Global generation of the direct images of relative pluricanonical systems

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Basic Question.

- Under what condition can one construct global holomorphic sections on semipositive vector bundles ?

- For a projective variety X with nonnegative Kodaira dimension, in general the canonical model (existence has been proven by B-C-H-M)

$$X_{can} := \text{Proj}\, R(X, K_X)$$

  does not encode the canonical ring $R(X, K_X)$ unless X is of general type. Can we encode the information of $R(X, K_X)$ by adding additional structure on X_{can}? (eg. Orbifold structure (F. Campana))

$f : X \longrightarrow Y$: **algebraic fiber space**, i.e.,

- X, Y are smooth projective varieties.
- f is projective surjective morphism with connected fibers.
- $K_{X/Y} := K_X \otimes f^* K_Y^{-1}$: the relative canonical bundle.

Semipositivity of the direct image of pluricanonical systems

The following theorem is fundamental in algebraic geometry.

Theorem 1 *(Kawamata, 1982) If* $\dim Y = 1$*, then for every* $m > 0$*,* $f_* K_{X/Y}^{\otimes m}$ *is semipositive in the sense that every quotient* $\mathcal{Q}$ *of* $f_* K_{X/Y}^{\otimes m}$*,* $\deg \mathcal{Q} \geqq 0$ *holds.* $\square$

The proof depends on the **variation of Hodge structure (VHS)** due to Griffiths and Schumidt. The reason why we do not have the semipositive curvature property of $f_*K_{X/Y}^{\otimes m}$ is that the proof depends on the **Finslar metric** :

$$\| \sigma \| := \left(\int_{X/Y} |\sigma|^{\frac{2}{m}} \right)^{\frac{m}{2}}$$

on $f_*K_{X/Y}^{\otimes m}$.

Viehweg's weak semipositivity

Definition 1 *Let Y be a quasi-projective reduced scheme, $Y_0 \subseteq Y$ an open dense subscheme and let $\mathcal{G}$ be locally free sheaf on Y, of finite constant rank. Then $\mathcal{G}$ is* **weakly positive** *over Y_0, if for an ample invertible sheaf $\mathcal{H}$ on Y and for a given number $\alpha > 0$ there exists some $\beta > 0$ such that $S^{\alpha \cdot \beta}(\mathcal{G}) \otimes \mathcal{H}^{\beta}$ is globally generated over Y_0.* □

Definition 2 *Let $\mathcal{F}$ be a locally free sheaf and let $\mathcal{A}$ be an invertible sheaf, both on a quasi-projective reduced scheme Y. We denote*

$$\mathcal{F} \succeq \frac{b}{a} \cdot \mathcal{A},$$

if $S^a(\mathcal{F}) \otimes \mathcal{A}^{-b}$ is weakly positive over Y, where a, b are positive integers.

□

Theorem 2 *(Viehweg 1995)* $f : X \longrightarrow Y$*: an algebric fiber space such that* $K_{X/Y}$ *is* <u>f*-semiample*</u> *over the complement of the discriminant locus* Y°.

1. *(***Weak positivitiy***)* $f_* K^m_{X/Y} (m > 0)$ *is weakly positive over* Y°.

2. *(***Weak semistability***) There exists* $e > 0$ *such that*

$$f_* K^{\otimes m}_{X/Y} \succeq \frac{1}{e \cdot r(m)} \cdot \det(f_* K^{\otimes m}_{X/Y}) \quad \text{on } Y^\circ. \ \square$$

AZD (Analytic Zariki Decomposition)

Definition 3 *Let X be a compact complex manifold and let L be a holomorphic line bundle on X. A singular hermitian metric h on L is said to be an analytic Zariski decomposition(**AZD**), if the followings hold.*

1. *Θ_h is a closed positive current,*

2. *for every $m \geq 0$, the natural inclusion*

$$H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h^m)) \to H^0(X, \mathcal{O}_X(mL))$$

   is an isomorphim. $\square$

Theorem 3 (**Main Theorem**) *$f : X \longrightarrow Y$: algebraic fiber space and let Y° be the complement of the discriminant locus.*

1. (**Global generation**)*There exist positive integers b and m_0 (depending on $f : X \longrightarrow Y$) such that for every $m \geqq m_0$, $b|m$,*
   $f_*K_{X/Y}^{\otimes m}$ is **globally generated** *over Y°.*

2. (**Weak semistability**) *There exist $e > 0$ and a singular hermitian metric $H_{m,e}$ on*

   $$K_{X/Y}^{\otimes m} \otimes (f^* \det f_* K_{X/Y}^{\otimes m})^{-e}$$

   with semipositive curvature current such that for every $y \in Y^\circ$ $H_{m,e}|X_y$ is an AZD of $K_{X/Y}^{\otimes m} \otimes (f^* \det f_* K_{X/Y}^{\otimes m})^{-e}|X_y$. □

Main issue

Semipositivity of $f_* K_{X/Y}^{\otimes m} \Rightarrow$ **Global generation of** $f_* K_{X/Y}^{\otimes m}$

Main Idea

- Detect the null direction of the semipositivity in terms of Monge-Ampère foliations.
- Realize $f_* K_{X/Y}^{\otimes m}$ as the pull back of the strictly positive sheaf on some quasiprojective scheme.

The main advantage of Theorem 3 is that we can construct section of $f_*K_{X/Y}^{\otimes m}$ without tensorize ample line bundles.

Kodaira dimension

$$\mathrm{Kod}(X) := \limsup_{m\to\infty} \frac{\log h^0(X, \mathcal{O}_X(mK_X))}{\log m} (= -\infty, 0, \cdots, \dim X)$$

Conjecture 1 *(***Iitaka's conjecture***) Let $f : X \longrightarrow Y$ be an algebraic fiber space. Then*

$$\mathrm{Kod}(X) \geqq \mathrm{Kod}(Y) + \mathrm{Kod}(X/Y)$$

holds, where $\mathrm{Kod}(X), \mathrm{Kod}(Y)$ *denote the Kodaira dimensions of X, Y resepectively and* $\mathrm{Kod}(X/Y)$ *denotes the Kodaira dimension of a general fiber of $f : X \longrightarrow Y$.* □

Corollary 1 *Iitaka's conjecture holds.* $\square$

Also the orbifold version of Iitaka's conjecture holds (see below).

X : smooth projective $m >> 1$

$$\Phi_{|m!K_X|} : X - \cdots \to Y$$

is a fibration such that $\dim Y = \mathrm{Kod}(X)$ and for a general fiber F, $\mathrm{Kod}(F) = 0$ holds.

This fibration is called the **Iitaka fibration**.

KLT version

Theorem 4 *Let $f : X \longrightarrow Y$ be an algebraic fiber space and let D be an effective $\mathbb{Q}$ divisor on X such that (X, D) is KLT. Let Y° denote the complement of the discriminant locus of f. We set*

$$Y_0 := \{y \in Y | y \in Y^\circ, (X_y, D_y) \text{ is a KLT pair}\}$$

- *Let a be a minimal positive integer such that mD is Cartier. Then there exist a positive integers b and m_0 such that for every $m \geqq m_0$, $b|m$, $m(K_{X/Y} + D)$ is Cartier and $f_*\mathcal{O}_X(m(K_{X/Y} + D))$ is globally generated over Y_0.*

- Let r denote *rank* $f_*\mathcal{O}_X(\lfloor m(K_{X/Y}+D)\rfloor)$ and let $X^r := X\times_Y X\times_Y \cdots\times_Y X$ be the r-times fiber product over Y and let $f^r : X^r \longrightarrow Y$ be the natural morphism. And let D^r denote the divior on X^r defined by $D^r = \sum_{i=1}^r \pi_i^* D$, where $\pi_i : X^r \longrightarrow X$ denotes the projection: $X^r \ni (x_1,\cdots,x_n)\mapsto x_i \in X$.

  There exists a canonically defined effective divisor Γ (depending on m) on X^r which does not conatin any fiber $X^r_y (y\in Y^\circ)$ such that if we we define the number δ_0 by

$$\delta_0 := \sup\{\delta \,|\, (X^r_y, D^r_y + \delta\Gamma_y) \text{ is KLT for all } y\in Y^\circ\},$$

  then for every $\varepsilon < \delta_0$

$$f_*\mathcal{O}_X(\lfloor m(K_{X/Y}+D)\rfloor) \succeq \frac{m\varepsilon}{(1+m\varepsilon)r}\cdot \det f_*\mathcal{O}_X(\lfloor m(K_{X/Y}+D)\rfloor)$$

  holds over Y_0.

- There exists a singular hermitian metric $H_{m,\varepsilon}$ on $(1+m\varepsilon)(K_{X^r/Y}+D^r)-\varepsilon\cdot f^*\det f_*\mathcal{O}_X(\lfloor m(K_{X/Y}+D)\rfloor)^{**}$ such that

  1. $\sqrt{-1}\,\Theta_{H_{m,\varepsilon}}\geqq 0$ holds on X in the sense of current.

  2. For every $y\in Y_0$, $H_{m,\varepsilon}|X^r_y$ is well defined and is an AZD of

     $(1+m\varepsilon)(K_{X^r/Y}+D^r)-\varepsilon\cdot(f^r)^*\det f_*\mathcal{O}_X(\lfloor m(K_{X/Y}+D)\rfloor)^{**}|X_y$

$\square$

Canonical measure (Generalized Kähler-Einstein metrics)

Let $f : X \longrightarrow Y$ be an Iitaka fibration such that $(f_*K_{X/Y}^{\otimes m!})^{**}$ is locally free on Y for some m (hence for every sufficiently large m), where $**$ denotes the double dual. We define the $\mathbb{Q}$-line bundle

$$L := \frac{1}{m!}(f_*K_{X/Y}^{\otimes m!})^{**}$$

on Y. We note that L is independent of a sufficiently large m. L carries the natural singular hermitian metric h_L defined by

$$h_L^{m!}(\sigma,\sigma) = \left(\int_{X/Y}|\sigma|^{\frac{2}{m!}}\right)^{m!}.$$

(L, h_L) : **Hodge $\mathbb{Q}$-line bundle**

Theorem 5 *(***Existence of canonical measures (Song-Tian, T-)***)*
In the above notations, there exists a unique singular hermitian metric on h_K on $K_Y + L$ and a nonempty Zariski open subset U in Y such that

1. *h_K is an AZD of $K_Y + L$.*

2. *h_K is real analytic on U.*

3. *$\omega_Y = \sqrt{-1}\,\Theta_{h_K}$ is a Kähler form on U.*

4. *$-\mathrm{Ric}_{\omega_Y} + \sqrt{-1}\,\Theta_{h_L} = \omega_Y$ holds on U.* $\square$

The above equation:

$$-\mathrm{Ric}_{\omega_Y} + \sqrt{-1}\,\Theta_{h_L} = \omega_Y \tag{1}$$

is similar to the Kähler-Einstein equation :

$$-\mathrm{Ric}_{\omega_Y} = \omega_Y .$$

The correction term $\sqrt{-1}\,\Theta_{h_L}$ represents the isomorphism :

$$R(X, K_X)^{(a)} = R(Y, K_Y + L)^{(a)}$$

for some positive integer a, where for a graded ring $R := \oplus_{i=0}^{\infty} R_i$, where for a graded ring $R := \oplus_{i=0}^{\infty} R_i$ and a positive integer b, we set

$$R^{(b)} := \oplus_{i=0}^{\infty} R_{bi}.$$

Canonical measure :

$$d\mu_{can} := f^*\left(\frac{\omega_Y^n}{n!} \cdot h_L^{-1}\right)$$

is called the canonical measure on X. $d\mu_{can}$ has the following properties.

- $d\mu_{can}$ is a bounded volume form on X which degenerates along subvarieties on X.
- $d\mu_{can}^{-1}$ is an AZD of K_X.
- $d\mu_{can}$ is unique and birationally invariant.

Relative canonical measure

Theorem 6 *Let $f : X \longrightarrow S$ be a projective family such that X, S are smooth and f has connected fibers. And let D be an effective divisor on X such that (X, D) is KLT. Suppose that $f_*\mathcal{O}_S\left(\lfloor m(K_{X/S} + D)\rfloor\right) \neq 0$ for some $m > 0$. Then there exists a singular hermitian metric h_K on $K_{X/Y} + D$ such that*

1. *Let us define $\omega_{X/S} := \sqrt{-1}\,\Theta_{h_K}$. Then $\omega_{X/S} \geqq 0$ holds on X.*

2. *For a general smooth fiber $X_s := f^{-1}(s)$ such that (X_s, D_s) is KLT, $h_K|X_s$ is $d\mu^{-1}_{can,(X_s,D_s)}$, where $d\mu_{can,(X_s,D_s)}$ denotes the canonical measure on (X_s, D_s). In particular $\omega_{X/S}|X_s$ is the canonical semipositive current on (X_s, D_s) constructed as in Theorem 3.* $\square$

Bergman kernel

- X: a complex manifold,

- (L, h_L) : a singular hermitian line bundle on X.

- Hilbert space:
  $A^2(X, K_X+L) := \{\sigma \in \Gamma(X, \mathcal{O}_X(K_X+L)) | (\sqrt{-1})^{n^2} \int_X h_L \sigma \wedge \overline{\sigma} < +\infty\}$

- inner product:
$$(\sigma, \sigma') := (\sqrt{-1})^{n^2} \int_X h_L \cdot \sigma \wedge \bar{\sigma}'$$

- $\{\sigma_i\}$: a complete orthonormal basis of $A^2(X, K_X + L)$

- $K(X, K_X + L, h_L) = \sum_i |\sigma_i|^2$: Bergman kernel of $K_X + L$ with respect to h_L.

- $$K(X, K_X{+}L, h_L)(x) = \mathrm{sup}\{|\sigma|^2(x); \sigma \in A^2(X, K_X{+}L, h_L), \| \sigma \| = 1\}$$

Dynamical construction of the canonical measure

- $f : X \longrightarrow Y$: Iitaka fibration
- (L, h_L): Hodge $\mathbb{Q}$-line bundle
- A: sufficiently ample line bundle on Y
- h_A: C^∞ hermitian metric on A
- a : least positive integer such that $aL \in \mathrm{Div}(Y)$

$$K_1 := \begin{cases} K(Y, K_Y + A, h_A), & \text{if } a > 1 \\ \\ K(Y, K_Y + A + L), h_A \cdot h_L), & \text{if } a = 1 \end{cases}$$

and $h_1 := 1/K_1$.

Inductively we define $\{K_m\}$ and $\{h_m\}$ by

$$K_m := \begin{cases} K(Y, mK_Y + \lfloor \frac{m}{a} \rfloor aL + A, h_{m-1}), & \text{if } a \not| m \\ \\ K(Y, m(K_Y + L) + A, h_{m-1} \cdot h_L^a), & \text{if } a | m \end{cases}$$

Theorem 7 (**Dynamical construction**) *Let X be a smooth projective variety of nonnegative Kodaira dimension and let $f : X \longrightarrow Y$ be the Iitaka fibration as above. Let m_0 and $\{h_m\}_{m \geq m_0}$ be the sequence of hermitian metrics as above and let n denote* $\dim Y$. *Then*

$$h_\infty := \liminf_{m \to \infty} \sqrt[m]{(m!)^n \cdot h_m}$$

is a singular hermitian metric on $K_Y + L$ such that

$$\omega_Y = \sqrt{-1}\,\Theta_{h_\infty}$$

holds, where ω_Y is the canonical Kähler current on Y as in Theorem 3 and $n = \dim Y$.

In particular $\omega_Y = \sqrt{-1}\Theta_{h_\infty}$ (in fact h_∞) is unique and is independent of the choice of A and h_A. $\square$

By Theorem 7, Theorem 5 follows from the following theorem.

Theorem 8 *(Berndtsson, T-) Let $f : X \longrightarrow Y$ be an algebraic fiber space and let (L, h_L) be a singular hermitian line bundle on X such that $\sqrt{-1}\Theta_{h_L} \geqq 0$. Then the singular hermitian metric h on $K_{X/Y} + L$ defined by*

$$h|X_y := K(X_y, K_{X_y} + L, h_L|X_y)^{-1}(y \in Y^{\circ})$$

has semipositive curvature on X, where Y° denotes the complement of the discriminant locus of f. $\square$

Scheme of the proof

- Plurisubharmonic variation of Canonical measures.

- Two Monge-Ampère foliations on the relative Iitaka fibrations and the base spaces induced by the $-$Ric of the relative canonical measure.

- Comparison of the two Monge-Ampère foliation in terms of the weak semistability

- Metrized canonical models are locally trivial along the leaves on the base.

- Leaves are closed and are the fibers of the moduli map to the moduli of metrized canonical models.

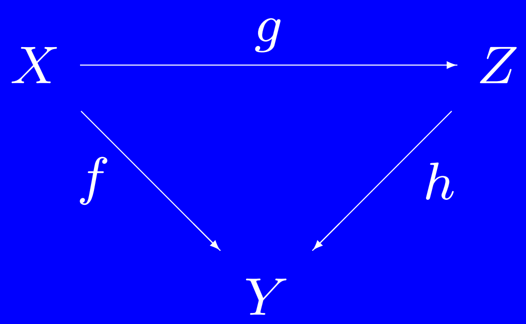
- The family of canonical measures defines a positive $\mathbb{Q}$-line bundle on the moduli space of the metrized canonical models.

Relative Iitaka fibration

$f : X \longrightarrow Y$ be an algebraic fiber space such that $\mathrm{Kod}(X/Y) \geqq 0$. Let Z be the image of the relative pluricanonical map

$$\Phi : X - \cdots \longrightarrow \mathbb{P}(f_* K_{X/Y}^{\otimes m!})$$

for $m >> 1$.



For a sufficiently large m we see that a general fiber F of $g : X - \cdots \rightarrow Z$ is connected and $\mathrm{Kod}(F) = 0$. We call $g : X - \cdots \rightarrow Z$**@a@relative Iitaka fibration**. By taking a suitable modification of X, we may assume that g is a morphism.

Let $f : X \longrightarrow Y$ be an algebraic fiber space and let $g : X \longrightarrow Z$ be a relative Iitaka fibration associated with $f_*K_{X/Y}^{\otimes m!}$. Taking a suitable modification we may and do assume the followings :

- g is a morphism,
- Z is smooth.
- $(g_*K_{X/Z}^{\otimes m!})^{**}$ is a line bundle on Z.

Let $h : Z \longrightarrow Y$ be the natural morphism.

Regularity of relative canonical measure

Let $f : X \longrightarrow Y$ be an algebraic fiber space such that $\mathrm{Kod}(X/Y) \geqq 0$and let $g : X \longrightarrow Z$ be the relative Iitaka fibration as above.

$$\begin{array}{ccc} X & \xrightarrow{\quad g \quad} & Z \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle h} & \\ & Y & \end{array}$$

By the dynamical construction and the generalized Kähler-Einstein equation,we have the following lemma.

Lemma 1 *Let $d\mu_{X/Y,can}$ is C^{ω} on a Zariski open subset of X. Also the relative canonical Kähler current $\omega_{Z/Y}$ is C^{ω} on a Zariski open subset of Z.* $\square$

Monge-Ampère foliation

- Ω: domain in $\mathbb{C}^n$.
- $f \in C^3(\Omega)$: plurisubharmonic function such that $dd^c f$ has constant rank say r on Ω.

Then

$$\mathcal{F} := \{\xi \in T\Omega | dd^c f(\xi, \bar{\xi}) = 0\}$$

defines a foliation on Ω such that the leaves are complex submanifolds of dimension $n - r$. This foliation $\mathcal{F}$ is said to be a Monge-Ampère foliation on Ω associated with $dd^c f$.

Two Monge-Ampère foliations on the relative canonical model

$\omega_{Z/Y}$ defines a Monge-Ampère foliation $\mathcal{F}_Z$ on the generic point of Z. Let us consider the singular hermitian line bundle $(\det f_* K_{X/Y}^{\otimes m!}, \det h_m)$, where

$$h_m(\sigma, \sigma') := \int_{X/Y} \sigma \cdot \overline{\sigma'} \cdot d\mu_{X/Y,can}^{-(m!-1)}.$$

$\Theta_{\det h_m}$ defines a Monge-Ampère foliation $\mathcal{F}_Y$ on Y on the generic point of Y. The following is the key observation.

Lemma 2 *$h_* \mathcal{F}_Z = \mathcal{F}_Y$ holds.* $\square$

Weak stability

- $f : X \longrightarrow Y$: algebraic fiber space with Kod$(X/Y) \geqq 0$.
- $r :=$ *rank* $f_*\mathcal{O}_X(mK_{X/Y})$.
- $X^r := X \times_Y X \times_Y \cdots \times_Y X$($r$-times),
- $f^r : X^r \longrightarrow Y$:the natural morphism.

- $f^r_* K^{\otimes m}_{X^r/Y} \simeq \otimes^r f_* K^{\otimes m}_{X/Y}$

- $\Gamma \in |K^{\otimes m}_{X^r/Y} \otimes (f^{r*} \operatorname{det} f_* K^{\otimes m}_{X/Y})^{-1}|$: corresponding to the inclusion :

  $$(f^r)^*(\operatorname{det} f_* \mathcal{O}_X(mK_{X/Y})) \hookrightarrow (f^r)^* f^r_* \mathcal{O}_{X^r}(mK_{X^r/Y}) \hookrightarrow \mathcal{O}_{X^r}(mK_{X^r/Y}).$$

- $$\delta_0 := \sup\{\delta \,|\, (X^r_y, \delta \cdot \Gamma_y) \text{ is KLT for all } y \in Y^\circ\},$$

- For every $\varepsilon < \delta_0$

$$f_*\mathcal{O}_X(mK_{X/Y}) \succeq \frac{m\varepsilon}{(1+m\varepsilon)r} \cdot \det f_*\mathcal{O}_X(mK_{X/Y})$$

  holds over Y°.

- There exists a singular hermitian metric $H_{m,\varepsilon}$ on $(1+m\varepsilon)K_{X^r/Y} - \varepsilon \cdot (f^r)^* \det f_*\mathcal{O}_X(mK_{X/Y})^{**}$ such that $\sqrt{-1}\,\Theta_{H_{m,\varepsilon}} \geqq 0$ holds on X^r in the sense of current.

- For every $y \in Y^\circ$, $H_{m,\varepsilon}|X^r_y$ is well defined and is an AZD of

$$(1+m\varepsilon)K_{X^r/Y} - \varepsilon \cdot (f^r)^* \det f_*\mathcal{O}_X(mK_{X/Y})^{**}|X_y.$$

- Weak semistability $\Rightarrow \Theta_{h^* \det h_m}|\mathcal{F}_Z \equiv 0$

  Since $h_K := (\omega^n_{Z/Y})^{-1} \cdot h_L$ is an AZD of $K_{Z/Y} + L$,

  $$(\omega^n_{Z/Y})^{-1} \cdot h_L = O(H_{m,\varepsilon} \otimes (h^* \det h_m)^{\varepsilon})$$

  holds. This implies that h_K **is more positive than** $(h^* \det h_m)^{\varepsilon}$**.** This implies the assertion.

- Along the leaves of $\mathcal{F}_Y$, $h : (Z, (L, h_L)) \longrightarrow Y$ is locally trivial.

  This is because $\sqrt{-1}\,\Theta_{h_m} \geqq 0$ and $\mathrm{trace}\,\Theta_{h_m} \equiv 0$ on $\mathcal{F}_Y$. Hence

  $$\Theta_{h_m} \equiv 0 \ \text{ along } \mathcal{F}_Y$$

  Then the parallel transport on $f_* K^{\otimes m}_{X/Y}$ trivialize $(Z, (L, h_L))$ along $\mathcal{F}_Y$.

Hence we have that $f_* \mathcal{F}_Z = \mathcal{F}_Y$

Metrized canonical models

- (X, D): KLT pair with Kod$(X, D) \geqq 0$.

- $R(X, K_X + D) := \oplus_{m=0}^{\infty} \Gamma(X, \mathcal{O}_X(\lfloor m(K_X + D) \rfloor))$: the log canonical ring of (X, D) : finitely generated.

- $Y := \operatorname{Proj} R(X, K_X + D)$: the canonical model of (X, D).

- $L := \frac{1}{m_0!} \left(f_* \mathcal{O}_X (m_0!(K_{X/Y} + D)) \right)^{**}$ $(m_0 >> 1)$: the Hodge $\mathbb{Q}$-line bundle.

- h_L: the Hodge metric on L.

- ω_Y : the canonical Kähler current.

- $h_K := n!(\omega_Y^n)^{-1} \cdot h_D(n = \dim Y)$: canonical metric on $K_Y + L$.

Definition 4 *The pair* $(Y, (L, h_L))$ *is called the* **metrized canonical model** *associated with the KLT pair* (X, D)*.* $\square$

The moduli space of metrized canonical models

Let $(Z_y, (L, h_L)|Z_y)$ be the canonical model Z_y of X_y and the metrized Hodge bundle. The Hodge metric comes from a variation of Hodge structure on the canonical cyclic cover $W_y^\circ \longrightarrow Z_y^\circ$.

$$\mathcal{M} = \{(Z_y, (L, h_L)|Z_y)\}/\sim$$

where the equivalece $\sim$ is defined by

$$\varphi : Z_y \longrightarrow Z_{y'}$$

covered by the biholomorphism $\tilde{\varphi} : W_y^\circ \longrightarrow W_{y'}^\circ$ which induces an isomorphism between flat bundles preserving the Hodge line bundles.

Theorem 9 *$\mathcal{M}$ has a structure of separable complex space and for $m >> 1$ (some multiple of)* $\det f_*\mathcal{O}_X(m!K_{X/Y})$ *decends to a polarization of $\mathcal{M}$. In particular $\mathcal{M}$ is quasiprojective.* $\square$

This theorem implies that the leaves of $\mathcal{F}_Y$ is the fiber of the classifying map :

$$\Phi : Y^\circ \longrightarrow \mathcal{M}.$$

Then some symmetric power $S^r(f_*\mathcal{O}_X(m!K_{X/Y}))$ decends to a vector bundle on $\mathcal{M}$. Then by the **weak semistability**, we see that for $m >> 1$ $S^r(f_*\mathcal{O}_X(m!K_{X/Y}))$ decends to a very ample vector bundle on $\mathcal{M}$. Then $f_*\mathcal{O}_X(rm!K_{X/Y})$ is globally generated on Y° for $m >> 1$.

This theorem gives an alternative proof of the following theorem.

Theorem 10 *(Viehweg) Let $\mathcal{M}_{pol,min}$ be the polarized minimal algebraic varieties with semiample canonical divisors, then $\mathcal{M}_{pol,min}$ is quasiprojective .* $\square$