From Ginzburg-Landau to vortex lattice problems

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The Ginzburg-Landau energy with magnetic field

$$G_\varepsilon(\psi, A) = \frac{1}{2} \int_\Omega |\nabla_A \psi|^2 + |\mathrm{curl}\, A - h_{\mathrm{ex}}|^2 + \frac{(1-|\psi|^2)^2}{2\varepsilon^2}$$

- $\Omega \subset \mathbb{R}^2$ simply connected
- $\psi : \Omega \to \mathbb{C}$ "order parameter"
- $|\psi|^2$= density of superconducting Cooper pairs, $|\psi| \sim 1$ superconducting phase, $|\psi| \sim 0$ normal phase, $\psi = 0$ *vortices*
- $A : \Omega \to \mathbb{R}^2$ vector potential $\nabla_A = \nabla - iA$
- $h = \mathrm{curl}\, A$ induced magnetic field
- $h_{\mathrm{ex}} > 0$ intensity of applied field
- $\varepsilon = \frac{1}{\kappa}$ "Ginzburg-Landau parameter": material constant
- limit $\varepsilon \to 0$ extreme type-II or strongly repulsive

The Ginzburg-Landau equations

$$(GL)\begin{cases} -\nabla_A^2\psi = \frac{\psi}{\varepsilon^2}(1-|\psi|^2) & \text{in } \Omega \\ -\nabla^\perp h = \psi \times \nabla_A \psi & \text{in } \Omega \\ h = h_{\rm ex} & \text{on } \partial\Omega \\ \nabla_A\psi \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

Invariance under $\mathbb{U}(1)$-gauge-transformations (“Abelian gauge theory")

$$\begin{cases} \psi \mapsto \psi e^{i\Phi} \\ A \mapsto A + \nabla\Phi \end{cases} \tag{1}$$

The physical quantities are gauge-invariant, such as: $|\psi|^2$, h, $j = \psi \times \nabla_A\psi$, G_ε.

motivations: superconductivity, superfluidity, Bose-Einstein condensates

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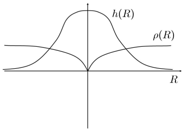
Vortices

- $|\psi|^2 \leq 1$ density of superconducting electrons
- $|\psi| = 0$ normal phase
- $|\psi| \sim 1$ superconducting phase
- vortices: zeros of ψ with nonzero degree

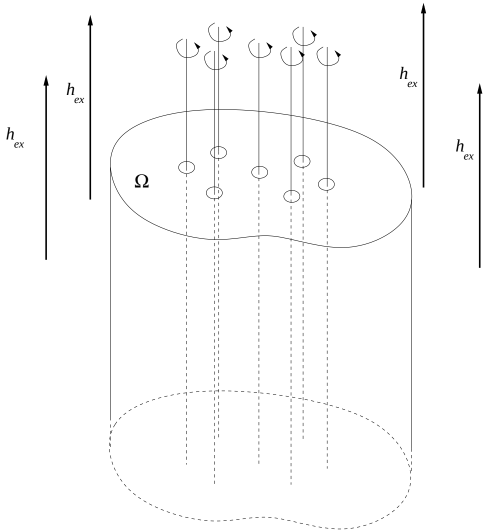


- $\psi = \rho e^{i\varphi}$

$$\frac{1}{2\pi}\int_{\partial B(x_0,r)} \frac{\partial \varphi}{\partial \tau} = d \in \mathbb{Z}$$

degree of the vortex

- In the limit $\varepsilon \to 0$ vortices become *point-like*, or more generally *codimension-2* singularities



h_{ex}
h_{ex}
h_{ex}
h_{ex}
Ω

Vorticity

φ not single-valued

introduce the vorticity-measure

$$\mu_\varepsilon := \mu(\psi, A) = \mathrm{curl}\,(\psi \times \nabla_A \psi) + \mathrm{curl}\, A$$

"Jacobian estimate" (see Jerrard-Soner)

$$\mathrm{curl}\,(\psi \times \nabla \psi) = det\ D\psi = \mathrm{curl}\,(\rho^2 \nabla \varphi) \simeq \mathrm{curl}\, \nabla \varphi = 2\pi \sum_i d_i \delta_{a_i} \quad \text{qd } \varepsilon \to 0$$

If (ψ, A) satisfies (GL2)

$$-\nabla^\perp h = \psi \times \nabla_A \psi$$

taking the curl

$$\begin{cases} -\Delta h + h = \mu \simeq 2\pi \sum_i d_i \delta_{a_i} & \text{in } \Omega \\ h = h_{\mathrm{ex}} & \text{on } \partial\Omega. \end{cases}$$

Also $|\nabla_A \psi| \simeq |\nabla h| \rightsquigarrow$ logarithmic divergence of $\int_\Omega |\nabla_A \psi|^2$

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Influence of the applied field and critical fields

- $h_{\mathrm{ex}} < H_{c_1}$ no vortex, $|\psi| \sim 1$ (Meissner effect)
- $H_{c_1} = O(|\log \varepsilon|)$ first critical field: first vortices appear, then number increases with h_{ex}
  $\longrightarrow$ Abrikosov lattices (triangular) vortices repell...
- $H_{c_2} = O(\frac{1}{\varepsilon^2})$ bulk superconductivity destroyed, surface superconductivity remains
- $H_{c_3} = O(\frac{1}{\varepsilon^2})$ superconductivity destroyed, normal state $\psi \equiv 0$

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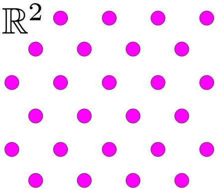

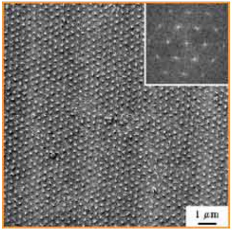

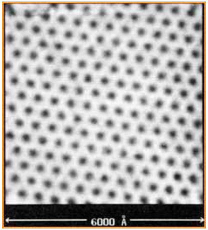

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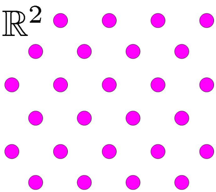

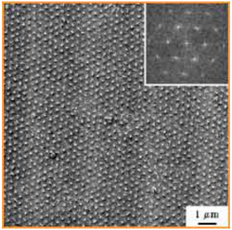

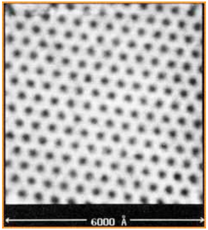

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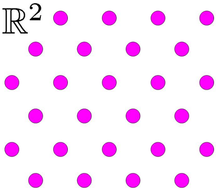

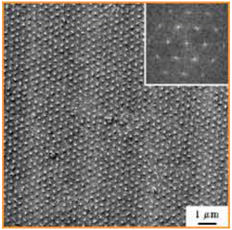

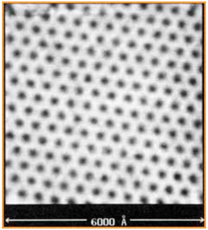

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Leading order results for minimizers (mean field description)

Theorem (Sandier-S)

Assume $h_{\rm ex} = \lambda|\log \varepsilon|$. *As* $\varepsilon \to 0$, $\frac{G_\varepsilon}{h_{\rm ex}^2}$ Γ*-converges w.r.to the convergence of* $\mu(u,A)/h_{\rm ex}$ *to*

$$E_\lambda(\mu) := \frac{1}{2\lambda}\int_\Omega |\mu| + \frac{1}{2}\int_\Omega |\nabla h_\mu|^2 + |h_\mu - 1|^2$$

where

$$\begin{cases} -\Delta h_\mu + h_\mu = \mu & \text{in } \Omega \\ h_\mu = 1 & \text{on } \partial\Omega. \end{cases}$$

Consequently, for minimizers of G_ε, *as* $\varepsilon \to 0$ *we have*

$$\frac{\mu_\varepsilon}{h_{\rm ex}} \rightharpoonup \mu_* = m\mathbf{1}_{\omega_\lambda} \qquad \frac{h}{h_{\rm ex}} \rightharpoonup h_*$$

$$\frac{G_\varepsilon(\psi, A)}{h_{\rm ex}^2} \to E_\lambda(\mu_*)$$

where μ_* *is the minimizer of* E_λ.

Minimization of E_λ: the obstacle problem

The minimization of E_λ is equivalent (by convex duality) to the *obstacle problem*

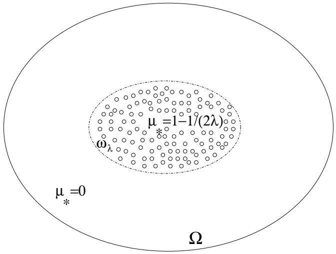
$$\min_{\substack{h-1\in H_0^1(\Omega)\\ h\geq 1-\frac{1}{2\lambda}}} \frac{1}{2}\int_\Omega |\nabla h|^2 + h^2$$

with $\mu_* = -\Delta h_* + h_*$

Coincidence set

$$\omega = \left\{ x \in \Omega / h_*(x) = 1 - \frac{1}{2\lambda} := m \right\}$$

μ_* is a uniform density $= m$ on $\omega \subset \Omega$.



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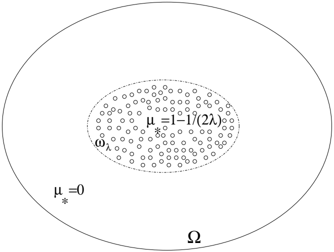

Dependence on λ

- $\lambda < \lambda_0$: $\omega_\lambda = \varnothing$, $\mu_* = 0$, no vortices
- $\lambda = \lambda_0$: $\omega_\lambda = \Lambda =$ finite set of points (assume $\Lambda = \{p\}$)
- $\lambda > \lambda_0$: $\omega_\lambda \neq \varnothing$
- $|\log \varepsilon| \ll h_{\mathrm{ex}} \ll \frac{1}{\varepsilon^2}$: $\quad \omega_\infty = \Omega, \quad \mu_* = 1$

$$H_{c_1} \sim \lambda_0 |\log \varepsilon|$$

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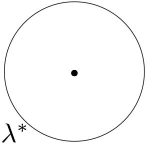

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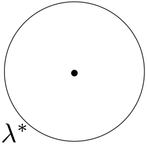

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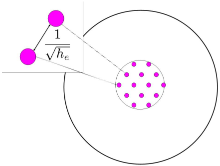

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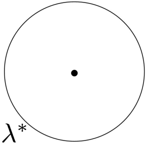

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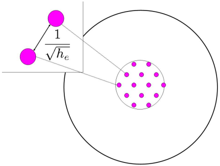

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A splitting of the energy

Let (ψ, A) satisfy (GL2). We are able to show

$$G_\varepsilon(\psi, A) = h_{\rm ex}^2 E_\lambda(\mu_*) + G_1(\psi, A)$$

where G_1 is roughly like

$$G_1(\psi, A) \simeq \frac{1}{2}\int |\nabla_A \psi|^2 + |h - h_{\rm ex}|^2 + \frac{(1-|\psi|^2)^2}{2\varepsilon^2} - \pi \sum_i d_i \log \frac{1}{\varepsilon\sqrt{h_{\rm ex}}}.$$

First part $\sim$ GL "free" energy without applied field
$\geq \pi \sum |d_i| \log \frac{1}{\varepsilon\sqrt{h_{\rm ex}}}$
energy in the vortex cores - lower bounds by "ball construction methods", Bethuel-Brezis-Hélein, Jerrard, Sandier, Sandier-S...

- When adding a vortex an "infinite" amount of energy is added, but also substracted
- $\rightsquigarrow$ remains a "renormalized energy"
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Behaviour of energy-minimizers at next order

We also have

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- density of vortices mh_{ex}, distances $\sim 1/\sqrt{mh_{\mathrm{ex}}} \to$ should blow-up to see the pattern
- after blow up at the scale $\sqrt{mh_{\mathrm{ex}}}$, around a point in ω, we get a configuration of points in the WHOLE plane with

$$-\Delta H = 2\pi \sum_i d_i \delta_{a_i} - 1 \quad \text{in } \mathbb{R}^2$$

Question: what's the interaction energy of the a_i's? Pbl: infinite-size domain

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Question: what's the interaction energy of the a_i's? Pbl: infinite-size domain

The renormalized energy

Given a configuration of points + degree (a_i, d_i) in the plane obtained this way, assuming all $d_i = 1$ and given H a solution to

$$-\Delta H = 2\pi \sum_i \delta_{a_i} - 1.$$

We consider for any R a cutoff function $\chi_R \in C_0^\infty(B_R)$ such that $0 \le \chi_R \le 1$ and $\chi_R \equiv 1$ in B_{R-1}, and $|\nabla \chi_R| \le 2$, and we define

$$W(\{a_i\}, H) = \liminf_{R\to\infty} \frac{1}{|B_R|} \lim_{\alpha\to 0} \Big(\frac{1}{2}\int_{B_R \setminus \cup_i B(a_i,\alpha)} \chi_R |\nabla H|^2 + \sum_i \chi_R(a_i)\pi \log \alpha\Big)$$

cf renormalized energy of Bethuel-Brezis-Hélein for finite number of vortices

$$\text{“}W(\{a_i\}) = \|2\pi \sum_i \delta_{a_i} - 1\|^2_{H^{-1}}\text{''}$$

$\mathcal{F}$ denotes the set of $(\{a_i\}, H)$ with $-\Delta H = 2\pi \sum_i \delta_{a_i} - 1$ in $\mathbb{R}^2$

Theorem (Lower bound)

Let ω denote the support of μ_*. Then for any $(\psi_\varepsilon, A_\varepsilon)$, there exists a probability measure P on $\mathcal{F}$ such that

$$\liminf_{\varepsilon \to 0} \frac{1}{m h_{\mathrm{ex}} |\omega|} G_1(\psi_\varepsilon, A_\varepsilon) \geq \int W(\{a_i\}, H) \, dP(\{a_i\}, H) \geq \inf_{a_i, H} W$$

and thus

$$G_\varepsilon(\psi_\varepsilon, A_\varepsilon) \geq h_{\mathrm{ex}}^2 E_\lambda(\mu_*) + m h_{\mathrm{ex}} |\omega| \inf_{a_i, H} W + o(h_{\mathrm{ex}})$$

Sharp lower bound up to higher order $o(h_{\mathrm{ex}})$ (=o(number of vortices)) = best possible

The matching upper bound

Theorem (Upper bound)

Assume $h_{\mathrm{ex}} \ll \frac{1}{\varepsilon^2}$. *For* $\varepsilon < \varepsilon_0$, *there exists* $(\psi_\varepsilon, A_\varepsilon)$ *such that*

$$G_\varepsilon(\psi_\varepsilon, A_\varepsilon) \leq h_{\mathrm{ex}}^2 E_\lambda(\mu_*) + m h_{\mathrm{ex}} |\omega| \inf_{a_i, H} W + o(h_{\mathrm{ex}})$$

Corollary

"For minimizers of G_ε, *blown-up of the vortices at scale* $\sqrt{m h_{\mathrm{ex}}}$ *around* x_ε *chosen at random converge* P*-a.s. to configurations of points in the plane minimizing* W*."*

Method and difficulties

- in order to derive W we need to control the number of vortices per unit volume after blow-up
  $\rightsquigarrow$ need very sharp (sharper than in the past!) lower bounds on the energy of each vortex with *possibly infinite number* of them
- the renormalized cost of a vortex in B_R tends to $-\infty$ when the vortex approaches ∂B_R $\rightsquigarrow$ need cut-off and letting $R \to \infty$
- the size of the blown-up domain ω tends to ∞. Through the ergodic theorem, we define an averaged notion of Γ-convergence which works for infinite domains when the energy is translation invariance. Alternate to a method of Alberti-Müller. Pbl: our energy density is not positive.
- to prove the upper bound we first need to be able to reduce to periodic configurations of points in the plane, i.e. show that minimizing W in $\mathbb{R}^2$ can be well-approximated by minimizing it over configurations of points on large tori
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The result for periodic configurations

- Let H be a solution to

$$-\Delta H = \delta_0 - 1$$

  on a torus of volume 1 of arbitrary shape.
- Fourier transform the explicit expression for W in that case to make it a function of the lattice (regularisation of $\sum_{p\in\Lambda} \frac{1}{|p|^2}$)
- its value becomes related to Dedekind eta function and Eisenstein series
- Minimizing W becomes equivalent to minimizing the Epstein zeta function $\zeta(s) = \sum_{p\in\Lambda} \frac{1}{|p|^s}$, $s > 2$, over lattices
- results from number theory (Cassels, Rankin, 60's) say that this is minimized by the triangular lattice

Theorem

The function W restricted to periodic configurations is minimized over all lattices of volume 1 by the triangular lattice

$\rightsquigarrow$ W allows to distinguish between lattices!

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Conclusion and perspectives

- we have characterized the location of vortices in all applied field regimes $h_{\mathrm{ex}} \ll \frac{1}{\varepsilon^2}$ up to the scale where we see individual vortices
- derived a limiting problem of interaction of points in the plane: the renormalized energy W
- W is a logarithmic type of interaction $\rightsquigarrow$ long range!
- this problem allows to distinguish between different kind of lattices and prefers the triangular one $\rightsquigarrow$ first justification of the Abrikosov lattice in this regime
- remains to study the renormalized energy W without assuming periodicity $\rightsquigarrow$ question of crystallisation...

Bibliography

- E. Sandier, S.S, *Vortices in the Magnetic Ginzburg-Landau Model*, Progress in Nonlinear Differential Equations, Birkhaüser, 2007.
- E. Sandier, S.S, forthcoming.

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