Large deviations for zeros of random holomorphic sections on Riemann surfaces

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Based on prior work and B.Shiffman

Purpose of talk

The purpose of this talk is to give a

Large deviations principle

for probabilities of

configurations of zeros

of random holomorphic sections of line bundles over Riemann surfaces.

Result: Zero configurations concentrate exponentially fast around the equilibrium measure

Empirical measure of zeros of a section

Let $L^N \to C$ be an ample line bundle of degree N over a compact Riemann surface C of any genus. Let $H^0(C, L)$ denote its holomorphic sections.

The empirical measure of zeros of $s \in H^0(C, L)$ is the probability measure on C defined by

$$Z_s := d\mu_\zeta := \frac{1}{N} \sum_{\{\zeta : s(\zeta)=0\}} \delta_\zeta$$

Here, δ_ζ is the Dirac point measure at $\zeta \in C$.

Configuration spaces

Configurations of N points on C are points of the Nth configuration space

$$C^{(N)} = Sym^N C := \underbrace{C \times \cdots \times C}_{N} / S_N.$$

Here, S_N is the symmetric group on N letters.

The zero set of a section s is equally specified by its empirical measure Z_s and by its divisor

$$\mathcal{D}(s) = \zeta_1 + \cdots + \zeta_N,$$

where $\mathcal{D}$ defines a map

$$\mathcal{D} : H^0(M, L^N) \to C^{(N)}.$$

Configuration space versus the space of measures

There is a natural embedding of any configuration space $C^{(N)}$ to the space $\mathcal{M}(C)$ of probability measures on C:

$$\mu : C^{(N)} \to \mathcal{M}(C),$$

$$\mu_{(\zeta_1+\cdots+\zeta_N)} := \mu_\zeta := \frac{1}{N}\sum_{j=1}^{N} \delta_{\zeta_j}.$$

We view $\mathcal{M}(C)$ as the 'large N limit' of $C^{(N)}$ and probability measures as generalized configurations.

How probable is a given configuration of zeros?

To make sense of this question, we need to introduce probability measures γ_N on the spaces $H^0(M, L^N)$.

We first discuss genus zero. In this case, the simplest and most natural γ_N are the Gaussian measures $\gamma_N(h, \nu)$ which depend on the choice of two objects:

- A C^∞ Hermitian metric h on L;
- A probability measure ν on C.

Hermitian inner products $G_N(h,\nu)$ and Gaussian measures $\gamma_N(h,\nu)$

The data (h,ν) induce Hermitian inner products $Hilb_N(h,\nu)$ on $H^0(\mathbb{CP}^1, \mathcal{O}(N))$,

$$||s||^2_{G_N(h,\nu)} := \int_{\mathbb{CP}^1} |s(z)|^2_{h^N} d\nu(z).$$

The associated Gaussian measure is the probability measure on $H^0(\mathbb{CP}^1, \mathcal{O}(N))$ formally given by

$$d\gamma_N(h,\nu) = e^{-||s||^2_{G_N(h,\nu)}} ds.$$

The Gaussian measures $\gamma_N(h, \nu)$

More precisely, let $\{S_j\}$ denote a $G_N(h, \nu)$ - ONB (orthonormal basis) of $H^0(\mathbb{CP}^1, \mathcal{O}(N))$. Express each section as

$$s = \sum_{j=1}^{d_N} c_j S_j, \quad \langle S_j, S_k \rangle_{G_N(h,\nu)} = \delta_{jk}.$$

Here, $d_N = \dim H^0(C, L^N) = N - g + 1$.

In the complex coordinates c_j,

$$d\gamma_N(h, \nu) := (2\pi)^{-d_N} e^{-||c||^2} dc.$$

Equivalently, the c_j are complex normal variables satisfying $\mathbf{E}(c_j) = 0 = \mathbf{E}(c_j c_k)$, $\mathbf{E}(c_j \overline{c_k}) = \delta_{jk}$.

Why Gaussian measure?

- It is defined in terms of the coefficients of s (relative to a basis). Studying zeros of Gaussian random sections is studying the map from coefficients c_j to zeros ζ_k of a section.

- One studies this map probabilistically since it is very complicated to study for individual sections.

- Gaussian measure essentially means: pick a section at random from the unit sphere in $H^0(\mathbb{CP}^1, \mathcal{O}(N))$ wrt $G_N(h, \nu)$.

- In higher genus, we use Abel-Jacobi theory to define probability measures (later).

Fubini-Study measures $dV_N(h,\nu)$ on $\mathbb{P}H^0(\mathbb{CP}^1, \mathcal{O}(N))$

This is an equivalent but more convenient probability measure: since zeros of s and cs are the same, it is natural to work on $\mathbb{P}H^0(\mathbb{CP}^1, \mathcal{O}(N))$. The inner product $G_N(h,\nu)$ then induces a Fubini-Study metric and volume form

$$dV_N(h,\nu) \text{ on } \mathbb{P}H^0(C, L^N).$$

$$\mathbb{P}H^0(\mathbb{CP}^1, \mathcal{O}(N)) \simeq (\mathbb{CP}^1)^{(N)}.$$

Write $P_\zeta = \Pi_{j=1}^N (z - \zeta_j)$.

We define a line bundle $\mathcal{Z}_N \to (\mathbb{CP}^1)^{(N)}$: the fiber of $\mathcal{Z}_N$ at $\zeta_1 + \cdots + \zeta_N$ is the line $\mathbb{C}P_\zeta$ of holomorphic sections of $\mathcal{O}(N)$ with zeros $\zeta = \zeta_1 + \cdots + \zeta_N$.

The Hermitian inner product $G_N(h, \nu)$ defines a Hermitian metric on $\mathcal{Z}_N$. $dV_N(h, \nu)$ is its curvature volume form. It extends to all of $\mathbb{P}H^0(\mathbb{CP}^1, \mathcal{O}(N))$.

Explicit formula

Let V denote a vector space of of dimension $d+1$. A Fubini-Study metric on $\mathbb{P}V$ corresponds to an inner product $||f||_G^2$ L^2. The associated Fubini-Study $(1,1)$ form is

$$\omega_{FS,G} = \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\log||f||_G^2.$$

The normalized volume form (of mass one) is given by

$$dV_{FS,G} = \frac{\bigwedge^d\left(\partial\bar{\partial}||f||_G^2\right)}{(||f||_G^2)^{d+1}}.$$

Joint probability current of zeros

The identification map $\mathcal{D} : \mathbb{P}H^0(\mathbb{CP}^1, \mathcal{O}(N)) \simeq (\mathbb{CP}^1)^{(N)}$ can be used to push forward the Fubini-Study measure $dV_N(h, \nu)$ to $(\mathbb{CP}^1)^{(N)}$. The image measure $\mathcal{D}_* dV_N(h, \nu)$ is called the JPC: joint probability current.

It is Fubini-Study measure on coefficients expressed in terms of zeros. I.e. we rewrite the probability measure on sections by changing coordinates

$$\text{coefficients} \;\to\; \text{zeros}.$$

It gives the probability density of zeros occurring at $\zeta_1 + \cdots + \zeta_N$.

Joint probability current of zeros (2)

More precisely, the JPC is the (N, N) current on $(\mathbb{CP}^1)^N$ defined by

$$\vec{K}_N^N(\zeta^1, \ldots, \zeta^N) = \mathbf{E}\left(Z_s(\zeta_1) \otimes Z_s(\zeta_2) \otimes \cdots \otimes Z_s(\zeta_N)\right).$$

A calculation which is easy in genus zero but hard in higher genera is that the JPC

$$= \frac{1}{Z_N(h)} \frac{|\Delta(\zeta_1, \ldots, \zeta_N)|^2 d^2\zeta_1 \cdots d^2\zeta_N}{\left(\int_{\mathbb{CP}^1} \prod_{j=1}^N |(z-\zeta_j)|^2 e^{-N\varphi(z)} d\nu(z)\right)^{N+1}}$$

where $Z_N(h)$ is the normalizing constant; in the affine chart $h = e^{-\varphi}$.

The Vandermonde in the numerator damps out probability of multiple or near multiple zeros.

Large N limit

We can now begin to state our problem more rigorously: find the large N asymptotics of $\vec{K}_N^N(\zeta^1, \ldots, \zeta^N)$. I.e. the probability that $\zeta_1 + \cdots + \zeta_N$ is the zero set of a polynomial.

But: how can we take the large N limit when the domain $(\mathbb{CP}^1)^{(N)}$ is changing with N?

Large N limit and large deviations

This problem is solved by the map $\mu : (\mathbb{CP}^1)^{(N)} \to \mathcal{M}(\mathbb{CP}^1)$ taking a configuration to its empirical measure,

$$d\mu_\zeta = \frac{1}{N}\sum_{j=1}^{N} \delta_{\zeta_j}.$$

We will push forward $dV_N(h,\nu)$ again under μ to obtain a sequence of – probability measures $\mathbf{Prob}_N$ on the space $\mathcal{M}(\mathbb{CP}^1)$ of probability measures. This is standard in large deviations and explains why we are using this theory.

$$\mathbf{Prob}_N = \mu_* \mathcal{D}_* dV_N(h,\nu) = \mu_* \vec{K}_N^N.$$

Main result

Our main results show that this sequence of measures $\mathbf{Prob}_N$ satisfies a large deviations principle with speed N^2 and with a **rate functional** I reflecting the choice of (h, ν). Roughly speaking, an LDP means that for any Borel subset $E \subset \mathcal{M}(\mathbb{CP}^1)$, as $N \to \infty$,

$$\frac{1}{N^2} \log \mathbf{Prob}_N\{\sigma \in \mathcal{M} : \sigma \in E\} \to - \inf_{\sigma \in E} I(\sigma).$$

Main result

Theorem **1** *Under a technical assumption on* ν*, the sequence of probability measures* $\{Prob_N\}$ *on* $\mathcal{M}(\mathbb{CP}^1)$ *satisfies a large deviations principle with speed* N^2 *and rate functional*
(1)
$$I^{h,K}(\mu) = -\frac{1}{2}\mathcal{E}_h(\mu) + \sup_K U_h^\mu + E(h) - \int \varphi dd^c \varphi.$$

Here,

$$\mathcal{E}_h(\mu) = \int_{\mathbb{CP}^1 \times \mathbb{CP}^1} G_h(z,w) d\mu(z) d\mu(w),$$

and $U_h^\mu(z) = \int_{\mathbb{CP}^1} G_h(z,w) d\mu(w)$ *is the Greens's potential. This rate functional is LSC, proper and convex, and there exists a unique measure* $\nu_{h,K} \in \mathcal{M}(\mathbb{CP}^1)$ *minimizing* $I^{h,K}$*, namely the Green's equilibrium measure of* K *with respect to* h

Informally...

The empirical measures $d\mu_\zeta$ of zeros of random sections are highly concentrated in a shrinking small ball around the equilibrium measure $d\nu_{h,K}$. The probability of the empirical measure being outside the ball decays exponentially fast. The rate is given by the speed and rate functional.

The Green's function is defined by

$$\begin{cases} dd^c G_h(z,w) + \delta_z = \omega_h, \\ \int_{\mathbb{CP}^1} G_h(z,w)\omega_h(w) = 0. \end{cases}$$

More precisely

Let $B(\sigma, \delta)$ denote the ball of radius δ around $\sigma \in \mathcal{M}(\mathbb{CP}^1)$ in the Wasserstein metric. Then

$$-\inf_{\mu \in B^o(\sigma,\delta)} I^{h,K}(\mu)$$

$$\leq \liminf_{N\to\infty} \frac{1}{N^2} \log \mathbf{Prob}_N(B(\sigma,\delta))$$

$$\leq \limsup_{N\to\infty} \frac{1}{N^2} \log \mathbf{Prob}_N(B(\sigma,\delta)) I(\mu)$$

$$\leq -\inf_{\mu \in \overline{B(\sigma,\delta)}} I^{h,K}(\mu).$$

Here,

$$d_W(\mu,\nu) = \sup_{f:||f||_{Lip}\leq 1} |\int f d\mu - \int f d\nu|.$$

Prior results

The model for results of this kind are results of Ben-Arous-Guionnet and Ben-Arous-Zeitouni on LDP for empirical empirical measures of eigenvalues of random matrices.

This is the first LDP result on zeros of random functions. There are overlaps in the methods but several new features.

Afortiori, the expected value $\mathbf{E}\, d\mu_\zeta \to \nu_{h,K}$. This was proved by: Shiffman-Zelditch '99 in the case where the curvature $(1,1)$ form of h is positive; Shiffman-Z in dimension one where $h = 0$ and $d\nu$ was an analytic measure on a domain $\Omega \subset \mathbb{C}$ or on its boundary; Bloom in all dimensions for domains in $\mathbb{C}^m$ and ν is a Bernstein-Markov measure; R. Berman for smooth h and BM $d\nu$ on general Kähler manifolds.

Heuristic proof

Write $h = e^{-\varphi}$ locally. Take $\frac{1}{N^2}$ log of the JPC:

$$-\frac{1}{N^2}\log\frac{|\Delta(\zeta_1,\ldots,\zeta_N)|^2}{\left(\int_{\mathbb{C}}\prod_{j=1}^N|(z-\zeta_j)|^2 e^{-N\varphi}d\nu\right)^{N+1}} = I_N^{h,\nu}(\mu_\zeta):$$

$$= -\Sigma_N(\mu_\zeta) + J_N^{h,\nu}(\mu_\zeta),$$

where (modulo less important terms)

$$\Sigma_N(\mu) = \int_{\mathbb{C}\times\mathbb{C}-\Delta}\log|z-w|d\mu(z)d\mu(w),$$

and

$$\begin{aligned} J_N^{h,\nu}(\mu_\zeta) &= \frac{N+1}{N^2}\log\left(\int_{\mathbb{C}}\prod_{j=1}^N|(z-\zeta_j)|^2 e^{-N\varphi(z)}d\nu(z)\right) \\ &\simeq \frac{N+1}{N^2}\log\int e^{N\int G_h(z,w)d\mu_\zeta(w)}d\nu(z) \\ &= \frac{N+1}{N}\log||e^{U_h^{\mu_\zeta}}||_{L^N(\nu)} \end{aligned}$$

Heuristic proof

For any μ,

$\lim_{N\to\infty} \frac{1}{N} \log \int e^{NU^\mu(z)} d\nu(z) = \log ||e^{U^\mu}||_{L^N(\nu)}$

$\uparrow \log ||e^{U^\mu}||_{L^\infty(\nu)} = \sup_K U^\mu$

monotonically as $N \to \infty$.

Thus, $\frac{1}{N^2} \log \vec{K}_N^N$ as a functional on measures tends to $-\frac{1}{2}\mathcal{E}_h(\mu) + \sup_K U_h^\mu$.

It is not quite obvious that the only minimum of this functional is the equilibrium measure, which minimizes $-\mathcal{E}_h(\mu)$ on $\mathcal{M}(K)$.

Higher genus compared to genus zero

The main problem is to calculate the JPC in higher genus. It involves the prime form and Abel-Jacobi theory.

On a Riemann surface C of genus $g \geq 1$, line bundles of degree N are parametrized by the Picard Variety $Pic^N(C)$, a complex torus of dimension g. For $\xi \in Pic^N(C)$, $s \in H^0(C, \xi)$ has N zeros but the dimension of the space is $N - g + 1$ is smaller than its number of zeros, and the JPC of zeros of sections of ξ is a very singular on $C^{(N)}$. Indeed, the configuration of zeros of sections of ξ lies on the codimension g fiber over ξ of the Abel-Jacobi map $AJ : C^{(N)} \to Pic^N$.

Higher genus

The solution is not to single out one line bundle ξ from each Pic^N, but rather to define the ensemble of random sections to be the collection

$$\mathcal{E}^N := \bigcup_{\xi \in Pic^N} \mathbb{P}H^0(C, \xi) \simeq C^{(N)} \tag{2}$$

of all $H^0(C, \xi)$ as ξ varies over Pic^N. The fiber of

$$\pi_N : \mathcal{E}^N \to Pic^N \tag{3}$$

over ξ is $\mathbb{P}H^0(C, \xi)$.

Higher genus JPC

We define the probability measure, roughly, by choosing a line bundle ξ at random from Pic^N with respect to a probability measure on this torus, and then choose a line of sections $[s] \in \mathbb{P}H^0(C, \xi)$ at random using Fubini-Study measure. The JPC is

$$\frac{1}{Z_N(h)} \frac{\prod_{k=1}^{r} \left| \prod_{j=1}^{g} E(P_j,\zeta_k) \cdot \prod_{j:k\neq j} E(\zeta_j,\zeta_k) \right|^2_{h_{\mathcal{D}} \otimes h^N}}{\det\left(B_N(\zeta_j,\zeta_k)\right)_{j,k=1}^{n}} d\zeta \wedge d\bar{\zeta}$$

$$\cdot$$

$$\left(\int_C \left| \prod_{j=1}^{g} E(P_j, z) \right|^2_{h_g} \cdot \left| \prod_{j=1}^{N} E(\zeta_j, z) \right|^2_{h^N} d\nu(z) \right)^{-N-1}$$

Here, E is the prime form, and P_j are the image of $\{\zeta_k\}$ under the AJ map. B_N is a Bergman kernel.

Final comments

- The first key step was to find a formula for the JPC in coordinates on configuration space.

- Any higher dimensional generalization?

- The JPC is rewritten in terms of the empirical measure and one takes the limit of $\frac{1}{N}$ log of the JPC to get the rate functional. The rate functional itself makes sense in all dimensions.

- Many of the basic results of potential theory are needed to analyze the rate functional, in particular the term $\Lambda_K(\mu) = \sup_K U_h^\mu$.