Boltzmann Equation in Bounded Domains

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In Honor of Cathleen S. Morawetz

1. Boltzmann Equation (1872)

$$\partial_t F + v \cdot \nabla_x F = Q(F, F) \tag{1}$$

where $F(t, x, v) \geq 0$ is the distribution function for the gas particles at time $t \geq 0$, position $x \in \Omega$, and $v \in \mathbf{R}^3$, $Q(F_1, F_2)$ (hard-sphere) is

$$\int_{\mathbf{R}^3} \int_{\mathbf{S}^2} |v - u| F_1(u') F_2(v') |\cos\theta| d\omega du$$

$$- \int_{\mathbf{R}^3} \int_{\mathbf{S}^2} |v - u| F_1(u) F_2(v) |\cos\theta| d\omega du \tag{2}$$

where

$$\begin{aligned} u' &= u + (v - u) \cdot \omega, v' = v - (v - u) \cdot \omega, \cos\theta \\ &= (u - v) \cdot \omega / |u - v|. \end{aligned}$$

The Maxwellian distribution: $\mu(v) = e^{-|v|^2/2}$.
Global Solutions Near Maxwellian: Let

$$\begin{aligned} F &= \mu + \sqrt{\mu} f, \\ \{\partial_t + v \cdot \nabla_x + L\} f &= \Gamma(f, f) \end{aligned}$$

where the standard linear Boltzmann operator is given by

$$Lf \equiv \nu f - Kf = -\frac{1}{\sqrt{\mu}}\{Q(\mu, \sqrt{\mu} f) + Q(\sqrt{\mu} f, \mu)\}, \tag{3}$$

and

$$\begin{aligned} \Gamma(f_1, f_2) &= \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} f_1, \sqrt{\mu} f_2) \equiv \Gamma_{\text{gain}}(f_1, f_2) \\ &\quad - \Gamma_{\text{loss}}(f_1, f_2). \end{aligned} \tag{4}$$

The collision invariant and $H-$Theorem implies that $L \geq 0$, and

$$\ker L = \{a(t,x)\sqrt{\mu}, \quad b(t,x) \cdot v\sqrt{\mu}, \quad c(t,x)|v|^2\sqrt{\mu}\}.$$

Here for fixed (t, x), the standard projection $\mathbf{P}$ onto the hydrodynamic (macroscopic) part is given by

$$\mathbf{P}f = \{a_f(t,x) + b_f(t,x) \cdot v + c_f(t,x)|v|^2\}\sqrt{\mu(v)}.$$

Energy Estimate:

$$\frac{1}{2}\frac{d}{dt}||f||^2 + \langle Lf, f\rangle = \int\int f\Gamma(f,f).$$

If $\langle Lf, f\rangle \geq \delta||f||^2$, global small solutions. But we only know

$$\langle Lf, f\rangle \geq \delta||\{\mathbf{I}-\mathbf{P}\}f||^2,$$

It suffices to estimate $\mathbf{P}f$ in terms of $\{\mathbf{I}-\mathbf{P}\}f$:

$$\{\partial_t + v\cdot\nabla_x\}\,\mathbf{P}f = \Gamma(f,f) - \{\partial_t + v\cdot\nabla + L\}\,\{\mathbf{I}-\mathbf{P}\}f$$

to get

$$\begin{aligned} \nabla_x c &= g_1 \\ \partial_t c + \nabla\cdot b &= g_2 \\ \partial_{x_i} b^j + \partial_{x_j} b^i &= g_3 \\ \partial_t b + \nabla_x a &= g_4 \\ \partial_t a &= g_5 \end{aligned}$$

where $g_l \backsim \partial\{\mathbf{I}-\mathbf{P}\}f+$high order. We deduce that

$$\Delta b \backsim \partial^2\{\mathbf{I}-\mathbf{P}\}f.$$

We hence have **ellipticity** for b.

2. Boundary Value Problem

One basic problem in the Boltzmann study: Uniqueness, Time-decay toward $e^{-|v|^2/2}$, with physical boundary conditions in a general domain Ω.

Let $\Omega = \{x : \xi(x) < 0\}$ is connected, and bounded with $\xi(x)$ smooth. The outward normal vector at $\partial\Omega$ is given by

$$n(x) = \frac{\nabla\xi(x)}{|\nabla\xi(x)|}. \tag{5}$$

We say Ω is real analytic if ξ is real analytic in x. We define Ω is strictly convex if there exists $c_\xi > 0$ such that

$$\partial_{ij}\xi(x)\zeta^i\zeta^j \geq c_\xi|\zeta|^2 \tag{6}$$

for all x such that $\xi(x) \leq 0$, and all $\zeta \in \mathbf{R}^3$. We say that Ω has a rotational symmetry, if there are vectors x_0 and ϖ, such that for all $x \in \partial\Omega$

$$\{(x - x_0) \times \varpi\} \cdot n(x) \equiv 0. \tag{7}$$

We denote the phase boundary in the space $\Omega \times \mathbf{R}^3$ as $\gamma = \partial\Omega \times \mathbf{R}^3$:

$$\begin{aligned}\gamma_+ &= \{(x,v) \in \partial\Omega \times \mathbf{R}^3 : \quad n(x) \cdot v > 0 \text{ outgoing}\},\\ \gamma_- &= \{(x,v) \in \partial\Omega \times \mathbf{R}^3 : \quad n(x) \cdot v < 0 \text{ incoming}\},\\ \gamma_0 &= \{(x,v) \in \partial\Omega \times \mathbf{R}^3 : \quad n(x) \cdot v = 0 \text{ grazing}\}.\end{aligned}$$

Given (t,x,v), let the trajectory (or the characteristics) for the Boltzmann equation:

$$\begin{aligned}[X(s),V(s)] &= [X(s;t,x,v),V(s;t,x,v)]\\ &= [x+(s-t)v,v] \qquad (8)\end{aligned}$$

with the initial condition: $[X(t;t,x,v),V(t;t,x,v)] = [x,v]$. For any (x,v) such that $x \in \bar{\Omega}, v \neq 0$, we define its **backward exit time** $t_{\mathbf{b}}(x,v) > 0$ to be the the last moment at which the back-time straight line $[X(s;0,x,v),V(s;0,x,v)]$ remains in Ω :

$$t_{\mathbf{b}}(x,v) = \sup\{\tau \geq 0 : x - \tau v \in \Omega\}.$$

Four Basic Types of Boundary Conditions:

(1) In-flow:

$$f|_{\gamma_-} = g(t,x,v) \tag{9}$$

(2) Bounce-back:

$$f(t,x,v)|_{\gamma_-} = f(t,x,-v) \tag{10}$$

(3) Specular reflection: for $x \in \partial\Omega$, let

$$R(x)v = v - 2(n(x)\cdot v)n(x), \tag{11}$$

and

$$\begin{aligned} f(t,x,v)|_{\gamma_-} &= f(x,v,v-2(n(x)\cdot v)n(x)) \\ &= f(x,v,R(x)v) \end{aligned} \tag{12}$$

(4) Diffusive reflection:

$$\begin{aligned} f(t,x,v)|_{\gamma_-} &= c_\mu\sqrt{\mu(v)}\int_{v'\cdot n(x)>0} f(t,x,v') \\ &\quad \sqrt{\mu(v')}\{n_x\cdot v'\}dv'. \end{aligned} \tag{13}$$

2.1 Previous Work

In 1977, Asano and Shizuta's announcement.

Desvillettes and Villani (2005) establish almost exponential decay rate with large amplitude, provided certain **a-priori** strong Sobolev estimates can be verified.

2.2 Difficulties

The validity of these a-priori estimates is completely open even local in time, in a bounded domain. As a matter of fact, such kind of strong Sobolev estimates are not be expected for a general non-convex domain.

Characteristic Boundary: This is because even for simplest kinetic equations with the differential operator $v \cdot \nabla_x$, the phase boundary $\partial\Omega \times \mathbf{R}^3$ is always characteristic but not uniformly characteristic at the grazing set

$$\gamma_0 = \{(x,v) : x \in \partial\Omega, \text{ and } v \cdot n(x) = 0\}.$$

The complication of the geometry makes it difficult to employ spatial Fourier transforms in x.

Bouncing Characteristics: particles interacting with the boundary repeatedly.

Strategy: To develop an unified $L^2 - L^\infty$ theory.

2.3 Main Results

We introduce the weight function

$$w(v) = (1+\rho|v|^2)^{\beta} e^{\theta|v|^2}. \tag{14}$$

where $0 \leq \theta < \frac{1}{4}, \rho > 0$ and $\beta \in \mathbf{R}^1$.

Theorem 1 *Assume* $w^{-2}\{1+|v|\}^3 \in L^1$. *There exists* $\delta > 0$ *such that if* $F_0 = \mu + \sqrt{\mu} f_0 \geq 0$, *and*

$$||wf_0||_\infty + \sup_{0\leq t\leq\infty} e^{\lambda_0 t}||wg(t)||_\infty \leq \delta,$$

with $\lambda_0 > 0$, *then there there exists a unique solution* $F(t,x,v) = \mu + \sqrt{\mu} f \geq 0$ *to the inflow boundary value problem (9) for the Boltzmann equation (1). There exists* $0 < \lambda < \lambda_0$ *such that*

$$\sup_{0\leq t\leq\infty} e^{\lambda t}||wf(t)||_\infty \leq C\{||wf_0||_\infty + \sup_{0\leq t\leq\infty} e^{\lambda_0 t}||wg(t)||_\infty\}.$$

Moreover, if Ω is strictly convex (6), and if $f_0(x,v)$ is continuous except on γ_0, and $g(t,x,v)$ is continuous in $[0,\infty)\times\{\partial\Omega\times\mathbf{R}^3\backslash\gamma_0\}$ satisfying

$$f_0(x,v) = g(x,v) \quad \textit{on } \gamma_-,$$

then f is continuous in $[0,\infty)\times\{\bar{\Omega}\times\mathbf{R}^3\setminus\gamma_0\}$.

Theorem 2 *Assume $w^{-2}\{1+|v|\}^3 \in L^1$. Assume the conservations of mass and energy are valid for f_0. Then there exists $\delta > 0$ such that if $F_0(x,v) = \mu + \sqrt{\mu} f_0(x,v) \geq 0$ and $||wf_0||_\infty \leq \delta$, there exists a unique solution $F(t,x,v) = \mu + \sqrt{\mu} f(t,x,v) \geq 0$ to the bounce-back boundary value problem (10) for the Boltzmann equation (1) such that*

$$\sup_{0\leq t\leq\infty} e^{\lambda t}||wf(t)||_\infty \leq C||wf_0||_\infty.$$

Moreover, if Ω is strictly convex (6), and if initially $f_0(x,v)$ is continuous except on γ_0 and

$$f_0(x,v) = f_0(x,-v) \textit{ on } \partial\Omega\times\mathbf{R}^3\setminus\gamma_0,$$

then f is continuous in $[0,\infty)\times\{\bar{\Omega}\times\mathbf{R}^3\setminus\gamma_0\}$.

Theorem 3 *Assume* $w^{-2}\{1+|v|\}^3 \in L^1$. *Assume that* ξ *is both strictly convex (6) and analytic, and the mass and energy are conserved for* f_0. *In the case of* Ω *has any rotational symmetry (7), we require that the corresponding angular momentum is conserved for* f_0. *Then there exists* $\delta > 0$ *such that if* $F_0(x,v) = \mu + \sqrt{\mu} f_0(x,v) \geq 0$ *and* $||wf_0||_\infty \leq \delta$, *there exists a unique solution* $F(t,x,v) = \mu + \sqrt{\mu} f(t,x,v) \geq 0$ *to the specular boundary value problem (12) for the Boltzmann equation (1) such that*

$$\sup_{0\leq t\leq\infty} e^{\lambda t}||wf(t)||_\infty \leq C||wf_0||_\infty.$$

Moreover, if $f_0(x,v)$ *is continuous except on* γ_0 *and*

$$f_0(x,v) = f_0(x, R(x)v) \text{ on } \partial\Omega$$

then f *is continuous in* $[0,\infty) \times \{\bar{\Omega} \times \mathbf{R}^3 \setminus \gamma_0\}$.

Theorem 4 *There is $\theta_0(\nu_0) > 0$ such that*

$$\theta_0(\nu_0) < \theta < \frac{1}{4}\ , \text{ and } \rho \text{ is sufficiently small} \tag{15}$$

for weight function w. Assume the mass conservation is valid for f_0. If $F_0(x,v) = \mu + \sqrt{\mu} f_0(x,v) \geq 0$ and $||wf_0||_\infty \leq \delta$ sufficiently small, then there exists a unique solution $F(t,x,v) = \mu + \sqrt{\mu} f(t,x,v) \geq 0$ to the diffuse boundary value problem (14) for the Boltzmann equation (1) such that

$$\sup_{0 \leq t \leq \infty} e^{\lambda t} ||wf(t)||_\infty \leq C||wf_0||_\infty.$$

Moreover, if ξ is strictly convex, and if $f_0(x,v)$ is continuous except on γ_0 with

$$\begin{aligned} f_0(x,v)|_{\gamma_-} \ = \ & c_\mu \sqrt{\mu} \int_{\{n_x \cdot v' > 0\}} f_0(x,v') \\ & \sqrt{\mu(v')}\{n(x) \cdot v'\} dv' \end{aligned}$$

then f is continuous in $[0,\infty) \times \{\bar{\Omega} \times \mathbf{R}^3 \setminus \gamma_0\}$.

2.4 Velocity Lemma

Lemma 5 *Let Ω be strictly convex defined in (6). Along the trajectories $\frac{dX(s)}{ds} = V(s), \frac{dV(s)}{ds} = 0$ in (8), define:*

$$\alpha(s) \equiv \xi^2(X(s)) + [V(s) \cdot \nabla\xi(X(s))]^2$$

$$-2\{V(s) \cdot \nabla^2\xi(X(s)) \cdot V(s)\}\xi(X(s)). \quad (16)$$

Let $X(s) \in \bar{\Omega}$ for $t_1 \leq s \leq t_2$. Then there exists constant $C_\xi > 0$ such that

$$e^{C_\xi(|V(t_1)|+1)t_1}\alpha(t_1) \leq e^{C_\xi(|V(t_1)|+1)t_2}\alpha(t_2);$$

$$e^{-C_\xi(|V(t_1)|+1)t_1}\alpha(t_1) \geq e^{-C_\xi(|V(t_1)|+1)t_2}\alpha(t_2). \quad (17)$$

This lemma implies that in a strictly convex domain (6), the singular set γ_0 can not be reached via the trajectory $\frac{dx}{dt} = v$, $\frac{dv}{dt} = 0$ from interior points inside Ω, and hence γ_0 does not really participate or interfere with the interior dynamics. No singularity would be created from γ_0 and it is possible to perform calculus for the back-time exit time $t_{\mathbf{b}}(x, v)$. History and Vlasov-Poisson (Hwang and Velazquez 2007)

2.5 L^2 Decay Theory

It suffices to establish the following finite-time estimate

$$\int_0^1 ||\mathbf{P}f(s)||_\nu^2 ds \leq M\left\{\int_0^1 ||\{\mathbf{I}-\mathbf{P}\}f(s)||_\nu^2 + \text{boundary contributions}\right\} \quad (18)$$

for any solution f to the linear Boltzmann equation with boundary conditions.

It is challenging to estimate L^2 of b

$$\Delta b = \partial^2\{\mathbf{I}-\mathbf{P}\}f,$$

with $b\cdot n(x) = 0$ (bounce-back and specular) or $b \equiv 0$ (inflow and diffuse) at $\partial\Omega$.

Hyperbolic (Transport) Feature: If not, the normalized $Z_k(t,x,v) \equiv \frac{f_k(t,x,v)}{\sqrt{\int_0^1 ||\mathbf{P} f_k(s)||_\nu^2 ds}}$ satisfies $\int_0^1 ||\mathbf{P} Z_k(s)||_\nu^2 ds \equiv 1$, and

$$\int_0^1 ||(\mathbf{I} - \mathbf{P}) Z_k(s)||_\nu^2 ds \leq \frac{1}{k}. \tag{19}$$

Denote a weak limit of Z_k to be Z, we expect that $Z = \mathbf{P} Z = 0$, by each of the four boundary conditions. The key is to prove that $Z_k \to Z$.

No Concentration in Interior Ω : By the averaging Lemma, we know that $Z_k(s) \to Z$.

No Concentration Near $\partial\Omega$: On the **Non-Grazing Set** $v \cdot n(x) \neq 0$: Since Z_k is a solution to a transport equation, non-grazing point can be reached via a trajectory from the interior of Ω.

Almost Grazing Set $v \cdot n(x) \sim 0$, thanks to the fact (19) no concentration for

$$Z_k \sim \mathbf{P} Z_k = \{a_k(t,x) + b_k(t,x) \cdot v + c_k(t,x)|v|^2\}\sqrt{\mu(v)}$$

2.6 L^∞ Decay Theory

We denote a weight function $h(t,x,v) = w(v)$ $f(t,x,v)$. Let $U(t)h$ solves

$$\{\partial_t + v\cdot\nabla_x + \nu - K_w\}h = 0, \tag{20}$$

where $K_w h = wK(\frac{h}{w})$. Consider $G(t)h$ solves

$$\{\partial_t + v\cdot\nabla_x + \nu\}h = 0, \tag{21}$$

$$U(t) = G(t) + \int_0^t G(t-s_1)K_w U(s_1)ds_1. \tag{22}$$

By Vidav's (1970), **(two iterations)**:

$$U(t) = G(t) + \int_0^t G(t-s_1)K_w G(s_1)ds_1 +$$

$$\int_0^t \int_0^{s_1} G(t-s_1)K_w G(s_1-s)K_w U(s)dsds_1. \tag{23}$$

To estimate the last double integral in terms of the L^2 norm of $f = \frac{h}{w}$.

2.6.1 Inflow boundary condition (9):

With the compact property of K_w, we are led to the main contribution in (28) roughly of the form

$$\int_0^t \int_0^{s_1} \int_{v',v''} |h(s, X(s; s_1, X(s_1; t, x, v), v'), v'')| \, dv' dv'' ds ds_1. \tag{24}$$

The v' integral is estimated by a change of variable introduced in Vidav (A-smoothing 1970, Liu-Yu 2007)

$$\begin{aligned} y &\equiv X(s; s_1, X(s_1; t, x, v), v') \\ &= x - (t - s_1)v - (s_1 - s)v'. \end{aligned} \tag{25}$$

Since $\det(\frac{dy}{dv'}) \neq 0$ almost always true, the v' and $v''-$integration in (29) can be bounded by

$$\int_{\Omega, v'' \text{ bounded}} |h(s, y, v'')| dy dv''$$

$$\leq C \left(\int_{\Omega, v'' \text{ bounded}} |f(s, y, v'')|^2 dy dv'' \right)^{1/2}.$$

For bounce-back, specular or diffuse reflections, the characteristic trajectories repeatedly interact with the boundary. Instead of $X(s;t,x,v)$, we should use the generalized characteristics, $X_{\mathbf{cl}}(s;t,x,v)$. To determine the change of variable

$$y \equiv X_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1; t, x, v), v'), \tag{26}$$

always almost have

$$\det \left\{ \frac{dX_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1; t, x, v), v')}{dv'} \right\} \neq 0. \tag{27}$$

2.6.2 The Bounce-Back Reflection:

Let $(t_0, x_0, v_0) = (t, x, v)$, $(t_{k+1}, x_{k+1}, v_{k+1})$ is $(t_k - t_{\mathbf{b}}(x_k, v_k), x_{\mathbf{b}}(x_k, v_k), -v_k)$; and

$$
\begin{aligned}
X_{\mathbf{cl}}(s; t, x, v) &= \sum_k \mathbf{1}_{[t_{k+1}, t_k)}(s)\{x_k + (s - t_k)v_k\}, \\
V_{\mathbf{cl}}(s; t, x, v) &= \sum_k \mathbf{1}_{[t_{k+1}, t_k)}(s) v_k.
\end{aligned}
$$

The bounce-back cycles $X_{\mathbf{cl}}(s; t, x, v)$ from a given point (t, x, v) is relatively simple.

2.6.3 Specular Reflection:

Let $(t_0, x_0, v_0) = (t, x, v)$, $(t_{k+1}, x_{k+1}, v_{k+1})$ is $(t_k - t_{\mathbf{b}}(x_k, v_k), x_{\mathbf{b}}(x_k, v_k), R(x_{k+1})v_k)$; and $X_{\mathbf{cl}}(s; t, x, v) = \sum_k \mathbf{1}_{[t_{k+1}, t_k)}(s)\{x_k + (s - t_k)v_k\}$, $V_{\mathbf{cl}}(s; t, x, v) = \sum_k \mathbf{1}_{[t_{k+1}, t_k)}(s)v_k$.

Difficulty: The specular cycles $X_{\mathbf{cl}}(s; t, x, v)$ reflect repeatedly with the boundary in general, and $\frac{dX_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1; t, x, v), v')}{dv'}$ is very complicated to compute and (32) is extremely difficult to verify. This is non-standard in the billiard literature (x and v are not symmetric!)

Small almost tangential bounces:

Let $(0, x_1, v_1)$ and $s_k = t_{\mathbf{b}}(x_k, v_k)$ for $k = 1, 2, 3, ...$ so that $\xi(x_1 - s_1 v_1) = 0$, $x_2 = x_1 - s_1 v_1 \in \partial\Omega$ and for $k \geq 2$: $\quad x_k = x_{k-1} - s_k v_k \in \partial\Omega$.

$$\xi(x_1 - \sum_{j=1}^{k} s_j v_j) = 0, \; v_k = R(x_k) v_{k-1}.$$

Proposition 6 *For any finite $k \geq 1$,*

$$\frac{\partial v_k^i}{\partial v_1^l} = \delta_{li} + \zeta(k) n^i(x_1) n^l(x_1) + O(\varepsilon_0), \quad (28)$$

where $\zeta(k)$ is an even integer so that

$$\det\left(\frac{\partial v_k^i}{\partial v_1^l}\right) = \{\zeta(k) + 1\} + O(\varepsilon_0) \neq 0.$$

It then follows that $\det\{\frac{dX_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1; t, x, v), v')}{dv'}\} \neq 0$ *for these special cycles.*

This crucial observation is then combined with analyticity of ξ to conclude that the set of $\det\{\frac{dX_{\mathbf{cl}}(s; X_{\mathbf{cl}}(s_1, x, v), v')}{dv'}\} = 0$ is arbitrarily small.

2.6.4 Diffuse Reflection:

Let $(t_0, x_0, v_0) = (t, x, v)$, $(t_{k+1}, x_{k+1}, v_{k+1})$ is $(t_k - t_{\mathbf{b}}(x_k, v_k), x_{\mathbf{b}}(x_k, v_k), v_{k+1})$; for $v_{k+1} \cdot n(x_{k+1}) > 0$; and $X_{\mathbf{cl}}(s; t, x, v)$ is

$$\sum_k \mathbf{1}_{[t_{k+1}, t_k)}(s)\{x_k + (s - t_k)v_k\},$$

$$V_{\mathbf{cl}}(s; t, x, v) = \sum_k \mathbf{1}_{[t_{k+1}, t_k)}(s) v_k.$$

Difficulty: Similar change of variable is expected with respect to one of such independent variables. However, the main difficulty in this case is the L^∞ control of $G(t)$ which satisfies (26). The most natural L^∞ estimate: for weight $w = \mu^{-\frac{1}{2}}$: (bad for linear theory)

$$h(t, x, v) = c_\mu \int_{v' \cdot n(x) > 0} h(t, x, v') \mu(v') \{v' \cdot n(x)\} dv'.$$

L^{∞} for weaker weight $\backsim \mu^{-\frac{1}{2}}$: The measure of those particles can not reach initial plane after $k-$bounces is small when k is large (non-zero!!)

Lemma 7 *Let the probability measure $d\sigma = d\sigma(x)$ is given by*

$$d\sigma(x) = c_{\mu}\mu(v')\{n(x)\cdot v'\}dv'. \qquad (29)$$

For any $\varepsilon > 0$, there exists $k_0(\varepsilon, T_0)$ such that for $k \geq k_0$, for all $(t,x,v), 0 \leq t \leq T_0$, $x \in \bar{\Omega}$ and $v \in \mathbf{R}^3$,

$$\int_{\prod_{l=1}^{k-1}\{v_l:\ v_l\cdot n(x_l)>0\}} \mathbf{1}_{\{t_k(t,x,v,v_1,v_2\ldots,v_{k-1})>0\}} \prod_{l=1}^{k-1} d\sigma(x_l) \leq \varepsilon.$$

We therefore can obtain an approximate representation formula for $G(t)$ by the initial datum, with only finite number of bounces.