Shock Reflection-Diffraction Phenomena
Transonic Flow
Free Boundary Problems

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Conference on Nonlinear Phenomena in Mathematical Physics
Dedicated to Cathleen Synge Morawetz on her 85th birthday

Fields Institute for Research in Mathematical Sciences, Toronto, Canada

September 18–20, 2008

Cathleen Synge Morawetz

On the non-existence of continuous transonic flows past profiles I

Comm. Pure Appl. Math. **9** (**1956**), 45–68

On the non-existence of continuous transonic flows past profiles II

Comm. Pure Appl. Math. **10** (**1957**), 107–131;

On the non-existence of continuous transonic flows past past profiles III

Comm. Pure Appl. Math. **11** (**1958**), 129–144.

Some of Cathleen's Work on Transonic Flow II

Cathleen Synge Morawetz

On a weak solution for a transonic flow problem
Comm. Pure Appl. Math. **38** (**1985**), 797–818.

On steady transonic flow by compensated compactness
Methods Appl. Anal. **2** (**1995**), 257–268.

Irene M. Gamba & Cathleen Synge Morawetz

A viscous approximation for a 2-D steady semiconductor or transonic gas dynamic flow: existence theorem for potential flow
Comm. Pure Appl. Math. **49** (**1996**), 999–1049.

Cathleen Synge Morawetz

Potential theory for regular and Mach reflection of a shock at a wedge Comm. Pure Appl. Math. **47** (**1994**), 593–624.

- Explored the nature of the shock reflection-diffraction patterns for weak incident shocks (strength b) and small wedge angles $2\theta_w$ through potential theory, a number of different scalings, a study of mixed equations and matching asymptotics for the different scalings;
- Recognized an important parameter

$$\beta = \beta(b, \theta_w, c_1, \gamma) \in [0, \infty)$$

  to determine some shock reflection-diffraction patterns, for c_1 the sound speed behind the incident shock for polytropic gas with adiabatic exponent $\gamma > 1$;
- Discovered new features and phenomena;
- ......

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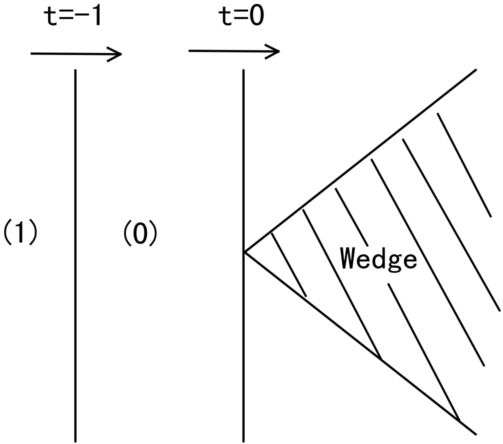
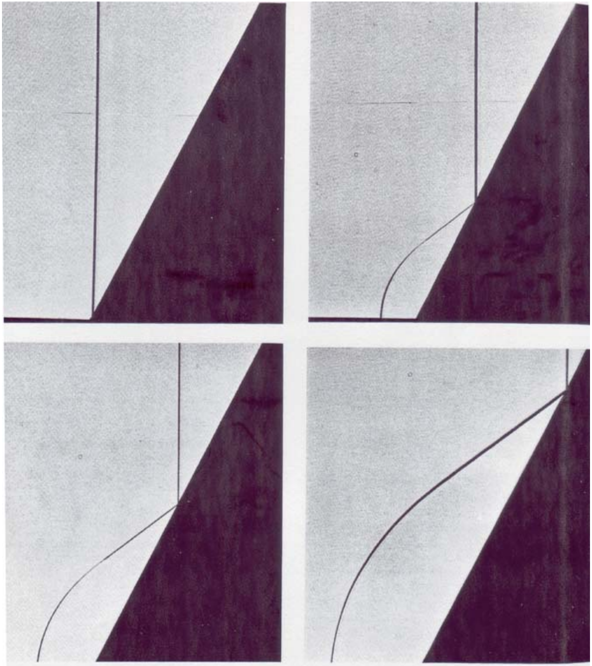
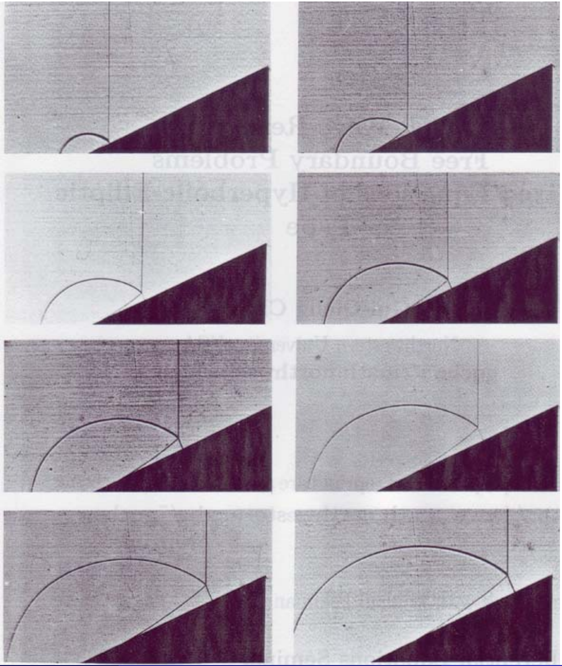


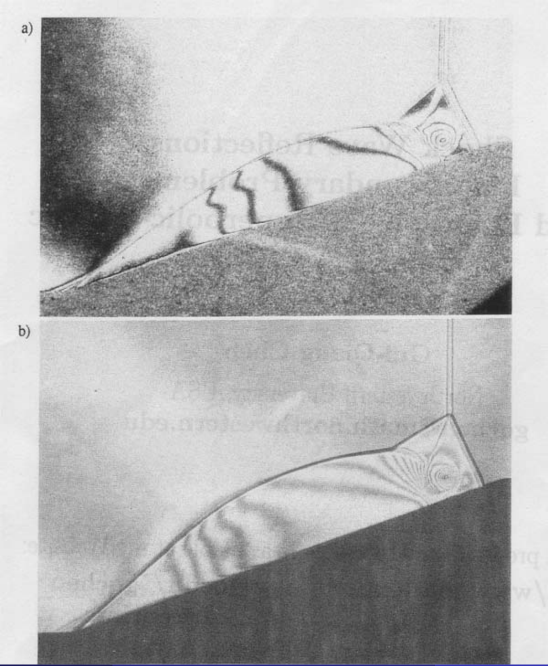
? **Shock Wave Patterns** around a Wedge (airfoils, inclined ramps, $\cdots$)

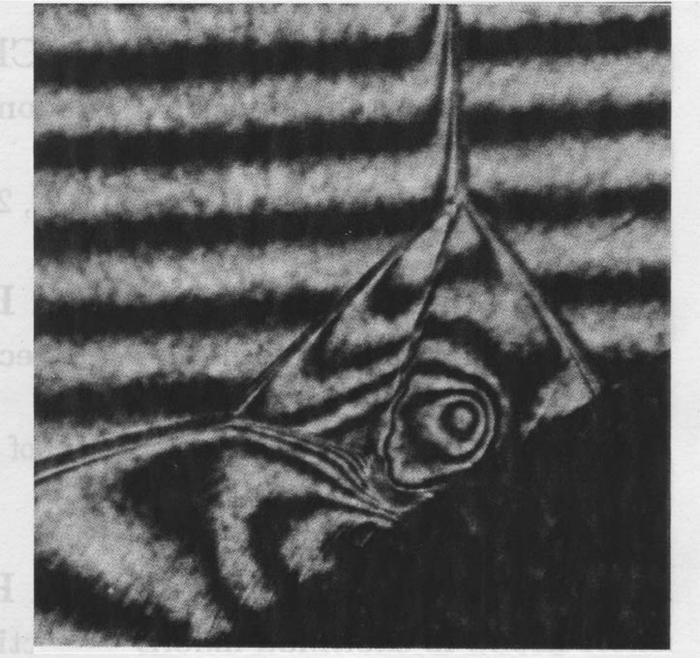
Complexity of Reflection-Diffraction Configurations Was First Identified and Reported by **Ernst Mach 1879**

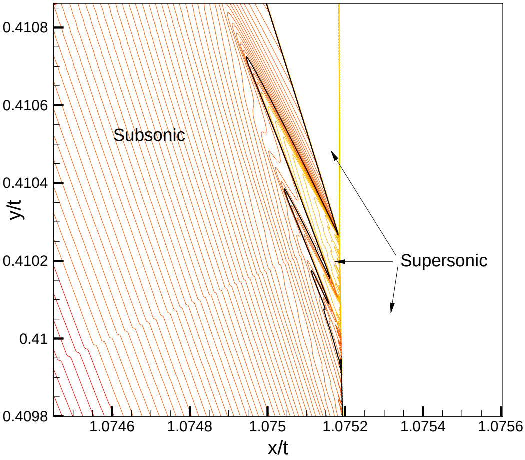
Experimental Analysis: **1940s$\Longrightarrow$ von Neumann, Bleakney, Bazhenova Glass, Takyama, Henderson,** $\cdots$






a)
b)





A New Mach Reflection-Diffraction Pattern:

A. M. Tesdall and J. K. Hunter: TSD, 2002

A. M. Tesdall, R. Sanders, and B. L. Keyfitz: NWE, 2006; Full Euler, 2008

B. Skews and J. Ashworth: J. Fluid Mech. 542 (2005), 105-114

Shock Reflection-Diffraction Patterns

- **Gabi Ben-Dor** Shock Wave Reflection Phenomena
  Springer-Verlag: New York, 307 pages, 1992.

  Experimental results before 1991
  Various proposals for transition criteria

- **Milton Van Dyke** An Album of Fluid Motion
  The parabolic Press: Stanford, 176 pages, 1982.

  Various photographs of shock wave reflection phenomena

- **Richard Courant & Kurt Otto Friedrichs**
  Supersonic Flow and Shock Waves
  Springer-Verlag: New York, 1948.

Scientific Issues

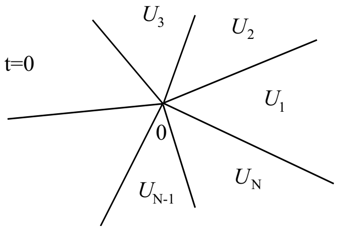
- **Structure of the Shock Reflection-Diffraction Patterns**
- **Transition Criteria among the Patterns**
- **Dependence of the Patterns on the Parameters**
  wedge angle θ_w, adiabatic exponent $\gamma \geq 1$
  incident-shock-wave Mach number M_s
- $\cdots\cdots$

Interdisciplinary Approaches:

- **Experimental Data and Photographs**
- **Large or Small Scale Computing**
  Colella, Berger, Deschambault, Glass, Glaz, Woodward,....
  Anderson, Hindman, Kutler, Schneyer, Shankar, ...
  Yu. Dem'yanov, Panasenko, ...
- **Asymptotic Analysis**
  Lighthill, Keller, Majda, Hunter, Rosales, Tabak, Gamba, Harabetian..
- **Morawetz: CPAM 1994**
- **Rigorous Mathematical Analysis??** **(Global Solutions)**
  Existence, Stability, Regularity, Bifurcation, $\cdots\cdots$

2-D Riemann Problem for Hyperbolic Conservation Laws

$$\partial_t U + \nabla_{\mathbf{x}} \cdot F(U) = 0, \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$$



Numerical Solutions: Glimm-Klingenberg-McBryan-Plohr-Sharp-Yaniv 1985
Lax-Liu 1998, Schulz-Rinne-Collins-Glaz 1993, Chang-Chen-Yang 1995, 2000
Kurganov-Tadmor 2002, $\cdots$

Books and Survey Articles: Chang-Hsiao 1989, Glimm-Majda 1991
Li-Zhang-Yang 1998, Zheng 2001, Chen-Wang 2002, Serre 2005, Chen 2005,$\cdots$

Theoretical Roles: Asymptotic States and Attractors
Local Structure and Building Blocks...

Full Euler Equations: $(t, \mathbf{x}) \in \mathbb{R}^3_+ := (0, \infty) \times \mathbb{R}^2$

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0 \\ \partial_t (\rho \mathbf{v}) + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p = 0 \\ \partial_t (\frac{1}{2} \rho |\mathbf{v}|^2 + \rho e) + \nabla_{\mathbf{x}} \cdot ((\frac{1}{2} \rho |\mathbf{v}|^2 + \rho e + p) \mathbf{v}) = 0 \end{cases}$$

Constitutive Relations: $p = p(\rho, e)$

- ρ–density, $\mathbf{v} = (v_1, v_2)^\top$–fluid velocity, p–pressure
- e–internal energy, θ–temperature, S–entropy

For a polytropic gas: $p = (\gamma - 1)\rho e, \ \ e = c_v \theta, \qquad \gamma = 1 + \frac{R}{c_v}$

$$p = p(\rho, S) = \kappa \rho^\gamma e^{S/c_v}, \qquad e = e(\rho, S) \frac{\kappa}{\gamma - 1} \rho^{\gamma - 1} e^{S/c_v},$$

- $R > 0$ may be taken to be the universal gas constant divided by the effective molecular weight of the particular gas
- $c_v > 0$ is the specific heat at constant volume
- $\gamma > 1$ is the adiabatic exponent, $\kappa > 0$ is any constant under scaling

Euler Equations for Potential Flow: $\mathbf{v} = \nabla\Phi$

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \nabla_{\mathbf{x}} \Phi) = 0, & \text{(conservation of mass)} \\ \partial_t \Phi + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2 + \frac{\rho^{\gamma-1}}{\gamma-1} = \frac{\rho_0^{\gamma-1}}{\gamma-1}, & \text{(Bernoulli's law);} \end{cases}$$

or, equivalently,

$$\partial_t \rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi, \rho_0) + \nabla_{\mathbf{x}} \cdot \big(\rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi, \rho_0)\nabla_{\mathbf{x}}\Phi\big) = 0,$$

with

$$\rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi, \rho_0) = \big(\rho_0^{\gamma-1} - (\gamma-1)(\partial_t\Phi + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2)\big)^{\frac{1}{\gamma-1}}.$$

Celebrated steady potential flow equation of aerodynamics:

$$\nabla_{\mathbf{x}} \cdot (\rho(\nabla_{\mathbf{x}}\Phi, \rho_0)\nabla_{\mathbf{x}}\Phi) = 0.$$

This approximation is well-known in transonic aerodynamics.

We will see the Euler equations for potential flow is actually EXACT in an important region of the solution to the shock reflection problem.

Discontinuities of Solutions

$$\partial_t U + \nabla_{\mathbf{x}} \cdot F(U) = 0, \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$$

An oriented surface Γ with unit normal $\mathbf{n} = (n_t, n_1, n_2) \in \mathbb{R}^3$ in the $(t, \mathbf{x})$-space is a discontinuity of a piecewise smooth entropy solution U with

$$U(t, \mathbf{x}) = \begin{cases} U_+(t, \mathbf{x}), & (t, \mathbf{x}) \cdot \mathbf{n} > 0, \\ U_-(t, \mathbf{x}), & (t, \mathbf{x}) \cdot \mathbf{n} < 0, \end{cases}$$

if the Rankine-Hugoniot Condition is satisfied

$$(U_+ - U_-, F(U_+) - F(U_-)) \cdot \mathbf{n} = 0 \qquad \text{along } \Gamma.$$

The surface $(\Gamma, \mathbf{n})$ is called a Shock Wave if the Entropy Condition (i.e., the Second Law of Thermodynamics) is satisfied:

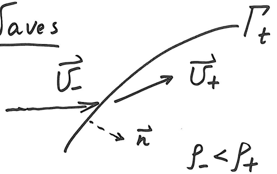
$$(\eta(U_+) - \eta(U_-), q(U_+) - q(U_-)) \cdot \mathbf{n} \geq 0 \qquad \text{along } \Gamma,$$

where $(\eta(U), q(U)) = (-\rho S, -\rho \mathbf{v} S)$.

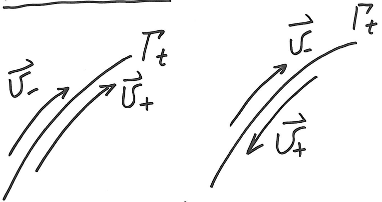
Shock Waves vs Vortex Sheets

Two Types of Discontinuities

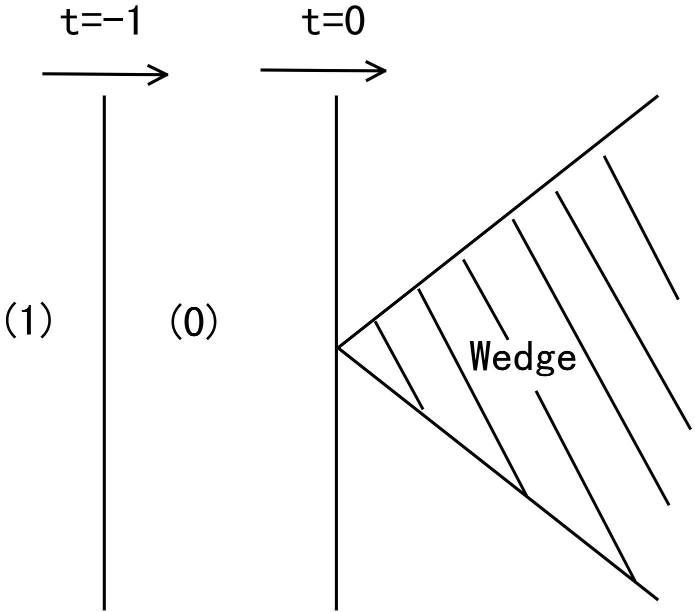
① Shock Waves



② Vortex Sheets



$$p_+ = p_-$$

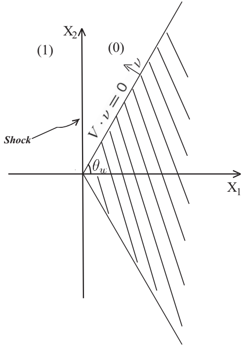

t=−1
t=0
(1)
(0)
Wedge

Initial-Boundary Value Problem: $0 < \rho_0 < \rho_1,\ v_1 > 0$

Initial condition **at** $t = 0$:

$$(\mathbf{v}, p, \rho) = \begin{cases} (0, 0, p_0, \rho_0), & |x_2| > x_1 \tan\theta_w, x_1 > 0, \\ (v_1, 0, p_1, \rho_1), & x_1 < 0; \end{cases}$$

Slip boundary condition **on the wedge bdry:** $\mathbf{v} \cdot \nu = 0$**.**



Invariant under the Self-Similar Scaling: $(t, \mathbf{x}) \longrightarrow (\alpha t, \alpha \mathbf{x}),\ \ \alpha \neq 0$

Seek Self-Similar Solutions

$$(\mathbf{v}, p, \rho)(t, \mathbf{x}) = (\mathbf{v}, p, \rho)(\xi, \eta), \quad (\xi, \eta) = (\tfrac{x_1}{t}, \tfrac{x_2}{t})$$

$$\begin{cases} (\rho U)_\xi + (\rho V)_\eta + 2\rho = 0, \\ (\rho U^2 + p)_\xi + (\rho U V)_\eta + 3\rho U = 0, \\ (\rho U V)_\xi + (\rho V^2 + p)_\eta + 3\rho V = 0, \\ (U(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}))_\xi + (V(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}))_\eta + 2(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}) = 0, \end{cases}$$

where $q = \sqrt{U^2 + V^2}$ and $(U, V) = (v_1 - \xi, v_2 - \eta)$ is the pseudo-velocity.

Eigenvalues: $\lambda_0 = \frac{V}{U}$ (repeated), $\lambda_\pm = \frac{UV \pm c\sqrt{q^2 - c^2}}{U^2 - c^2}$,
where $c = \sqrt{\gamma p / \rho}$ is the **sonic speed**

When the flow is pseudo-subsonic: $q < c$, the system is

hyperbolic-elliptic composite-mixed

Euler Equations in Self-Similar Coordinates

Entropy: $S = c_v\, ln(p\rho^{\gamma})$

Pseudo-velocity Angle: $\lambda_0 = V/U = \tan\Theta$

Pseudo-velocity Magnitude: $q = \sqrt{U^2 + V^2}$

$$\begin{cases} S_\xi + \lambda_0 S_\eta = 0, \\ \rho q(q_\xi + \lambda_0 q_\eta) + p_\xi + \lambda_0 p_\eta = -\rho(U + \lambda_0 V), \\ (U^2 - c^2)p_{\xi\xi} + 2UVp_{\xi\eta} + (V^2 - c^2)p_{\eta\eta} + A_1 p_\xi + A_2 p_\eta + \cdots = 0, \\ (U^2 - c^2)\lambda_{0\xi\xi} + 2UV\lambda_{0\xi\eta} + (V^2 - c^2)\lambda_{0\eta\eta} + A_1\lambda_{0\xi} + A_2\lambda_{0\eta} + \cdots = 0. \end{cases}$$

When the flow is pseudo-subsonic: $q < c$, the system consists of

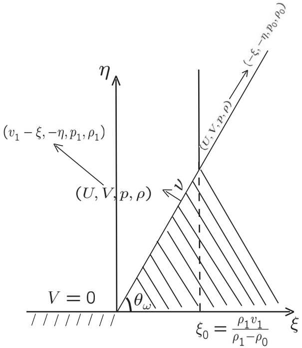
- 2-transport equations
- 2-nonlinear equations of hyperbolic-elliptic mixed type

Slip boundary condition on the wedge boundary:

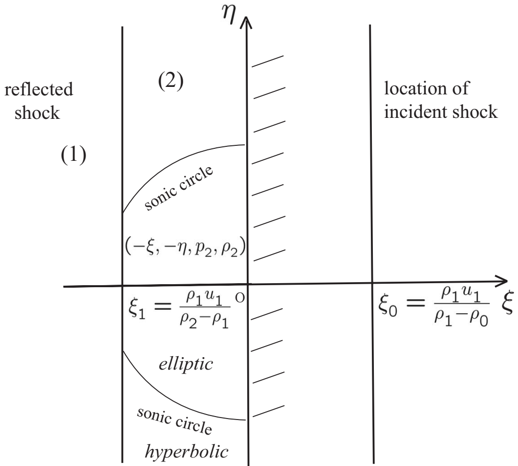
$$(U, V) \cdot \nu = 0 \qquad \textbf{on} \quad \partial D$$

Asymptotic boundary condition as $\xi^2 + \eta^2 \to \infty$:

$$(U + \xi, V + \eta, p, \rho) \to \begin{cases} (0, 0, p_0, \rho_0), & \xi > \xi_0, \eta > \xi \tan\theta_w, \\ (v_1, 0, p_1, \rho_1), & \xi < \xi_0, \eta > 0. \end{cases}$$



When $\theta_w = \pi/2$, the reflection becomes the normal reflection, for which the incident shock normally reflects and the reflected shock is also a plane.



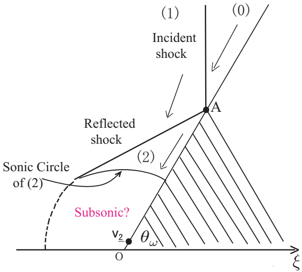
Local Theory for Regular Reflection (cf. Chang-C. 1986)

$\exists\, \theta_d = \theta_d(M_s, \gamma) \in (0, \frac{\pi}{2})$ such that, when $\theta_W \in (\theta_d, \frac{\pi}{2})$, there exist two states $(2) = (U_2^a, V_2^a, p_2^a, \rho_2^a)$ and $(U_2^b, V_2^b, p_2^b, \rho_2^b)$ such that $|(U_2^a, V_2^a)| > |(U_2^b, V_2^b)|$ and $|(U_2^b, V_2^b)| < c_2^b$.

Stability as $\theta_W \to \frac{\pi}{2}$: Choose $\varphi_2 = \varphi^a$.

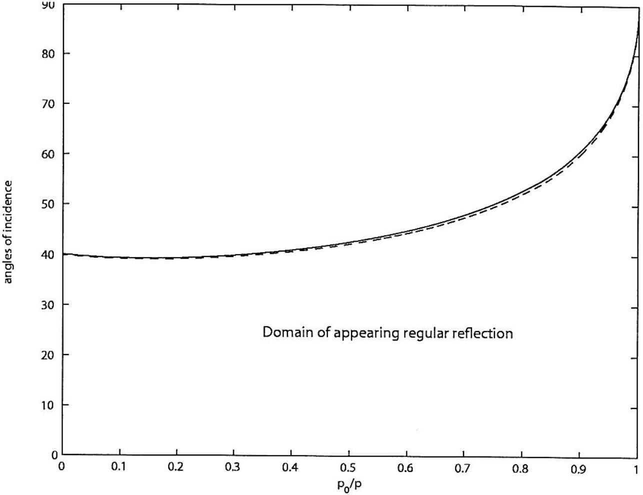
Detachment Conjecture: There exists a Regular Reflection Configuration when the wedge angle $\theta_W \in (\theta_d, \frac{\pi}{2})$.

Sonic Conjecture: There exists a Regular Reflection Configuration when $\theta_W \in (\theta_s, \frac{\pi}{2})$, for $\theta_s > \theta_d$ such that $|(U_2^a, V_2^a)| > c_2^a$ at A.

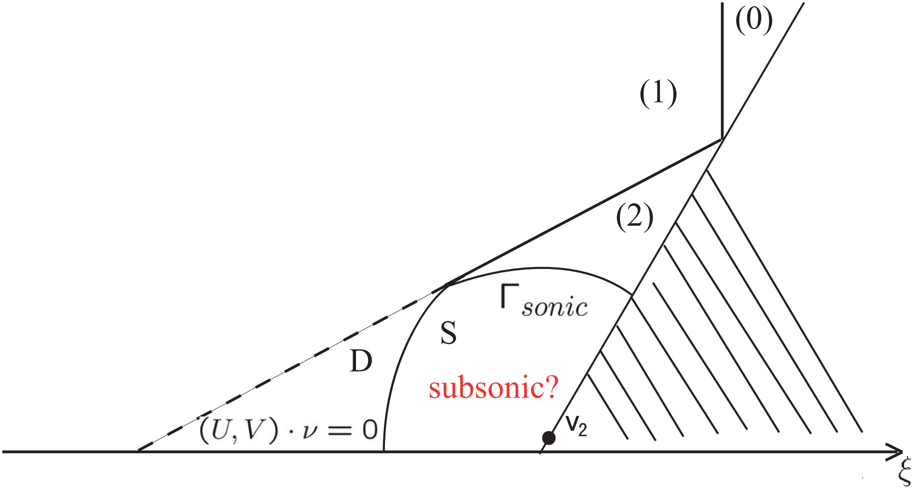


Detachment Criterion vs Sonic Criterion $\theta_c > \theta_s$: $\gamma = 1.4$

Courtesy of W. Sheng and G. Yin: ZAMP, 2008



Global Theory?



Euler Eqs. under Decomposition: $(U, V) = \nabla\varphi + W,\ \nabla \cdot W = 0$

$$\begin{cases} \nabla \cdot (\rho\nabla\varphi) + 2\rho + \nabla \cdot (\rho\nabla W) = 0, \\ \nabla(\frac{1}{2}|\nabla\varphi|^2 + \varphi) + \frac{1}{\rho}\nabla p = (\nabla\varphi + W) \cdot \nabla W + (\nabla^2\varphi + I)W, \\ (\nabla\varphi + W) \cdot \nabla\omega + (1 + \Delta\varphi)\omega = 0 \Longleftrightarrow \nabla \cdot ((\nabla\varphi + W)\omega) + \omega = 0, \\ (\nabla\varphi + W) \cdot \nabla S = 0. \end{cases}$$

where $\quad S = c_v \ln(p\rho^{-\gamma})$–**Entropy**; $\quad \omega = \mathrm{curl}\, W = \mathrm{curl}(U, V)$–**Vorticity**

When $\omega = 0, S = const.$, and $W = 0$ on a curve transverse to the fluid direction, then, in the region of the fluid trajectories past the curve,

$$W = 0,\ S = const. \quad \Rightarrow \quad W = 0,\ p = const.\, \rho^\gamma$$

Then we obtain the Potential Flow Equation (by scaling):

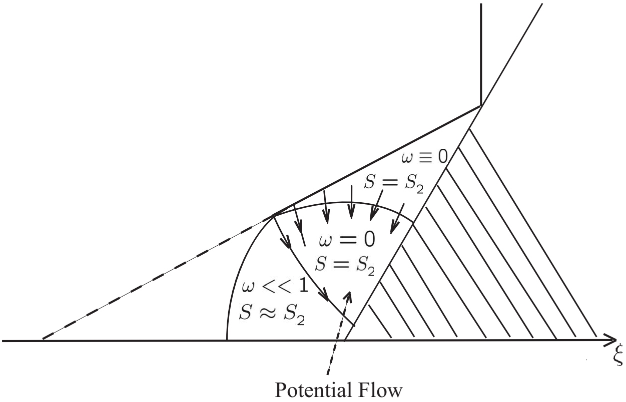
$$\begin{cases} \nabla \cdot (\rho\nabla\varphi) + 2\rho = 0, \\ \frac{1}{2}(|\nabla\varphi|^2 + \varphi) + \frac{\rho^{\gamma-1}}{\gamma - 1} = const. > 0. \end{cases}$$

J. Hadamard: Lecons sur la Propagation des Ondes, Hermann: Paris 1903

Potential Flow Dominates the Regular Reflection, provided that $\varphi \in C^{1,1}$ across the Sonic Circle

Potential Flow Equation

$$\begin{cases} \nabla \cdot (\rho \nabla \varphi) + 2\rho = 0, \\ \frac{1}{2}|\nabla\varphi|^2 + \varphi + \frac{\rho^{\gamma-1}}{\gamma-1} = \frac{\rho_0^{\gamma-1}}{\gamma-1} \end{cases}$$



Potential Flow Equation

$$\nabla \cdot (\rho(\nabla\varphi, \varphi, \rho_0)\nabla\varphi) + 2\rho(\nabla\varphi, \varphi, \rho_0) = 0$$

- Incompressible: $\rho = const. \Longrightarrow \Delta\varphi + 2 = 0$
- Subsonic (Elliptic):

$$|\nabla\varphi| < c_*(\varphi, \rho_0) := \sqrt{\frac{2}{\gamma+1}(\rho_0^{\gamma-1} - (\gamma-1)\varphi)}$$

- Supersonic (Hyperbolic):

$$|\nabla\varphi| > c_*(\varphi, \rho_0) := \sqrt{\frac{2}{\gamma+1}(\rho_0^{\gamma-1} - (\gamma-1)\varphi)}$$

Linear Models

Lavrentyev-Bitsadze Equation: $\partial_{xx} u + \mathrm{sign}(x)\partial_{yy} u = 0$;

Tricomi Equation: $u_{xx} + x u_{yy} = 0$ (Hyperbolic Degeneracy at $x = 0$);

Keldysh Equation: $x u_{xx} + u_{yy} = 0$ (Parabolic Degeneracy at $x = 0$).

Nonlinear Models

- Transonic Small Disturbance Equation:

$$\big((u - x)u_x + \frac{u}{2}\big)_x + u_{yy} = 0$$

or, for $v = u - x$,

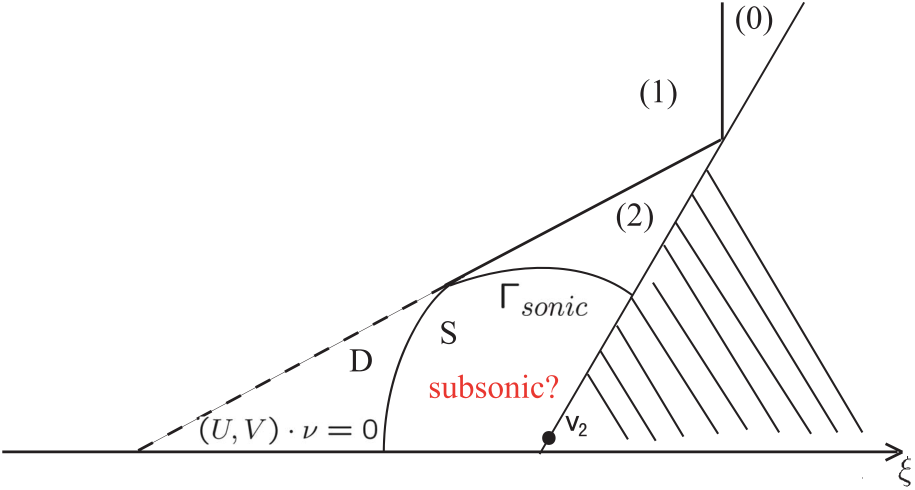
$$v\, v_{xx} + v_{yy} + \text{l.o.t.} = 0.$$

Morawetz, Hunter, Canic-Keyfitz-Lieberman-Kim, $\cdots$

- Pressure-Gradient Equations, Nonlinear Wave Equations

Y. Zheng, Canic-Keyfitz-Kim-Jegdic, $\cdots$

Global Theory?



Setup of the Problem for $\psi := \varphi - \varphi_2$ in Ω

- $\operatorname{div}\left(\rho(\nabla\psi, \psi, \xi, \eta, \rho_0)(\nabla\psi + \mathbf{v}_2 - (\xi, \eta)\right) + l.o.t. = 0 \qquad (*)$
- $\nabla\psi \cdot \nu|_{wedge} = 0$
- $\psi|_{\Gamma_{sonic}} = 0 \Longrightarrow \psi_\nu|_{\Gamma_{sonic}} = 0$
- Rankine-Hugoniot Conditions on Shock S:

$$[\psi]_S = 0$$

$$[\rho(\nabla\psi, \psi, \xi, \eta, \rho_0)(\nabla\psi + \mathbf{v}_2 - (\xi, \eta)) \cdot \nu]_S = 0 \qquad (**)$$

Free Boundary Problem

- $\exists\, S = \{\xi = f(\eta)\}$ such that $f \in C^{1,\alpha}$, $f'(0) = 0$ and

$$\Omega_+ = \{\xi > f(\eta)\} \cap D = \{\psi < \varphi_\mathbf{1} - \varphi_\mathbf{2}\} \cap \mathbf{D}$$

- ψ satisfy the free boundary condition $(**)$ along S
- $\psi \in C^{1,\alpha}(\overline{\Omega_+}) \cap C^2(\Omega_+) \begin{cases} \text{solves (*) in } \Omega_+, \\ \text{is subsonic in } \Omega_+ \end{cases}$

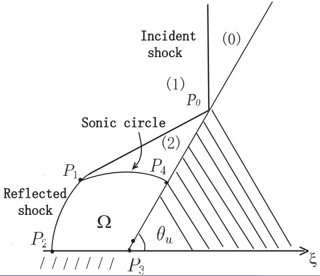
with $(\psi, \psi_\nu)|_{\Gamma_{sonic}} = 0, \qquad \nabla\psi \cdot \nu|_{wedge} = 0$

Theorem (Global Theory for Shock Reflection-Diffraction C.–Feldman: PNAS 2005; Ann. Math. accepted on Oct. 3, 2006)

$\exists\ \theta_c = \theta_c(\rho_0, \rho_1, \gamma) \in (0, \frac{\pi}{2})$ *such that, when* $\theta_W \in (\theta_c, \frac{\pi}{2})$, *there exist* (φ, f) *satisfying*

- $\varphi \in C^\infty(\Omega) \cap C^{1,\alpha}(\bar{\Omega})$ *and* $f \in C^\infty(P_1P_2) \cap C^2(\{P_1\})$;
- $\varphi \in C^{1,1}$ *across the sonic circle* P_1P_4
- $\varphi \longrightarrow \varphi_{NR}$ *in* $W^{1,1}_{loc}$ *as* $\theta_W \to \frac{\pi}{2}$.

$\Rightarrow \Phi(t,\mathbf{x}) = t\varphi(\frac{\mathbf{x}}{t}) + \frac{|\mathbf{x}|^2}{2t}$, $\rho(t,\mathbf{x}) = \left(\rho_0^{\gamma-1} - (\gamma-1)(\Phi_t + \frac{1}{2}|\nabla\Phi|^2)\right)^{\frac{1}{\gamma-1}}$ *form a solution of the IBVP.*

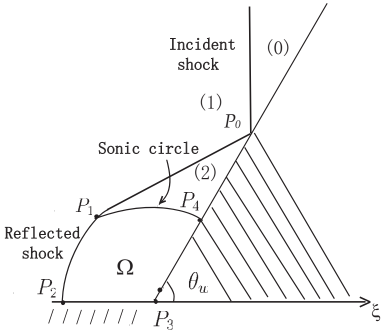


Theorem (Optimal Regularity; Bae–C.–Feldman 2007):

$\varphi \in C^{1,1}$ but NOT in C^2 across P_1P_4;

$\varphi \in C^{1,1}(\{P_1\}) \cap C^{2,\alpha}(\bar{\Omega} \setminus (\{P_1\} \cup \{P_3\})) \cap C^{1,\alpha}(\{P_3\}) \cap C^{\infty}(\Omega)$;

$f \in C^2(\{P_1\}) \cap C^{\infty}(P_1P_2)$.



Optimal Regularity and Sonic Conjecture

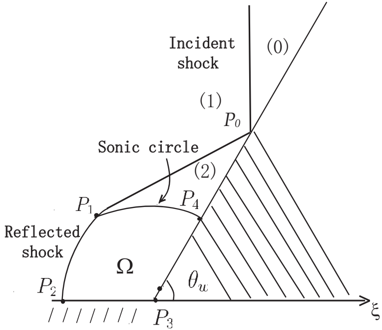
Theorem (Optimal Regularity; Bae–C.–Feldman 2007):

$\varphi \in C^{1,1}$ but NOT in C^2 across P_1P_4;

$\varphi \in C^{1,1}(\{P_1\}) \cap C^{2,\alpha}(\bar{\Omega} \setminus (\{P_1\} \cup \{P_3\})) \cap C^{1,\alpha}(\{P_3\}) \cap C^{\infty}(\Omega)$;

$f \in C^2(\{P_1\}) \cap C^{\infty}(P_1P_2)$.

$\Longrightarrow$ The optimal regularity and the global existence hold up to the sonic wedge-angle θ_s for any $\gamma \geq 1$ for $u_1 < c_1$; $u_1 \geq c_1$.
(the von Neumann's sonic conjecture)



- **Cutoff Techniques by Shiffmanization**
  $\Rightarrow$ Elliptic Free-Boundary Problem with Elliptic Degeneracy on Γ_{sonic}
- **Domain Decomposition**
  Near Γ_{sonic}
  Away from Γ_{sonic}
- **Iteration Scheme**
  C.–Feldman, J. Amer. Math. Soc. 2003
- $C^{1,1}$ **Parabolic Estimates near the Degenerate Elliptic Curve** Γ_{sonic};
- **Corner Singularity Estimates**
  In particular, when the Elliptic Degenerate Curve Γ_{sonic} Meets the Free Boundary, i.e., the Transonic Shock
- **Removal of the Cutoff**
  Require the Elliptic-Parabolic Estimates

?? Extend the Large-Angle to the Sonic-Angle θ_s??

Elliptic Degeneracy

- **Linear**

$$2x\psi_{xx} + \frac{1}{c_2^2}\psi_{yy} - \psi_x \sim 0$$

$$\psi \sim Ax^{3/2} + h.o.t. \qquad \text{when } x \sim 0$$

- **Nonlinear**

$$\begin{cases} \left(2x - (\gamma+1)\psi_x\right)\psi_{xx} + \frac{1}{c_2^2}\psi_{yy} - \psi_x \sim o(x^2) \\ \Psi|_{x=0} = 0 \end{cases}$$

Ellipticity: $\psi_x \leq \frac{2x}{\gamma+1}$

Apriori Estimate: $|\psi_x| \leq \frac{4x}{3(\gamma+1)}$

$$\psi \sim \frac{x^2}{2(\gamma+1)} + h.o.t. \qquad \text{when } x \approx 0$$

Large Angles $\Longrightarrow$ Sonic Angle θ_s: Method of Continuity

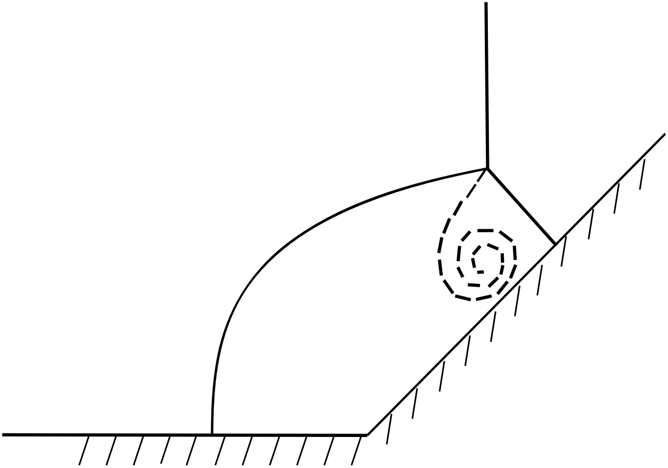
It suffices to prove that, for any $\theta_c \in (\theta_s, \frac{\pi}{2})$, a global regular reflection solution exists for all wedge angles $\theta_w \in [\theta_c, \frac{\pi}{2})$.

- **Design** $\mathcal{M} \subset [\theta_c, \frac{\pi}{2})$ to be **a set of all wedge angles** for which **a global regular reflection solution** exists and satisfies **certain properties**;
- **Show that the set $\mathcal{M}$ is nonempty**

  $\Longleftarrow$ Existence result for the large-angle case;
- **Prove that $\mathcal{M}$ is relatively open in $[\theta_c, \frac{\pi}{2})$**;
- **Prove that $\mathcal{M}$ is relatively closed in $[\theta_c, \frac{\pi}{2})$**;

$\Longrightarrow \mathcal{M} = [\theta_c, \frac{\pi}{2})$.

$\Longrightarrow$ von Neumann's Sonic Conjecture



? Right space for vorticity ω?

? Chord-arc $z(s) = z_0 + \int_0^s e^{ib(s)} ds$, $b \in BMO$?

Some of Other Recent Related Efforts and Developments

S.-X. Chen: Local Stability of Mach Configurations $\cdots$

D. Serre: Multi-D Shock Interaction for a Chaplygin Gas

S. Canic, B. Keyfitz, K. Jegdic, E. H. Kim:

Self-Similar Solutions of 2D Conservation Laws
Semi-Global Solutions for Shock Reflection Problems $\cdots$

V. Elling and T.-P. Liu:

Supersonic Flow onto a Solid Wedge (Prandtl Conjecture)
Counterexamples to the Sonic and Detachment Criteria $\cdots$

Y. Zheng+al: Solutions to Some 2-D Riemann Problems $\cdots$

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Steady Transonic Flow: Morawetz 1956–58, 1985–95

Transonic Nozzles: C.–Feldman, S. Chen, J. Chen, Xin-Yin, Yuan,...
Wedge or Conical Body: S. Chen, B. Fang, C.–Fang, $\cdots$
Transonic Flow past an Obstacle: Gamba-Morawetz, Gamba,
C.-Dafermos-Slemrod-Wang, C.-Slemrod-Wang, $\cdots$

$\cdots\cdots$

- **Free Boundary Techniques**
- **Mixed and Composite Eqns. of Hyperbolic-Elliptic Type**
  Degenerate Elliptic Techniques
  Degenerate Hyperbolic Techniques
  Transport Equations with Rough Coefficients
- **Regularity Estimates when a Free Boundary Meets a Degenerate Curve**
- **Boundary Harnack Inequalities**
- **Spaces for Compressible Vortex Sheets**
- **More Efficient Numerical Methods** $\cdots$
- $\cdots\cdots$

Happy 85th Birthday

To Cathleen