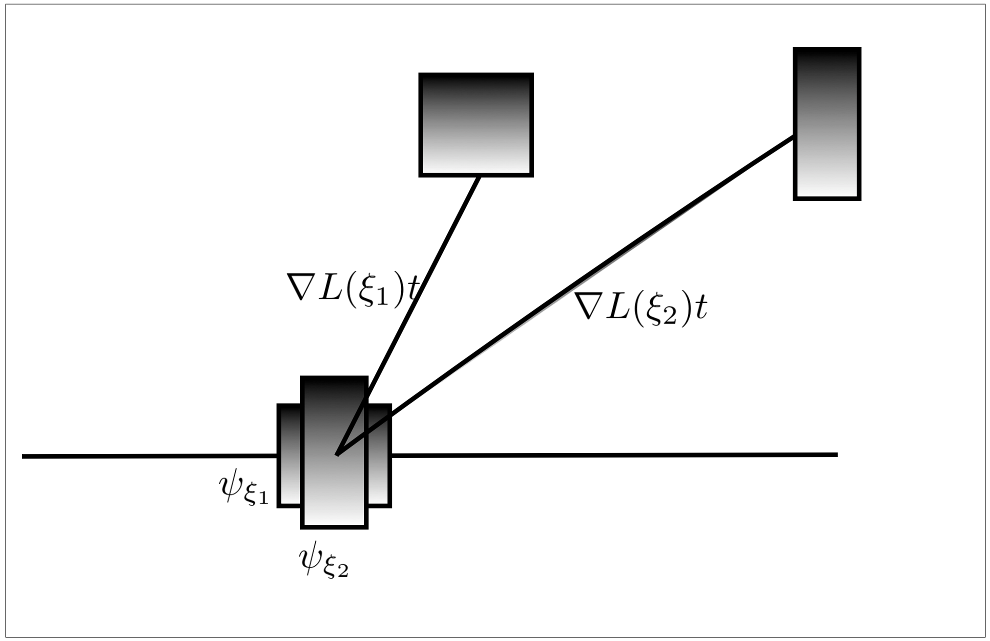
$$\partial_t u = iL(\frac{1}{i}\nabla)u$$

the solution after time $t$ is given by

$$\hat{u} = e^{iL(\xi)t}\psi \approx e^{iL(\xi_1)t}e^{i\nabla L(\xi_1)\cdot(\xi-\xi_1)t}\hat{\psi}$$

Thus the solution u is basically supported around $\nabla L(\xi_1)t$.

# Group velocity $\nabla L(\xi)$




Thus if we have two time resonant frequencies $\xi_1$ and $\xi_2$ they might be located at different places in space. And if the solutions are spatially localized, these time resonant frequencies wont even have a chance to feel the effects of one another.

## Space Time Resonances

Two wave packets adapted to a region $I$ are spatially resonant if after time $t$ they occupy essentially the same region in space, i.e., they have the same group velocity.

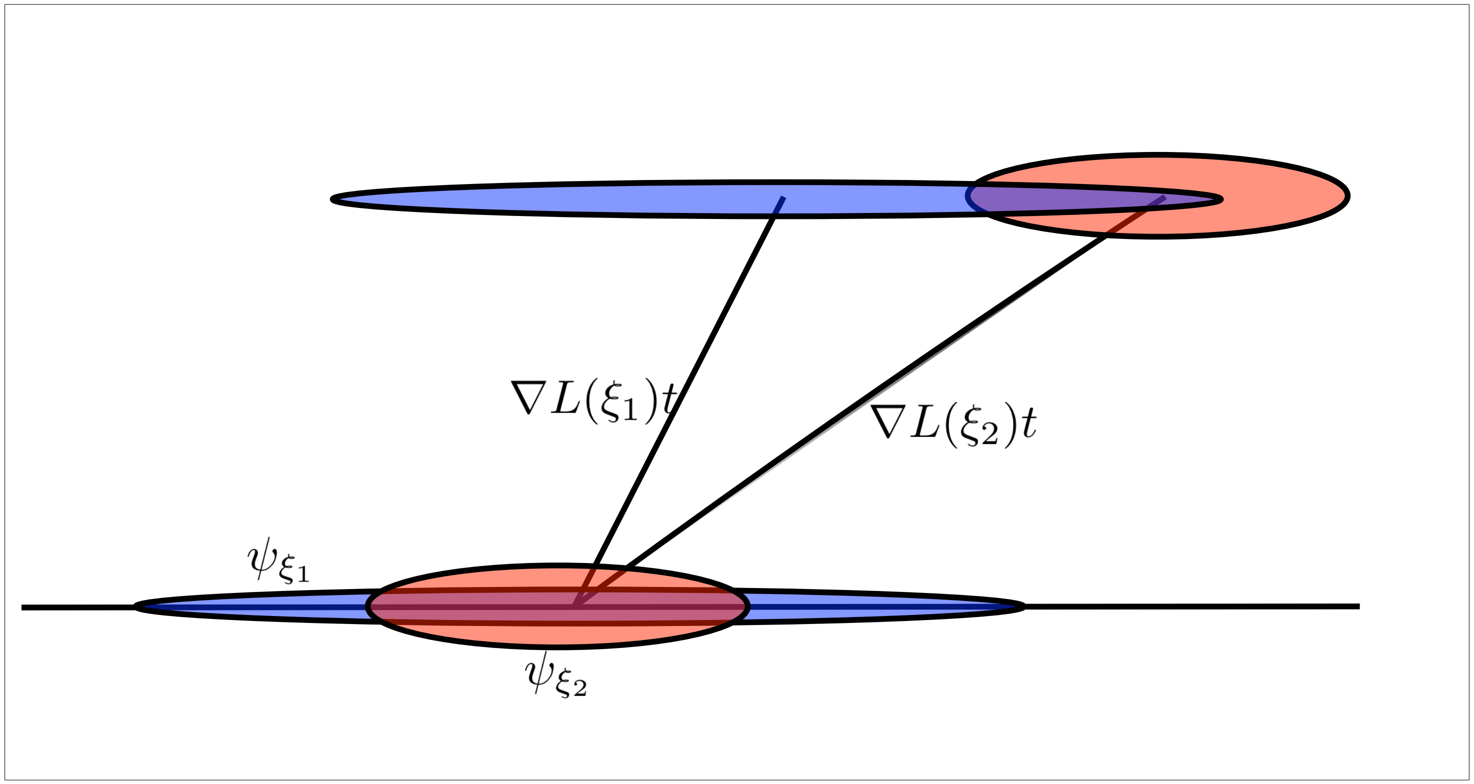
- Space resonance $\mathcal{S} = \{(\xi_1, \xi_2); \nabla L(\xi_1) = \nabla L(\xi_2)\}$.
- Time resonance $\mathcal{T} = \{(\xi_1, \xi_2); L(\xi_1) + L(\xi_2) = L(\xi_1 + \xi_2)\}$.
- Space Time resonance $\mathcal{R} = \mathcal{T} \cap \mathcal{S}$.

How do space resonances impact the behavior of solutions to the nonlinear equation

$$\partial_t u = iL(\frac{1}{i}\nabla)u + u^2$$

## Space resonances

If two frequencies $\xi_1$ and $\xi_2$ are time resonant but not space resonant, then




they resonate through the tail. Thus if the tail is sufficiently small the growth from resonance will be nullified by the smallness of the tail. The smallness of the tail can be measured by using weighted norms.

## How to prove global existence and scattering?

$$\partial_t u = iL(\frac{1}{i}\nabla)u + u^2$$

Let $\phi = -L(\xi) + L(\eta) + L(\xi - \eta)$ be the quadratic frequency interaction.

$$\hat{w}(\xi, s) = \hat{u}_0(\xi) - i \int_0^t \Big( \int e^{i\phi(\xi,\eta)s} \hat{w}(\eta, s)\hat{w}(\xi - \eta, s)d\eta\Big) ds$$

• On the space time resonant set $\mathcal{R}$ we apply stationary phase to see if the set is small enough so that growth due to resonant behavior can be dominated by the size of the resonant set.

$$\int_0^t \int e^{i\phi(\xi,\eta)s} \hat{w}(\eta, s)\hat{w}(\xi - \eta, s)d\eta ds \approx \int^t \frac{1}{s^\alpha} ds$$

## How to prove global existence and scattering? (continued)

• On the set $\{\phi \neq 0\}$, i.e., no time resonances, integrate by parts with respect to $s$. This introduces a factor of $\frac{1}{\phi}$ and changes the equation to cubic.

$$\int_0^t \int e^{i\phi(\xi,\eta)s} \hat{w}(\eta, s)\hat{w}(\xi - \eta, s) d\eta ds \approx \int_0^t \Big( \int e^{i\phi(\xi,\eta)s} \frac{1}{\phi} \mathcal{O}(\hat{w}^3) d\eta ds$$

• On the set $\{\phi \neq 0\}$ but $\{\nabla_\eta \phi \neq 0\}$, i.e., time resonances but no space resonances, integrate by parts with respect to $\eta$. This introduces an additional decay of $\frac{1}{t}$ and a factor of order $\frac{1}{|\nabla_\eta \phi|}$.

$$\int_0^t \int e^{i\phi(\xi,\eta)s} \hat{w}(\eta, s)\hat{w}(\xi - \eta, s) d\eta ds \approx \int_0^t \frac{1}{s} \int e^{i\phi(\xi,\eta)s} \frac{1}{|\nabla_\eta \phi|} \mathcal{O}(\hat{w} \nabla \hat{w}) d\eta ds$$

• The factors $\frac{1}{\phi}$ and $\frac{1}{|\nabla_\eta \phi|}$ are viewed as bilinear operators. These operators are usually of Coifman Meyer type and are harmless.

## Schrödinger equations

EXAMPLE 1 $\quad i\partial_t u - \Delta u = u^2 \qquad x \in \mathbb{R}^3.$

The phase: $\quad \phi = -|\xi|^2 + |\xi - \eta|^2 + |\eta|^2.$

Time resonance $\mathcal{T}$: $\quad \phi = 0 \Rightarrow \eta \cdot (\xi - \eta) = 0.$

Space resonance $\mathcal{S}$: $\quad \nabla_\eta \phi = 0 \Rightarrow \eta = \dfrac{\xi}{2}.$

Space Time resonance $\mathcal{R}$: $\quad \xi = \eta = 0.$

## Schrödinger equations

- Stationary phase to control the behavior near $\mathcal{R}$ ($\phi = 0$ and $\nabla_\eta \phi = 0$):

$$\int_0^t \int e^{i\phi(\xi,\eta)s} \hat{w}(\eta,s)\hat{w}(\xi-\eta,s)d\eta ds \approx \int^t \frac{1}{s^{\frac{3}{2}}}\hat{w}(0,s)^2 ds$$

This gives a bound on $|\hat{w}|_{L^\infty_{t,\xi}}$ which takes the place of bound the profile in $L^1$. The dispersive estimate that we use for the linear equation

$$|e^{-i\Delta t} f|_{L^\infty} \leq \frac{1}{t^{\frac{3}{2}}}|\hat{f}|_{L^\infty} + \frac{1}{t^{\frac{7}{4}}}|x^2 f|_{L^2}$$

- Energy estimates are straight forward since we are in $3$ dimensions.
- Weighted $L^2$ estimates can be achieved by computing $\partial_\xi^\ell \hat{w}$ for $\ell = 1, 2$.

## Schrödinger equations

This gives a simple proof of existence and scattering for small data in $3$ dimensions.

EXAMPLE 2 $\quad i\partial_t u - \Delta u = \bar{u}^2 \qquad x \in \mathbb{R}^3.$

$\phi = -|\xi|^2 - |\xi - \eta|^2 - |\eta|^2$. This has the same space time resonant sets as example 1.

EXAMPLE 3 $\quad i\partial_t u - \Delta u = |u|^2 \qquad x \in \mathbb{R}^3.$

$\phi = -|\xi|^2 + |\xi - \eta|^2 - |\eta|^2 = 2\xi \cdot \eta$. The resonant set is too big: stationary phase imply small data can not behave as linear

## Schrödinger equations

EXAMPLE 4 $\quad i\partial_t u - \Delta u = u\Lambda u \qquad x \in \mathbb{R}^2.$

$\phi = -|\xi|^2 + |\xi - \eta|^2 + |\eta|^2$. Space Time resonance $\mathcal{R}$: $\quad \xi = \eta = 0.$

For $\Lambda = 1$ stationary phase in a neighborhood of $\mathcal{R}$ imply that solutions diverge logarithmically from the linear behavior.

If $\Lambda$ vanishes on $\mathcal{R}$, e.g. $\hat{\Lambda} = \xi$ then we get rid of the logarithmic divergence. However in this case we lose energy estimates.

## Schrödinger equations

EXAMPLE 4 (continued)

By changing $\hat{\Lambda} = \xi$ to $\hat{\Lambda} = \frac{\xi}{\sqrt{1+|\xi|^2}}$ we can close the argument.

- The energy estimate requires the use of *space time resonance* splitting since the decay is borderline.
- Delort has an existence proof for $\hat{\Lambda} = \xi$. His proof uses smoothing estimates which so far we can not incorporate in our method.

## Schrödinger equations

EXAMPLE 5 $\quad i\partial_t u - \Delta u = |u|^2 u \qquad x \in \mathbb{R}.$

$$\hat{w}(\xi,t) = \hat{u}_0(\xi) + i\int_0^t e^{i\phi s}\bar{\hat{w}}(-\xi+\eta+\sigma,s)\hat{w}(\eta,s)\hat{w}(\sigma,s)d\eta d\sigma ds$$

$\phi = -|\xi|^2 - |\xi-\eta-\sigma|^2 + |\eta|^2 + |\sigma|^2.$

$\phi = \partial_\eta\phi = \partial_\sigma\phi = 0$ if $\xi = \eta = \sigma$. From stationary phase applied to

$$\hat{w}(\xi,t) = \hat{u}_0 + ic\int^t \frac{1}{s}|\hat{w}(\xi,s)|^2\hat{w}(\xi,s) + \mathcal{O}(s^{-\frac{3}{2}})ds.$$

This leads to a phase correction on $w$ for long time behavior $e^{-ic\int^t \frac{1}{s}|\hat{w}(\xi,s)|^2 ds}\hat{w}$. ( T. Ozawa 91)

## Absence of space time resonances

Example 4. illustrates how the structure of the nonlinearity can kill space time resonances. The most well known example of this is the "null condition" for wave equations.

EXAMPLE 6 $\Box u = Q(u, u) \qquad x \in \mathbb{R}^3.$

Small solutions exists globally provided the nonlinearity has a null structure.

D. Christodoulou *conformal coordinates*

S. Klainerman *vector fields*

# Wave equations

EXAMPLE 6 (continued) The symbol of the quadratic interaction is given by $\hat{Q} = \xi_1\eta_2 - \xi_2\eta_1$

Writing $(\partial_t \pm i|\xi|)\hat{u} = \hat{u}_\pm$ and expressing the equation as a system for $u_\pm$ we obtain a quadratic term with a symbol $\frac{\xi_1\eta_2-\xi_2\eta_1}{|\eta||\xi-\eta|}$

$$\phi = -|\xi| + |\xi-\eta| + |\eta|, \qquad \nabla_\eta\phi = \frac{\eta}{|\eta|} - \frac{\eta-\xi}{|\eta-\xi|}$$

$$2\frac{\xi_1\eta_2-\xi_2\eta_1}{|\eta||\xi-\eta|} = \big(\partial_{\eta_1}\phi\big)\left(\frac{\eta_2}{|\eta|} - \frac{\eta_2-\xi_2}{|\eta-\xi|}\right) - \big(\partial_{\eta_2}\phi\big)\left(\frac{\eta_1}{|\eta|} - \frac{\eta_1-\xi_1}{|\eta-\xi|}\right)$$

## A model for water waves with surface tension

$$\begin{cases} \partial_t v = & \Lambda u \qquad\qquad \Lambda = |\xi| \\ \partial_t u = & \Delta v + |\nabla u|^2 - |\Lambda u|^2 \end{cases}$$

This is equivalent to: $\partial_t^2 u = -\Lambda^3 u + \partial_t \left( |\nabla u|^2 - |\Lambda u|^2 \right).$

We have $\pm$ waves: $e^{\pm it|\xi|^{\frac{3}{2}}}.$

## A model for water waves with surface tension

To compute space time resonances we compute the phase for $\pm$ waves.

$\mathcal{T}$ is given by $\phi = |\xi|^{\frac{3}{2}} \pm |\eta|^{\frac{3}{2}} \pm |\xi - \eta|^{\frac{3}{2}} = 0$

$\mathcal{R}$ is given by $\nabla_\eta \phi = \frac{3}{2}\left( \pm \frac{\eta}{|\eta|^{\frac{1}{2}}} \pm \frac{\eta - \xi}{|\eta - \xi|^{\frac{1}{2}}} \right) = 0.$

## A model for water waves with surface tension

The symbol of the quadratic interaction terms are given by:

$++$ and $--$ are given by: $$\pm\Big(|\eta|^{\frac{3}{2}} + |\xi-\eta|^{\frac{3}{2}}\Big)\Big(\eta\cdot(\xi-\eta) + |\eta||\xi-\eta|\Big)$$

$+-$ and $-+$ are given by: $$\pm\Big(-|\eta|^{\frac{3}{2}} + |\xi-\eta|^{\frac{3}{2}}\Big)\Big(\eta\cdot(\xi-\eta) + |\eta||\xi-\eta|\Big)$$

## A model for water waves with surface tension

For $++$ or $--$ interaction $\nabla_\eta \phi = 0$ if $\xi = 2\eta$ which make the phase $\phi \sim |\xi|^{\frac{3}{2}} \neq 0$ unless $\xi = 0$. This is the easy case

For $+-$ or $-+$ interaction $\nabla_\eta \phi = 0$ if $\xi = 0$ which make the phase $\phi = 0$. However in this case the quadratic interaction vanishes thus we have a null form structure

For this model problem we do not have even local energy estimates. However if we cut off the high frequencies we obtain global existence and linear asymptotic behavior.