On the mixed state in anisotropic superconductors

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Work in progress with Stan Alama & Etienne Sandier

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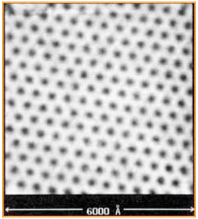

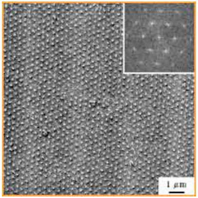

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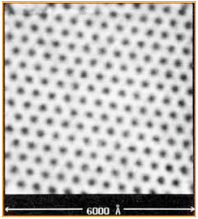

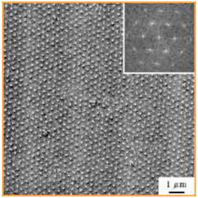

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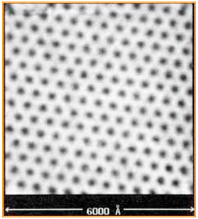

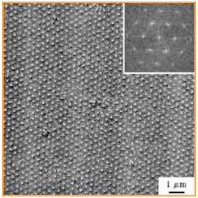

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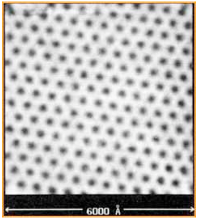

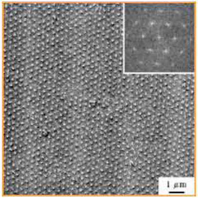

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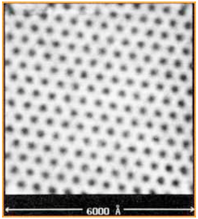

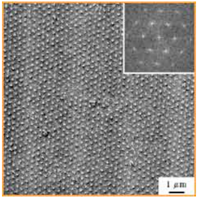

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- Vortex core radius $\sim \varepsilon$, separated by distance $\sim h_{ex}^{-1/2} \sim |\ln \varepsilon|^{-1/2}$

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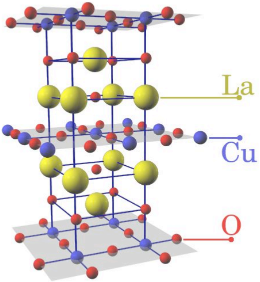
electrons pass easily in the CuO_2 planes, must tunnel through insulating gaps.



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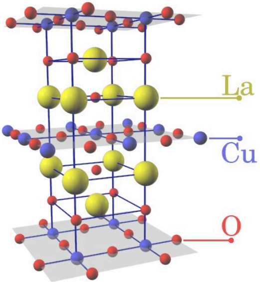

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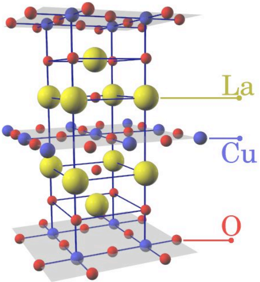

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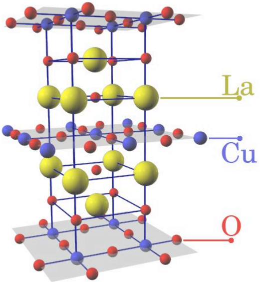

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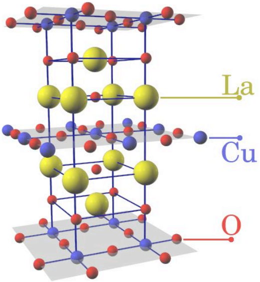

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Question: How are the lower critical field H_{c1} and the orientation of the vortex lattices affected by anisotropy?

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- Still expect a dense lattice of vortex lines for $h_{ex} \sim H_{c1} = O(|\ln \varepsilon|)$

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- In any plane P, if (u, A) is Floquet-Periodic on $\Omega \subset P$, magnetic flux is quantized:

$$\int_\Omega h \cdot n \, dS = 2\pi D,$$

where $D = \deg\left(\frac{u}{|u|}, \partial\Omega\right)$, the winding number of the phase of u.

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$$G_\varepsilon(u, A; Q) \gtrsim \frac{1}{2} \left| \int_Q h \right|_g |\ln \varepsilon| + \frac{1}{2} \int_Q |h - h_{ex}|^2.$$

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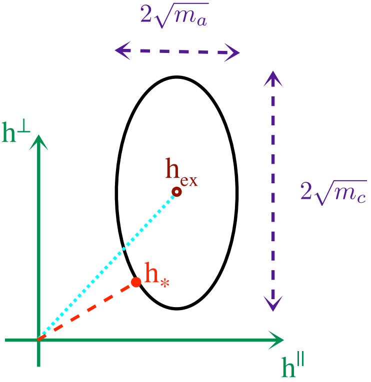

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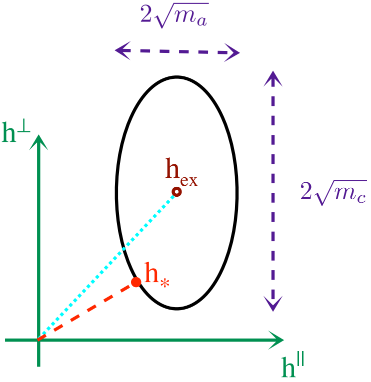

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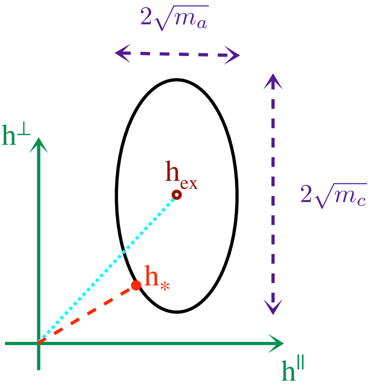
$$H_{c1}(\theta) = \frac{|\ln\varepsilon|}{2\sqrt{\frac{\cos^2\theta}{m_{ab}} + \frac{\sin^2\theta}{m_c}}}$$



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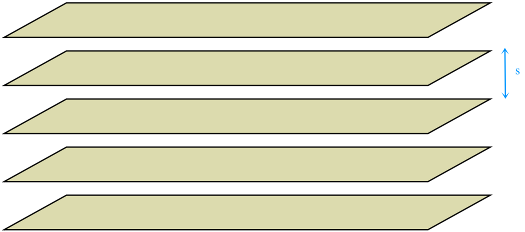
Let $Q = [0,1]^3 \subset \mathbb{R}^3$, and P_n an array of N horizontal planes with spacing $s = 1/N$, $P_n = [0,1]^2 \times \{z = z_n\}$, $z_n = ns$.



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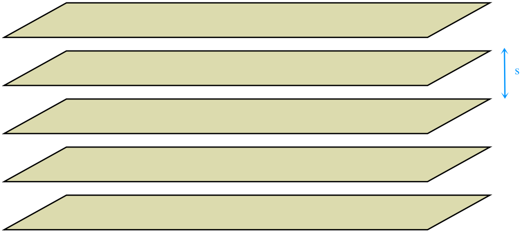

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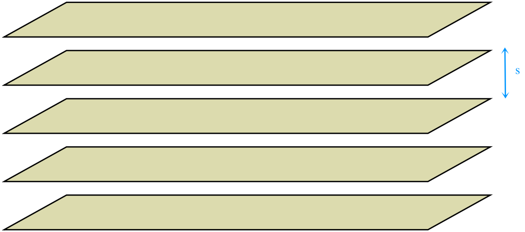

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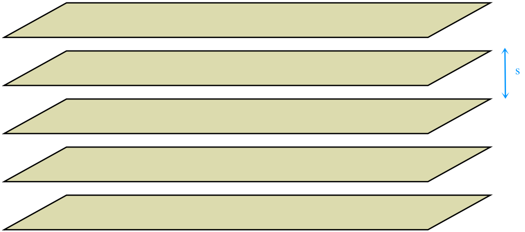

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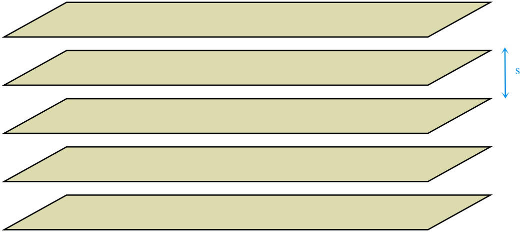

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- We will be interested in how energy-minimizing Q-periodic configurations depend on the angle between h_{ex} and the planes.

The Lawrence–Doniach functional

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LD(u_n, A) = s \sum_{n=1}^{N} \int_{P_n} \left\{ \frac{1}{2} |\nabla' u_n - i A' u_n|^2 + \frac{1}{4\varepsilon^2} \left(1 - |u_n|^2\right)^2 \right\} dx\, dy \\
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Mathematical results on LD

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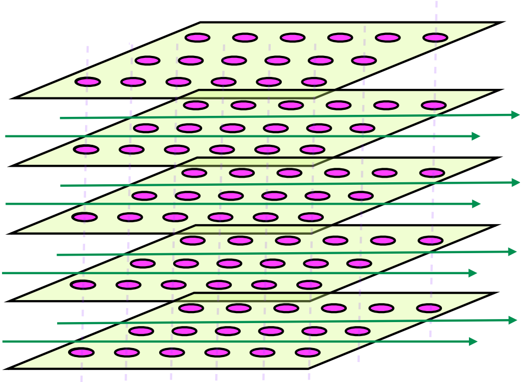
1. Decoupled lattices.
Vertical array of pancake vortices overlaps horizontal array of interlayer vortices



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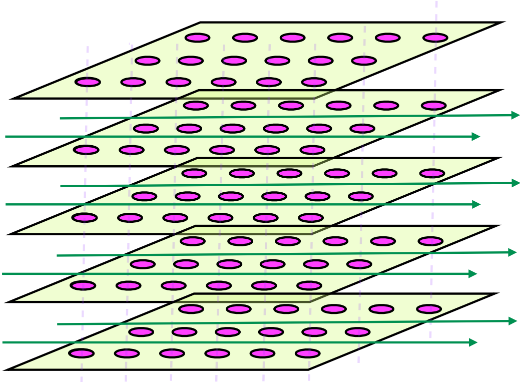

Work in progress: this happens when $\frac{1}{s^2} \ll |h_{ex}| \ll \frac{1}{\varepsilon^2}$

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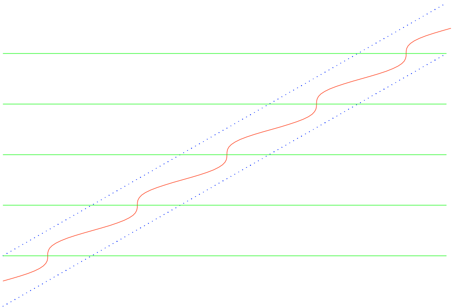

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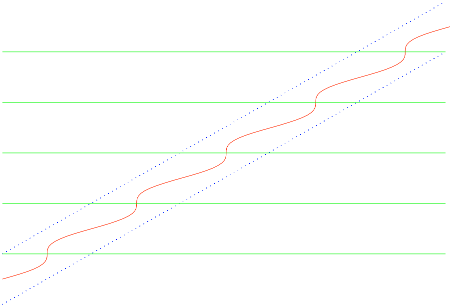

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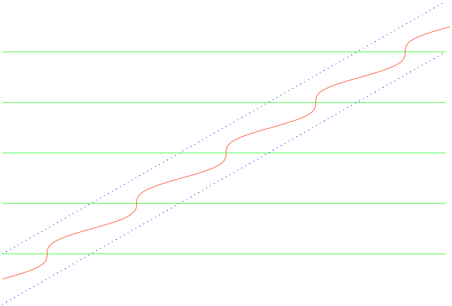

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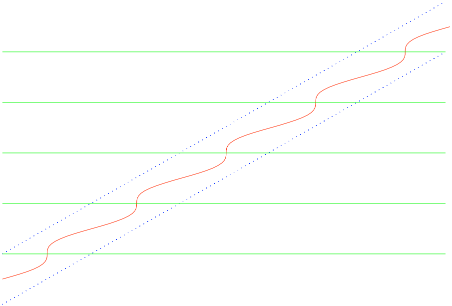

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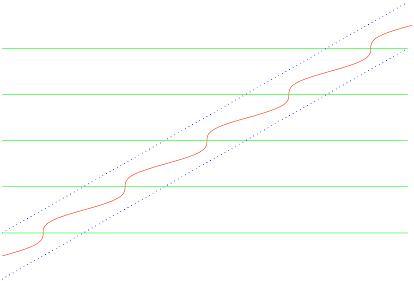

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Vortex energy from D "pancake vortices" (core radius ε) in each plane P_n:

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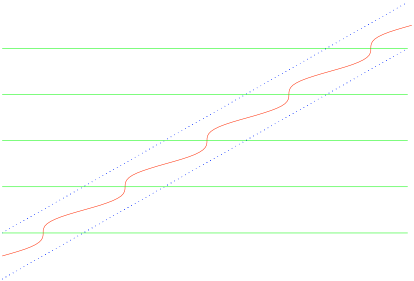

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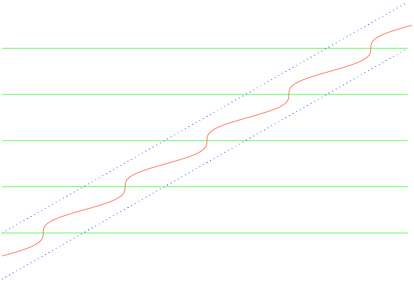

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Assume $\vec{h}_{ex} = (H^{||}\vec{e}_1 + H^{\perp}\vec{e}_3)|\ln\varepsilon|$, $s = \varepsilon^{\alpha}$, (u_n, A) minimizes LD, $h = \nabla \times A$. Then,

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- $\lim_{\varepsilon\to 0} \frac{LD(u_n,A)}{|\ln\varepsilon|^2} = \frac{1}{2}(1-\alpha)\left|\int_Q H^{\perp}\right| + \frac{1}{2}\alpha\left|\int_Q H\right| + \frac{1}{2}\int_Q |H - H|^2.$

Staircase vortices: Theorem

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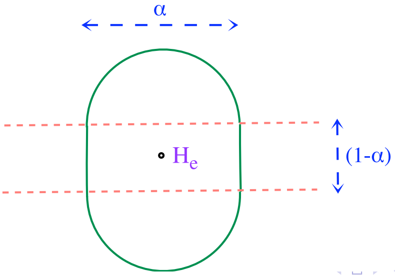
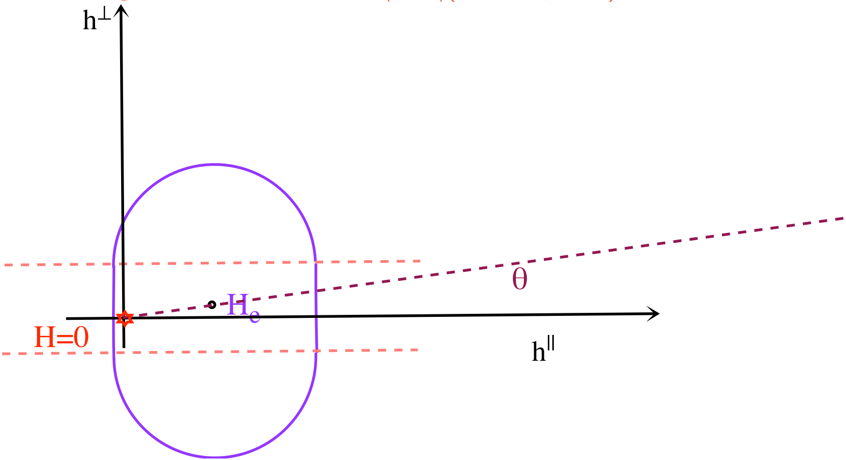
- Equivalently, $H \in K_\alpha$ is the point of the convex set K_α (below) which is closest to the origin.



Staircase vortices: limiting field H, examples

Fix an angle θ, consider $\vec{H}_{ex} = |H_{ex}|(\cos\theta, 0, \sin\theta)$



When $\vec{H}_{ex}$ is close enough to zero, then $0 \in K_\alpha$, so $\vec{H} = 0 \implies$ we are below the critical field $H_{C1}(\theta)$.

Staircase vortices: limiting field H, examples

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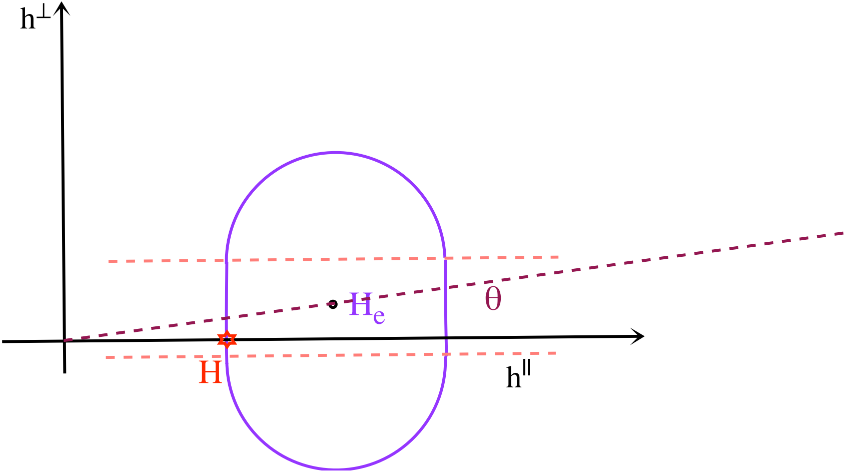

When 0 lies outside K_α, $\vec{H} \in K_\alpha$ chooses the closest point. The angle is small enough so $\vec{H}$ is chosen along the $h^{||}$-axis

Staircase vortices: limiting field H, examples

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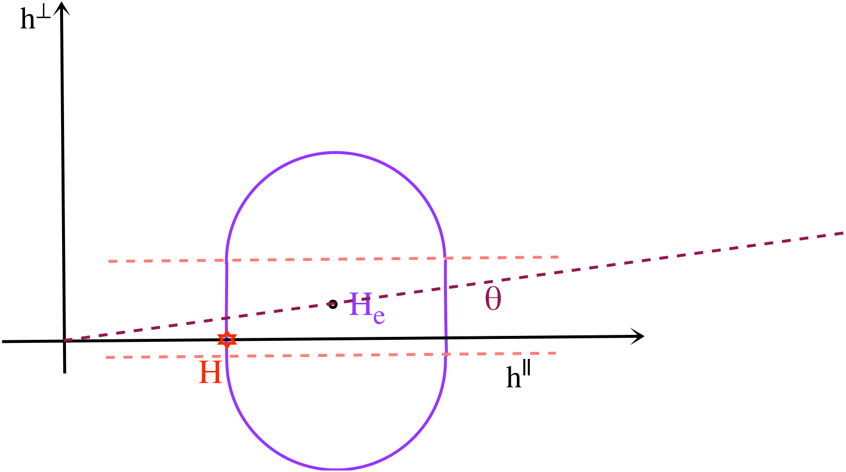

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This will never happen for Aniso GL!

Staircase vortices: limiting field H, examples

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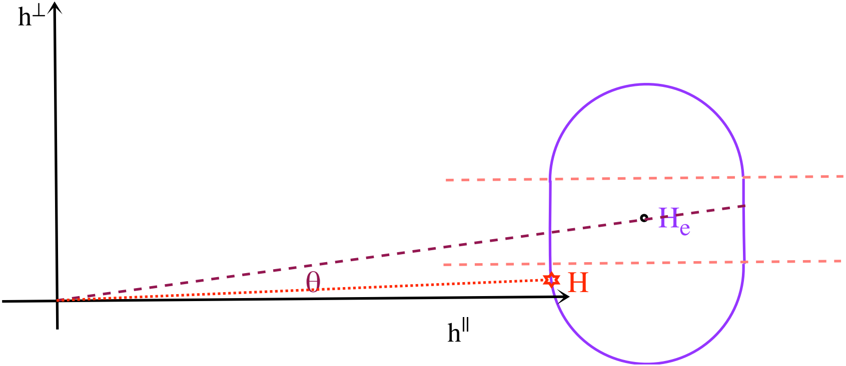

Even for a small angle, eventually the closest point $\vec{H}$ has an $h^{\perp}$ component.

The angle of the induced field H is generally not the same as the applied field $\vec{H}_{ex}$.