A Second Main Theorem for Moving Hypersurface Targets with Effective Truncation

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Notations:

$$f : \mathbf{C}^m - \to \mathbf{CP}^n$$

nonconstant meromorphic map, with reduced representation $f = [f_0 : ... : f_n]$, i.e. $\mathrm{codim} V(f_0, ..., f_n) \geq 2$.

$$T_f(r) := \int_{S(r)} \log ||f|| \sigma - \int_{S(1)} \log ||f|| \sigma$$

characteristic function, with
$||f||(z) := \max\{|f_0(z)|, ..., |f_n(z)|\}$,
$\sigma := d^c \log ||z||^2 \wedge (dd^c \log ||z||)^{m-1}$

$$\mathcal{K}_f$$

field of small meromorphic functions $\phi : \mathbf{C}^m - \to \mathbf{P}^1$, i.e. such that $T_\phi(r) = o(T_f(r))$

$$Q_j = \sum_{I \in \tau_{d_j}} a_{jI} x^I,\ j = 1, ..., q$$

homogenous polynomials in $\mathcal{K}_f[x_0, ..., x_n]$ of degree $d_j,\ q \geq n+1$

$$\mathcal{K}_{\{Q_j\}_{j=1}^q} :=$$

$$\mathbf{C} < \frac{a_{jI_1}}{a_{jI_2}} : a_{jI_2} \neq 0, I_1, I_2 \in \tau_{d_j}, j = 1, ..., q >$$

$$N_f^{[L]}(r,Q) := N^{[L]}_{Q(f_0,\dots,f_n)}(r)$$

truncated counting function of f w.r.t. Q, where $Q \in \mathcal{K}_f[x_0,...,x_n]$, $L \in \mathbf{N} \cup \{\infty\}$
In more details: If $\phi : \mathbf{C}^m - \rightarrow \mathbf{P}^1$ is meromorphic, $a \in \mathbf{C}^m$, $\phi = [G : F]$ a reduced representation, we put $\nu_\phi(a) := \nu_F(a) =$ vanishing order of F in a. If still $\mathcal{V} = (dd^c||z||^2)^{m-1}$ we put

$$n^{[L]}(t,\phi) = \int_{|\nu_\phi| \cap B(t)} \min(L, \nu_\phi)\mathcal{V}\,,$$

$$N_\phi^{[L]}(r) = \int_1^r \frac{n^{[L]}(t,\phi)}{t^{2m-1}} dt\,.$$

As usual, by the notation "$\| P$" we mean the assertion P holds for all $r \in [1,+\infty)$ excluding a Borel subset E of $(1,+\infty)$ with $\int\limits_E dr < +\infty$.

We say that a set $\{Q_j\}_{j=1}^{q}$ $(q \geq n+1)$ of homogeneous polynomials in $\mathcal{K}_f[x_0, \ldots, x_n]$ is **in (weakly) general position** if there exists $z \in \mathbf{C}^m$ in which all coefficient functions of all Q_j, $j = 1, ..., q$ are holomorphic and such that for any $1 \leq j_0 < \ldots < j_n \leq q$ the system of equations

$$\begin{cases} Q_{j_i}(z)(x_0, \ldots, x_n) = 0 \\ 0 \leq i \leq n \end{cases}$$

has only the trivial solution $(x_0, \ldots, x_n) = (0, \ldots, 0)$ in $\mathbf{C}^{n+1}$. In this case this is true for generic $z \in \mathbf{C}^m$.

Second Main Theorem. Let f be a non-constant meromorphic map of $\mathbf{C}^m$ into $\mathbf{P}^n$. Let $\{Q_j\}_{j=1}^q$ be homogeneous polynomials in weakly general position in $\mathcal{K}_f[x_0, \ldots, x_n]$ with $\deg Q_j = d_j \geq 1$. Assume that f is algebraically nondegenerate over $\mathcal{K}_{\{Q_j\}_{j=1}^q}$. Then for any $\varepsilon > 0$, there exist positive integers L_j $(j = 1, \ldots, q)$, depending only on n, ε and d_j $(j = 1, \ldots, q)$ in an explicit way such that

$$\|(q - n - 1 - \varepsilon)T_f(r) \leq \sum_{j=1}^{q} \frac{1}{d_j} N_f^{[L_j]}(r, Q_j).$$

Estimates With the notation of the Second Main Theorem above, we have

$$L_j \leq \frac{d_j \cdot \begin{pmatrix} n+N \\ n \end{pmatrix} t_{p_0+1} - d_j}{d} + 1\,,$$

where d is the least common multiple of the d_j's and

$$N = d \cdot [2(n+1)(2^n - 1)(nd+1)\epsilon^{-1} + n + 1],$$

$$M = \left(\begin{array}{c} n+N \\ n \end{array} \right),\ L = \left(\begin{array}{c} q \\ n \end{array} \right)$$

$$p_0 = [\frac{(M^2.L-1).\log(M^2.L)}{\log(1+\frac{\epsilon}{2MN})} + 1]^2,$$

and

$$t_{p_0+1} < \left(M^2.L + p_0\right)^{M^2.L-1},$$

where we denote $[x] := \max\{k \in \mathbf{Z} : k \leq x\}$ for a real number x. Furthermore, in the case of fixed hypersurfaces ($Q_j \in \mathbf{C}[x_0, \ldots, x_n], j = 1, \ldots, q$), we have $t_p = 1$ for all positive integers p, so we get a better estimate:

$$L_j \leq \frac{d_j \cdot \left(\begin{array}{c} n+N \\ n \end{array} \right) - d_j}{d} + 1.$$

The **First Main Theorem** is the following classical result: Let f be a nonconstant meromorphic map of $\mathbf{C}^m$ into $\mathbf{P}^n$. Let Q be a homogeneous polynomials in $\mathcal{K}_f[x_0, \dots, x_n]$ with $\deg Q = d \geq 1$. Assume $Q(f_0 : ... : f_n) \not\equiv 0$. Then

$$N_f(r, Q) \leq d \cdot T_f(r) + o(T_f(r)) \, .$$

The **defect** is defined by

$$\delta_f(Q) = \liminf_{r \to \infty} \left(1 - \frac{N_f(r, Q)}{d \cdot T_f(r)}\right) .$$

Corollary: Under the assumptions of the Second Main Theorem, we have

$$\sum_{j=1}^{q} \delta_f(Q_j) \leq n + 1 \, .$$

We remark that our Second Main Theorem is not strong enough to get the same defect relation for the truncated defects $\delta_f^{[L_j]}(Q_j) := \liminf_{r \to \infty} (1 - \frac{N_f^{[L_j]}(r,Q)}{d \cdot T_f(r)})$ (unless we estimate by $n + 1 + \epsilon$), since the L_j depend on ϵ.

Related results:

I) Results without truncation:

Theorem(Ru, Annals Math '08(?), Amer.J.Math '04) Let $V \subset \mathbf{P^N}$ be a smooth complex projective variety of dimension $n \geq 1$. Let $\{D_j\}_{j=1}^q$ be hypersurfaces in $\mathbf{P^N}$ of degree $\deg D_j = d_j \geq 1$, located in general position in V. Let $f : \mathbf{C} \to V$ be an algebraically non-degenerate holomorphic map. Then, for every $\varepsilon > 0$,

$$\|(q - n - 1 - \varepsilon)T_f(r) \leq \sum_{j=1}^{q} \frac{1}{d_j} N f(r, D_j)\,.$$

This result is from '08, the special case where $V = \mathbf{P^n}$ is from '04. The latter had been conjectured by Shiffman in '79.

We also remark that that for moving hyperplanes (i.e. $d_1 = ... = d_q = 1$), and counting functions non truncated our Second Main Theorem had been proved by Ru-Stoll in '91.

2) Results with truncation:

Theorem(An-Phuong, Houston J.Math. '08(?)) Let $\{D_j\}_{j=1}^q$ be hypersurfaces in $\mathbf{P}^\mathbf{n}$, of degree $\deg D_j = d_j \geq 1$, in general position. Let $f : \mathbf{C} \to \mathbf{P}^\mathbf{n}$ be an algebraically non-degenerate holomorphic map. Let d be the least common multiple of the d_j's. Let $1 > \varepsilon > 0$, and let

$$L \geq 2d[2^n(n+1)n(d+1)\varepsilon^{-1}]^n .$$

Then

$$\|(q-n-1-\varepsilon)T_f(r) \leq \sum_{j=1}^{q} \frac{1}{d_j} N^{[L]} f(r, D_j) .$$

We remark that Yan-Chen '08(?) Acta Math. Sinica had this result with a non-effective truncation. We also remark that a version of our Second Main Theorem for moving hypersurfaces with non-effective truncation was before the above result. All these results base on the result of Ru '04, which brought a technique of Corvaja-Zannier, (Amer.J.Math. '04) to Nevanlinna theory (cf. below).

3) Related results for hyperbolicity and meromorphically normal families:

Hyperbolicity of the complement of hypersurfaces only makes sense for fixed hypersurfaces. Using the Second Main Theorem for fixed hypersurfaces (Ru '04) gives:
An entire curve $f : \mathbf{C} \to \mathbf{P}^n \setminus \bigcup_{i=1}^{q} D_i$ is algebraically degenerate, for $q \geq n+2$ hypersurfaces in general position. This result was however already due to Green.

On the other hand, the question of algebraic degeneracy of entire curves omitting slowly moving hypersurfaces makes sense, and there we get, as a (trivial) corollary of our Second Main Theorem:

Corollary: Let f be a nonconstant meromorphic map of $\mathbf{C}^m$ into $\mathbf{P}^n$. Let $\{Q_j\}_{j=1}^{q}$ be $q \geq$

$n+2$ homogeneous polynomials in weakly general position in $\mathcal{K}_f[x_0, \ldots, x_n]$ with $\deg Q_j = d_j \geq 1$. Assume that f omits the divisors $Q_j = 0$, in the sense that $N_f(r, Q_j) = o(T_f(r))$ for $j = 1, ..., q$ (or $= 0$ if the coefficients of Q_j are in reduced representation). Then f is algebraically degenerate over $\mathcal{K}_{\{Q_j\}_{j=1}^q}$.

This result becomes meaningless if we enlarge the field of coefficients from $\mathcal{K}_f$ to the field $\mathcal{M}(\mathbf{C}^m)$ of all meromorphic functions on $\mathbf{C}^m$. However, as it was observed by Tu and Li in '05, theorems of normal family type can still make sense and hold even if hyperplanes move "rapidly", and even if there is some intersection with the hyperplanes, if there are at least $2n+1$ of them. Without mentioning all results which have been obtained so far, let me just mention the following recent result in the "rapidly" moving hypersurface context:

Theorem (Quang-Tan, Ann. Polonici Math. '08): Let $\mathcal{F}$ be a family of meromorphic mappings of a domain $D \subset \mathbf{C}^m$ into $\mathbf{P}^n$ and let $Q_1, ..., Q_q$ $(q \geq 2n+1)$ be q moving hypersurfaces in $\mathbf{P}^n$ in (weakly) general position such that:

i) For any fixed compact subset K of D, the $2(m-1)$-dimensional Lebesgue areas of $f^{-1}(Q_j) \cap K (1 \leq j \leq n+1)$ counting multiplicities for all $f \in \mathcal{F}$ are bounded above.

ii) For any fixed compact subset K of D, the $2(m-1)$-dimensional Lebesgue areas of $f^{-1}(Q_j) \cap K (n+2 \leq j \leq q)$ ignoring multiplicities for all $f \in \mathcal{F}$ are bounded above.

Then $\mathcal{F}$ is a meromorphically normal family on D.

4) Uniqueness theorems: There have been many uniqueness theorems for hyperplane targets (first fixed and then also moving). For fixed hypersurface targets we have

Theorem(Dulock-Ru '08) Let D_j, $1 \le j \le q$ be hypersurfaces of degree d_j in $\mathbf{P}^n$ in general position. Let $d_0 = \min\{d_1, ..., d_q\}$, $d = \mathrm{lcm}\{d_1, ..., d_q\}$, and $M = 2d[2^{n+1}(n+1)n(d+1)]^n$. Suppose $f, g : \mathbf{C} \to \mathbf{P}^n$ are algebraically non degenerate holomorphic mappings such that $f(z) = g(z)$ for all $z \in S$, where $S = \bigcup_{j=1}^{q}\{f^{-1}(D_j) \cup g^{-1}(D_j)\}$. Then if $q > (n+1) + 2Mn/d_0 + 1/2$, then $f \equiv g$.

For moving hypersurface targets, we can get (so far...):

Theorem: Let $f, g : \mathbf{C}^m - \to \mathbf{P}^n$ meromorphic maps which are algebraically non degenerate over $\mathcal{K}_{\{Q_j\}_{j=1}^q}$ and satisfy:
i) dim $\{Q_i(f) = Q_j(f) = 0\} \leq m - 2$ for all $1 \leq i < j \leq q$.
ii) $f = g$ on $\cup_{i=1}^q f^{-1}(Q_j)$.
Then if $q > n + 3/2 + 2L/d$ then $f \equiv g$, where d is the least common multiple of the degrees of Q_j and L the maximal L_j of our Second Main Theorem.

5) Related results in diophantine approximation and function fields:

Theorem(Corvaja-Zannier, Amer.J.Math. '04) For $\nu \in S$, let $f_{i\nu}$, $i = 1, ..., n-1$, be polynomials in $k[x_1, ... x_n]$ of degrees $\delta_{i\nu} > 0$. Put $\delta_\nu = \max_i \delta_{i\nu}$ and $\mu = \min_{\nu \in S} \sum_{i=1}^{n-1} \delta_{i\nu}/\delta_\nu$. Fix $\varepsilon > 0$ and consider the Zariski closure $\mathcal{H}$ in $\mathbf{P}^n$ of the set of solutions $x \in \mathcal{O}_S^n$ of

$$\prod_{\nu \in S} \prod_{i=1}^{n-1} |f_{i\nu}(x)|_\nu^{1/\delta_\nu} \leq H(x)^{\mu - n - \varepsilon}.$$

Suppose that, for $\nu \in S$, x_0 and the $\bar{f}_{i\nu}$, $i = 1, ..., n-1$ define a variety of dimension 0. Then $\dim \mathcal{H} \leq n-1$. Moreover, if $\mathcal{H}'$ is a component of $\mathcal{H}$ of dimension $n-1$, there exists $\nu \in S$ such that the $\bar{f}_{i\nu}$ determine in $\mathcal{H}'$ a variety of dimension 1.

Theorem(An-Wang, J.Number Theory '07) Let $\{Q_j\}_{j=1}^q$ be homogenous polynomials of degree d_j in $K[x_0, ..., x_n]$ in general position and S be a finite set of prime divisors of V. Then given $\varepsilon > 0$, there exists an effective countably union $\mathcal{U}_\varepsilon$ of proper algebraic subsets of $\mathbf{P}^n(K)$ and an effectively computable constants c_ε and c'_ε, depending only on ε and the given hypersurfaces, such that for any $x \in \mathbf{P} \setminus \mathcal{U}_\varepsilon$ either

$$h(x) \leq c_\varepsilon$$

or

$$\sum_{i=1}^{q} \sum_{\mathcal{P} \in S} d_i^{-1} \lambda_{\mathcal{P}, Q_i}(x) \leq (n+1+\varepsilon) h(x) + c'_\varepsilon \,.$$

Furthermore, the degree of the algebraic subsets in $\mathcal{U}_\varepsilon$ can be bounded by

$$2^{n+1} nd(d+1)(n+1)(2\varepsilon^{-1}+1) \,,$$

where $d = lcm(d_1, ..., d_q)$.

We also remark that Ru-Wang announced a result on the function field analogon of Ru's theorem in Annals of Math. '08, with effective bounds.

Description of the main proof ideas:

The proof of our Second Main Theorem and of the estimate of truncation consists of 3 main steps:

Step 1: We obtain the following estimate:

$$\int\limits_{S(r)} \log \prod_{j=1}^{q} |Q_j(f)|\sigma \geq (q-n-1)d \cdot T_f(r) - \frac{\varepsilon}{2} T_f(r)$$

$$-o(T_f(r)) + \frac{1}{A} \int\limits_{S(r)} \min_J \log \prod_{j=1}^{M} |\psi_j^J(f)|\sigma$$

This is obtained by the filtration of Corvaja-Zannier as in Ru's paper.

The additional difficulties come both from the facts that the concept "in general position" in our paper is more general than in Corvaja-Zannier's and Ru's paper and that the field $\mathcal{K}_f$ is not algebraically closed in general, so we cannot use any more Hilbert's Nullstellensatz.

Instead we have to use explicit results on resultants respectively discriminant varieties for universal families of configurations of q hypersurfaces in $\mathbf{P}^n$. This allows us to deal with such hypersurfaces with "variable" coefficients, namely in $\mathcal{K}_f$, but by specialization to the fibers to have nevertheless complex solutions of these configurations of hypersurfaces.

Another problem related to the fact that $\mathcal{K}_f$ is not algebraically closed in general is that the proof of the fact that admissible families of polynomials in $\mathcal{K}_f[x_0, ..., x_n]$ give regular families does not follow any more directly from Hilbert's Nullstellensatz, but needs another time resultants, as well as results on parameter systems in Cohen-Macauley rings.

Step 2:

We estimate the "error term" of step 1 against a Wronskian and a negligeable term:

$$\| \int\limits_{S(r)} \min_J \log \prod_{j=1}^{M} |\psi_j^J(f)|\sigma \geq$$

$$\frac{1}{t_p} \int\limits_{S(r)} \log|W^{\alpha}|\sigma - \frac{\varepsilon}{2} T_f(r) - o(T_f(r))$$

This step, which is an easy application of the lemma of logarithmic derivative for the Wronskian in the case of constant coefficients, becomes much more complicated for coefficients in $\mathcal{K}_f$: We use a technic from moving hyperplanes and generalize it. Another complication occurs since we have to obtain reduced representations of the coefficient functions of the polynomials giving the moving hypersurfaces.

Step 3:
In the third part, truncation is obtained. Here the concept "resultants of homogenous polynomials" and Wronskians are used again, now to estimate the corresponding divisors. The use of this tool, which is not necessary in the case of fixed hypersurfaces, is necessary in the case of moving hypersurfaces because of our very general notion of general position, in order to control what happens over the divisor where the resultant vanishes, this means where the hypersurfaces are not in general position.

Some more details: Resultants

Let $\{Q_j\}_{j=0}^n$ be a set of homogeneous polynomials of common degree $d \geq 1$ in $\mathcal{K}_f[x_0, \ldots, x_n]$

$$Q_j = \sum_{I \in \tau_d} a_{jI} x^I, \quad a_{jI} \in \mathcal{K}_f \quad (j = 0, \ldots, n).$$

Let $T = (\ldots, t_{kI}, \ldots) \quad (k \in \{0, \ldots, n\},\ I \in \tau_d)$ be a family of variables. Set

$$\widetilde{Q}_j = \sum_{I \in \tau_d} t_{jI} x^I \in \mathbf{Z}[T, x], \quad j = 0, \ldots, n.$$

Let $\widetilde{R} \in \mathbf{Z}[T]$ be the resultant of $\widetilde{Q}_0, \ldots, \widetilde{Q}_n$. This is a polynomial in the variables $T = (\ldots, t_{kI}, \ldots) \quad (k \in \{0, \ldots, n\},\ I \in \tau_d)$ with integer coefficients, such that the condition $\widetilde{R}(T) = 0$ is necessary and sufficient for the existence of a nontrivial solution $(x_0, \ldots, x_n) \neq (0, \ldots, 0)$ in $\mathbf{C}^{n+1}$ of the system of equations

$$\begin{cases} \widetilde{Q}_j(T)(x_0, \ldots, x_n) = 0 \\ 0 \leq i \leq n \end{cases}.$$

Then if

$$\{Q_j = \widetilde{Q}_j(a_{jI})(x_0, \ldots, x_n)\,,\ j = 0, \ldots, n\}$$

is an admissible set,

$$R := \widetilde{R}(\ldots, a_{kI}, \ldots) \not\equiv 0\,.$$

Furthermore, since $a_{kI} \in \mathcal{K}_f$, we have $R \in \mathcal{K}_f$.
proposition There exists a positive integer s and polynomials $\{\widetilde{b}_{ij}\}_{0\le i,j\le n}$ in $\mathbf{Z}[T, x]$, which are (without loss of generality) zero or homogenous in x of degree $s-d$, such that

$$x_i^s \cdot \widetilde{R} = \sum_{j=0}^{n} \widetilde{b}_{ij}\widetilde{Q}_j \quad \text{for all } i \in \{0, \ldots, n\}.$$

Some more details: Error terms

Let f be a nonconstant meromorphic map of $\mathbf{C}^m$ into $\mathbf{P}^n$. Denote by $\mathcal{C}_f$ the set of all non-negative functions $h : \mathbf{C}^m \setminus A \longrightarrow [0, +\infty] \subset \overline{\mathbf{R}}$, which are of the form

$$\frac{|g_1| + \ldots + |g_k|}{|g_{k+1}| + \ldots + |g_l|}\,,$$

where $k, l \in \mathbf{N}$, $g_1, \ldots, g_l \in \mathcal{K}_f \setminus \{0\}$ and $A \subset \mathbf{C}^m$, which may depend on $g_1, \cdots, g_l$, is an analytic set of codimension at least two. By

Jensen's formula and the First Main Theorem we have

$$\int_{S(r)} \mathrm{log}|\phi|\sigma = o(T_f(r)) \quad \text{as } r \to \infty$$

for $\phi \in \mathcal{K}_f \setminus \{0\}$. Hence, for any $h \in \mathcal{C}_f$, we have

$$\int_{S(r)} \mathrm{log} h\sigma = o(T_f(r)) \quad \text{as } r \to \infty.$$

It is easy to see that sums, products and quotients of functions in $\mathcal{C}_f$ are again in $\mathcal{C}_f$. We would like to point out that, in return, given any functions $g_1, \cdots, g_l \in \mathcal{K}_f \setminus \{0\}$, any expression of this form is in fact a well defined function (with values in $[0, +\infty]$) outside an analytic subset A of codimension at least two, even though all the $g_1, \cdots, g_l$ can have common pole or zero divisors in codimension one.