Diffusions and Nevanlinna theory

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[How Brownian motion recognize a point a to be an omitted value of meromophic function]

Jensen's formula:

f : a holomorphic function on $\mathbb{C}$ with $f(o) \neq a (\in \mathbb{C})$.

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta}) - a| d\theta = \sum_{f(\zeta)=a,\ |\zeta|<r} \log\frac{r}{|\zeta|}.$$

Let Z_t be BM(C) with $Z_0 = o$.

$\tau_r = \inf\{t > 0 : |Z_t| > r\}$.

$M_t := \log|f(Z_{t\wedge\tau_r}) - a|$: a local martingale.

$M_t^+ - M_0^+ =$ a martingale $+\frac{1}{2}L_t$

: bounded submartingale.

$M_t^- - M_0^- =$ a local martingale $+\frac{1}{2}L_t$

: local submartingale.

$$E[M_T^+] - M_0^+ = \frac{1}{2} E[L_T],$$

$$E[M_T^-] - M_0^- + \lim_{\lambda \to \infty} \lambda P\left(\sup_{0 < s < T} M_s^- > \lambda \right) = \frac{1}{2} E[L_T].$$

for $\forall$ stopping time T.

LHS of Jensen's formula $= E[M_{\tau_r}^+] - E[M_{\tau_r}^-]$.

We have

$$
\begin{aligned}
&\lim_{\lambda\to\infty} \lambda P(\sup_{0<s<\tau_r} \log^- |f(Z_s) - a| > \lambda) \\
&= \lim_{\lambda\to\infty} \lambda P(\sup_{0<s<\tau_r} \log |f(Z_s) - a|^{-1} > \lambda) \\
&= \sum_{f(\zeta)=a,\ |\zeta|<r} \log \frac{r}{|\zeta|}.
\end{aligned}
$$

f omits a iff

$$\lim_{\lambda\to\infty} \lambda P(\sup_{0<s<\tau_r} \log |f(Z_s) - a|^{-1} > \lambda) = 0.$$

For meromorphic function f we use $\log[f(z), a]^{-1}$ instead of $\log|f(z) - a|^{-1}$.

$[w, a]$: chordal distance on $\mathrm{P}^1(\mathrm{C})$.

$$[w, a] = \begin{cases} \dfrac{|w - a|}{\sqrt{|w|^2 + 1}\sqrt{|a|^2 + 1}} & (\text{ if } a \neq \infty), \\ \\ \dfrac{1}{\sqrt{|w|^2 + 1}} & (\text{ if } a = \infty), \end{cases}$$

Ito's formula with the previous argument

$$
\begin{aligned}
&E[\log[f(Z_{\tau_r}), a]^{-1}] - \log[f(o), a]^{-1} \\
&\quad + \lim_{\lambda\to\infty} \lambda P(\sup_{0<s<\tau_r} \log[f(Z_s), a]^{-1} > \lambda) \\
&\quad\quad = E[\int_0^{\tau_r} \frac{|f'(Z_s)|^2}{(1+|f(Z_s)|^2)^2} ds].
\end{aligned}
$$

This is First Main Theorem in the classical Nevanlinna theory :

$$
m(r, a) - m(0, a) + N(r, a) = T(r),
$$

$$m(a,r) = \int_0^{2\pi} \log[f(re^{i\theta}), a]^{-1} \frac{d\theta}{2\pi}$$

(proximity function),

$$N(a,r) = \sum_{f(\zeta)=a, |\zeta|<r} \log \frac{r}{|\zeta|}$$

(counting with multiplicity, counting function),

$$T(r) = \int_{|z|<r} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} g_r(o,z) dv(z)$$

(Ahlfors-Shimizu charcteristic function).

[defect]

$$\delta(a) := \liminf_{r \to \infty} \frac{m(a,r)}{T(r)}.$$

f **omits** $a \Rightarrow \delta(a) = 1$.

We wish to seek bounds of $\displaystyle\sum_{a \in \mathrm{P}^1} \delta(a)$.

[Second Main Theorem]

Let $a_1, a_2, \dots, a_q \in \mathrm{C} \cup \{\infty\}$ distinct points, f : nonconstant meromorphic function on C. $\exists E \subset [0, \infty)$ s.t. $|E| < \infty$ and

$$\sum_{k=1}^{q} m(a_i, r) + N_1(r) \leq 2T(r) + O(\log T(r) + \log r) \tag{1}$$

for $r \notin E$.

[cor:defect relation]

$$\sum_{k=1}^{q} \delta(a_k) \leq 2.$$

[Holomorphic diffusion]

Definition 1 A diffusion process X on M is called a holomorphic diffusion if $Re\ f(X)$ is a local martingale on R for any holomorphic function $f \in \mathcal{O}(U)$ ($\forall U$:open $\subset M$).

ex.1 Complex Brownian motion on C^m. i.e.
$$X_t = (X_t^{(1)}, \ldots, X_t^{(m)}),\ ; X_t^{(k)} = x_t^{(k)} + \sqrt{-1} y_t^{(k)}$$
$x_t^{(1)}, \ldots, x_t^{(m)}, y_t^{(1)}, \ldots, x_t^{(m)}$: indep BMs on R^1.

ex.2 Brownian motions on Kähler manifolds. Generator is half of Laplacian defined from the Kähler metric.

Important properties: 1. From the definition we have that if f is a holomorphic function on M,

$$f(X_t) = Z(\langle Re\, f(X)\rangle_t)$$

for some complex Brownian motion Z on $\mathbb{C}$. If M is a Kähler manifold and X_t is Brownian motion associated with the Kähler metric, $\langle Re\, f(X)\rangle_t = \int_0^t ||df(X_s)||^2 ds$.

2. If $\dim M = 1$, M is a Riemann surface and X is a time-changed diffusion of another holomorphic diffusion.

3. If $f : M \to N$ is a holomorphic map and X is a holomorphic diffsion on M, then $f(X)$ is a holomorphic martingale i.e. $Re\, h(f(X))$ is a local martingale for $\forall h \in \mathcal{O}(U)$ ($\forall U \subset N$:open). If M is Kähler, X is Brownian motion on M and f is a meromorphic function on M i.e. holomorphic map from M to $\mathrm{P}^1(\mathrm{C})$, then there exists Brownian motion W on $\mathrm{P}^1(\mathrm{C})$ w.r.t.Fubini-Study metric s.t. $f(X_t) = W(\rho_t)$ with $\rho_t = \int_0^t ||df||^2(X_s)ds$ where $||df||^2$ is the energy density of f.

[Construction of holomorphic diffusion]

We take a consevative holomorphic diffusion X on M constructed by the theory of Dirichlet form due to Fukushima-Okada.

$$\mathcal{E}(u,v) = \frac{1}{2}\int_M du \wedge d^c v \wedge \theta, \quad u,v \in C_o^\infty(M)$$

where $d^c = \dfrac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial), \theta$ is a closed positive current of type $(m-1, m-1)$. Assume that there exists a Radon measure dm on M s.t. $(\mathcal{E}, C_o^\infty(M))$ is closable on $L^2(dm)$ (Say dm is admissible). Then a holomorphic diffusion corresponds uniquely to $(\mathcal{E}, D(\mathcal{E}))$.

The associated generator is denoted by L. L is a self-adjoint operator on $L^2(dm)$ s.t.

$$-(Lu, v) = \mathcal{E}(u, v) \quad u, v \in C_o^\infty(M).$$

Formally
$Lu = \frac{1}{2}\frac{dd^c u \wedge \theta}{dm}$ and its diffusion semigroup: e^{tL} satisfies $e^{tL}\phi(x) = E_x[\phi(X_t)]$ $(\phi \in C_b(M))$.

Carré du champ operator:

$$\Gamma(u, u) := \frac{1}{2}\frac{du \wedge d^c u \wedge \theta}{dm}.$$

Proposition 2 (Ito's formula) Assume $u \in C^2(M)$.

$$u(X_t) - u(X_0) = B\left(\int_0^t \Gamma(u,u)(X_s)ds\right) + \int_0^t Lu(X_s)ds,$$

where B_t is standard Brownian motion on $\mathbb{R}$.

Rem.

If M has a Kähler form ω and X is the Brownian motion associated with the Kähler metric, then X corresponds to the above Dirichlet space with

$$\theta = \omega^{m-1}, \quad dm = const.\omega^m,$$

and L is half of Laplacian w.r.t.the Kähler metric and $\Gamma(u,u) = |\nabla u|^2$.

If X is Brownian motion on M,

$$u(X_t) - u(X_0) = B\left(\int_0^t |\nabla u|^2 (X_s) ds\right) + \frac{1}{2}\int_0^t \Delta u(X_s) ds.$$

We consider two types of formulation of Nevanlinna theory:

1. **based on Green's functions: X_t Brownian motion stopped at τ_r.**
2. **based on diffusion semigroup: general holomophic diffusion X_t.**

[A natural generalization of classical nevanlinna theory]

M : a complete Kähler manifold with $dim_{\mathbf{C}} M = m$.

v : a nonnegative, smooth and subharmonic exhaustion function on M. (always exists. Greene-Wu)

(X_t, P_x) : Brownian motion on M

$\tau_r = \inf\{t > 0 : v(X_t) > r\}$ Fix $o \in M$: ref. point.

f : a nonconstant meromorphic function on M

i.e.

f : a holomorphic map $M \to \mathrm{P}^1(\mathrm{C})$.

Definition 3 **Assume $a \in \mathrm{P}^1$ and $f(o) \neq a$.**

$$m(r,a) = E_o[\log[f(X_{\tau_r}),a]^{-2}], \tag{2}$$

$$N(r,a) = \lim_{\lambda\to\infty} \lambda P_o(\sup_{0<s<\tau_r} \log[f(X_s),a]^{-2} > \lambda), \tag{3}$$

$$T(r) = E_o[\int_0^{\tau_r} ||df||^2(X_s)ds]. \tag{4}$$

Rem.

$$m(r,a) = \int_{\partial B(r)} \log[f(z),a]^{-2} d\pi_r^o(z)$$

$$T(r) = c_m \int_{B(r)} g_r(o,z) f^*\omega_o \wedge \omega^{m-1}$$

where $B(r) = \{x \in M : v(x) < r\}$, $d\pi_r^o$: $B(r)$: harmonic measure on $\partial B(r)$ w.r.t. o, $g_r(o,z)$: Green function on $B(r)$ with Dirichlet boundary condition on $\partial B(r)$, ω : Kähler form on M, ω_o Fubini-Study metric on $\mathrm{P}^1(\mathrm{C})$. $c_m = 2\pi^m/(m-1)!$.

Since $\log[f(z), a]^{-2}$ is a δ-subharmonic function, $\Delta_M \log[f(z), a]^{-2}$ can be regarded as a signed measure denoted by $d\mu$. This signed measure $d\mu$, which is called a Riesz charge of $\log[f(z), a]^{-2}$, has a unique Jordan decomposition $d\mu = d\mu_1 - d\mu_2$. We note that μ_2 is supported by $f^{-1}(a)$. We define counting function of the points $f^{-1}(a)$ by

$$N(r, a) = \frac{1}{2} \int_{B(r) \cap f^{-1}(a)} g_r(o, z) d\mu_2(z).$$

[FMT] Apply Ito's formula to $\log[W_t, a]^{-2}$ with $W_{\rho_t} = f(X_t)$.

Assume $f(o) \neq a$.

$$m(r,a) - m(0,a) + N(r,a) = T(r).$$

Proposition 4 If M has Liouville property and f is nonconstant, then $T(r) \to \infty$ $(r \to \infty)$ and log-Cap$(f(M)^c) = 0$. (Casorati-Weierstrass thm)

$$R(x) = \inf_{\xi \in T_x M,\ ||\xi||=1} Ric(\xi, \xi)$$

$$N(r, Ric) = -E_o[\int_0^{\tau_r} R(X_s)ds].$$

$$N_1(r) := \lim_{\lambda \to \infty} \lambda P_o(\sup_{0 \le t \le \tau_r} \log^- ||df||^2(X_t) > \lambda).$$

Theorem 5 [A. JMSJ'08] $a_1, a_2, \ldots, a_q \in \mathrm{P}^1(\mathrm{C})$ distinct points. For any $\epsilon > 0$, $\exists E_\epsilon \subset [0, \infty)$ s.t. $|E_\epsilon| < \infty$ and

$$\sum_{j=1}^{q} m(r, a_j) + N_1(r)$$
$$\leq 2T(r) + 2N(r, Ric) + \log C(o, r, \epsilon)$$
$$+E[\log ||\nabla v||^2(X_{\tau_r})] + O(\log T(r))$$

for $r \notin E_\epsilon$.

$$C(x,r,\epsilon) = \frac{C_1(x,r)C_3(x,r,\epsilon)}{C_2(x,r)^{(1+\epsilon)^2}}$$

$\alpha < r, \quad x \in B(\alpha)$ **fixed.**

$C_1(x,r) = \sup_{z \in \partial B(\alpha)} g_r(x,z)/(r-\alpha).$

There exists $r' < r$ **s.t.** $\inf_{r'<t<r} \inf_{x \in B(t)} ||\nabla v||(x) > 0.$

$$C_2(x,r) = \inf_{y \in \partial B(r')} g_r(x,y)\left(\int_{v(x)}^{r} e^{-\int_{v(x)}^{t} 2\mu(z)dz} dt\right)^{-1},$$

where $\mu(t)$ is defined by

$$\mu(t) = \begin{cases} 0 & \text{for } 0 \leq t < r', \\ \mu^{(0)}(t) & \text{for } r' \leq t < r \end{cases}$$

and

$$\mu^{(0)}(t) = \frac{1}{2} \sup_{x \in \partial B(t)} \frac{\Delta_M v}{||\nabla v||^2}(x).$$

$$C_3(x, r, \epsilon) = \exp(2(1+\epsilon) \int_{v(x)}^{r} \mu(z) dz).$$

[Algebraic hypersurfaces in C^n]

Let M be an algebraic hypersurface of degree k nonsingular at infinity in C^n. i.e. $M = \{h = 0\}$ s.t. $h = h^{(k)} + h^{(k-1)} + \cdots + h^{(0)}$ where $h^{(j)}$ is a homogeneous polynomial of degree j and $\{h^{(k)} = 0\}$ is nonsingular in $\mathrm{P}^{n-1}(\mathrm{C})$.

Kähler metric: the induced metric from C^n. $v(x) = r(x)$: Euclidean distance between x and o.
$B(R) = \{r(x) < R\}$. $g_R(x, y)$: Green's function on $B(R)$ with Dirichlet condition.

Proposition 6

$$c(x_o)\log\frac{R}{r(x)} \le g_R(x_o,x) \le c'(x_o)\log\frac{R}{r(x)} \quad (n=2),$$

$$c(x_o)(r(x)^{4-2n} - R^{4-2n})$$
$$\le g_R(x_o,x) \le c'(x_o)(r(x)^{4-2n} - R^{4-2n}) \quad (n \ge 3).$$

Rem. i) If a complex hypersurface N in $\mathbb{C}^n$ has a Green function estimate as above, then N is algebraic.

ii) Any algebraic submanifold has Liouville property.

$\rightsquigarrow T(r) \uparrow \infty$ as $r \uparrow \infty$.

[SMT]

Theorem 7 For any $\epsilon > 0$ $\exists E_\epsilon \subset [0,\infty)$ s.t. $|E_\epsilon| < \infty$ and

$$\sum_{j=1}^{q} m(r, a_j) + N_1(r)$$
$$\leq 2T(r) + (2(k-1) + \epsilon(2n-3)) \log r + O(1)$$

for $r \notin E_\epsilon$.

[Defect relation]

We can also see

$$T(r) \geq const. \log r$$

if f is nonconstant.

Define $c(f) = \liminf_{r \to \infty} \frac{T(r)}{\log r}$ $(\leq \infty)$. Note $c(f) > 0$ if f is nonconstant.

$$\sum_{i=1}^{q} \delta(a_i) \leq 2 + \frac{2(k-1)}{c(f)}.$$

[Second formulation]

Let M be a complex manifold, X a holomorphic diffusion on M defined by the Dirichlet form

$$\mathcal{E}(u,v) = \frac{1}{2}\int_M du \wedge d^c v \wedge \theta, \quad u,v \in C_o^\infty(M)$$

with an admissible measure dm.

Assume X is conservative i.e. $e^{tL}1 = 1$.

Let f be a nonconstant meromorphic function on M(i.e. a nonconstant holomorphic map f from M to one dimensional complex projective space $\mathrm{P}^1(\mathrm{C})$). Then $f(X)$ is a holomorphic martingale and time changed process of Brownian motion on $\mathrm{P}^1(\mathrm{C})$. Then there exists an increasing process $[f(X), \overline{f(X)}]_t$ s.t.

$$f(X_t) = W([f(X), \overline{f(X)}]_t),$$

where W_t is Brownian motion on $\mathrm{P}^1(\mathrm{C})$.

Define Characteristic function:

$$\tilde{T}_x(t) := E_x[[f(X), \overline{f(X)}]_t].$$

Since $[f(X), \overline{f(X)}]_t$ **is a PCAF of** X**, there exists a measue** $d\mu_f$ **satisfying that**

$$\lim_{t \to 0} \frac{1}{t} \int_M \tilde{T}_x(t)\phi(x)dm(x) = \int_M \phi(x)d\mu_f(x)$$

for $\forall \phi \in C_o^\infty(M)$**.**
We call $d\mu_f$ **an energy measure of** f **w.r.t.** X**.**

If M is Kählerian and X is Brownian motion associated with the Kähler metric, then

$$\tilde{T}_x(t) = \frac{1}{2} E_x[\int_0^t ||df||^2 (X_s) ds]$$
$$= \frac{1}{2} \int_0^t \int_M p(s, x, y) ||df||^2 (y) dm(y) ds,$$

where $||df||^2$ is the energy density of f with respect to the Kähler metric. Hence $d\mu_f = \frac{1}{2} ||df||^2 dm$.

Note that this is an analogy of classical Ahlfors-Shimizu characteristic function :

$$T(r) = \frac{1}{2}\int_{|z|<r} ||df||^2 g_r(o,z)dxdy$$
$$= \frac{1}{2}E[\int_0^{\tau_r} ||df||^2(Z_s)ds]$$

where $M = \mathrm{C}$, Z_t is a complex Brownian motion on C and $g_r(x,y)$ is a Green's function of Laplacian on $\{|z| < r\}$ with Dirichlet boundary condition.

Let us consider a class of meromorphic functions

$$FC(M,X) := \{f : \text{ a nonconst. meromorphic funct. on } M \mid \tilde{T}_x(t) < \infty \ (\forall t > 0) \text{ for } m - a.e.\ x \in M\}$$

Rem. If f is of finite energy i.e. $\mu_f(M) < \infty$, then $f \in FC(M,X)$.

Counting function :

$$\tilde{N}_x(t,a) = \lim_{\lambda\to\infty} \lambda P_x\Big(\sup_{0\le s\le t} \log[f(X_s),a]^{-2} > \lambda \Big),$$

where $[w,a]$: chordal distance on $\mathrm{P}^1(\mathrm{C})$. Compare with the counting function in classical Nevanlinna theory:

$$N(r,a) = \sum_{f(\zeta)=a,\ |\zeta|<r} 2\log\frac{r}{|\zeta|}$$

$$= \lim_{\lambda\to\infty} \lambda P\Big(\sup_{0<s<\tau_r} \log|f(Z_s)-a|^{-2} > \lambda \Big),$$

where Z_t : BM(C) with $Z_0 = o$ and $\tau_r = \inf\{t>0 : |Z_t| > r\}$.

Define

$$\tilde{m}_x(t,a) = E_x[\log[f(X_t),a]^{-2}]$$

for $f(x) \neq a$.

[FMT] If $f \in FC(M,X)$ and $f(x) \neq a$,

$$\tilde{m}_x(t,a) - \tilde{m}_x(0,a) + \tilde{N}_x(t,a) = \tilde{T}_x(t)$$

for $0 \leq t < \infty$.

A desired property:

$$\tilde{N}_x(t, a) = 0 \text{ if } f \text{ omits } a.$$

does not always hold.

[Ass(A)]

$$\lim_{\lambda\to\infty} \lambda P_x(\sup_{0\le s\le t} \log[f(X_s), a]^{-2} > \lambda) = 0 \ (t > 0)$$

holds for a.e.-x and any $f \in FC(M, X)$ and $a \in \mathrm{P}^1(\mathrm{C}) \setminus f(M)$.

Introduce a class of f:

$$SG(M, X) = \{f \in Hol_*(M, \mathrm{P}^1(\mathrm{C})) \mid \int_1^\infty e^{-\epsilon r^2} \mu_f(B(r))dr < \infty \text{ for } \forall \epsilon > 0\}.$$

$$B(r) = \{x \in M \mid \rho(x) < r\},$$

ρ: an exhaustion function s.t. $\Gamma(\rho, \rho)$ is bounded.

Kähler case:

Ass(R): There exists a nonnegative increasing function k on $[0, \infty)$ s.t.

$$R(x) \geq -k(r(x)^2) \text{ and } k(t) = o(t) \text{ as } t \to \infty.$$

$\left(R(x) = \inf_{\xi \in T_x M, \ ||\xi||=1} Ric(\xi, \xi), \ \rho(x) = r(x)\right.$: Riemannian distance function)

$$\text{Ass(R)} + f \in SG(M, X) \Rightarrow f \in FC(M, X)$$

$$\text{Ass(R)} + f \in SG(M, X) \Rightarrow \text{Ass(A) for } f \in SG(M, X)$$

Rem. Ass(R) $\Rightarrow$ stochastic completeness i.e. Brownian motion on M is conservative.

[Ass(B)] There exists a function Φ independent of f s.t.

$$dd^c \log \Gamma(f, f) \wedge \theta \geq -\Phi(x) dm(x)$$

on $M \setminus \{x \in M \mid \Gamma(f, f)(x) = 0\}$ for any nonconstant $f \in \mathcal{O}(U)$ ($\forall U$:open $\subset M$).

If M is Kähler and X is the associated Brownian motion, then Ass(B) holds with $\Phi(x) = -2R(x)$.

Define

$$\tilde{N}_1(t,x) = \lim_{\lambda\to\infty} \lambda P_x(\sup_{0\le s\le t} \log^- \Gamma(f,f) > \lambda).$$

Theorem 8 [SMT] **Assume (A) + (B). If $f \in FC(M,X)$ and $a_1,\ldots,a_q$ distinct points in $\mathrm{P}^1(\mathrm{C})$, then**

$$\sum_{j=1}^{q} \tilde{m}_x(t,a_j) + \tilde{N}_1(t,x)$$

$$\le 2\tilde{T}_x(t) + E_x[\int_0^t \Phi(X_s)ds] + O(\log \tilde{T}_x(t))$$

holds except for t in an exceptional set of finite length and a.e. $x \in M$.

Theorem 9 **Assume (A) + (B). If $f \in FC(M, X)$,**

$$\#(\mathrm{P}^1(\mathrm{C}) \setminus f(M)) \leq 2 + \limsup_{t\to\infty} \frac{E_x[\int_0^t \Phi(X_s)ds]}{\tilde{T}_x(t)}.$$

Moreover if X is recurrent and $\Phi \in L^1(dm)$,

$$\#(\mathrm{P}^1(\mathrm{C}) \setminus f(M)) \leq 2 + \frac{\int_M \Phi(x)dm(x)}{\mu_f(M)}.$$

Corollary 10 **If Brownian motion is recurrent w.r.t. a complete Kähler metric g and Ass(R) are satisfied w.r.t. g, then for $f \in SG(M, X)$**

$$\#(\mathbf{P}^1(\mathbf{C}) \setminus f(M)) \leq 2 - \frac{4\int_M R(x)dv(x)}{\int_M ||df||^2 dv}.$$

[Recurrence of diffusion]

Definition 11 X is recurrent if

$$\limsup_{t\to\infty} 1_U(X_t) = 1$$

holds with probability 1 for any open set $U \subset M$.

[Equivalent condition to recurrence] One of the following condions is equivalent to recurrence.
i) There exist no constant bounded L-subharmonic functions.

ii) Let $p(t,x,y)$ be the transition kernel of X.

$$\int_0^\infty p(t,x,y)dt = \infty.$$

iii) [Grigor'yan, Sturm]

$$\int_1^\infty \frac{r}{m(B(r))} dr = \infty.$$

[Example 1. algebraic variety]

Consider a special algebraic variety $M = \overline{M} \setminus D$ where $\overline{M}$ is a projective algebraic manifold and D is an anlytic hypersurface in $\overline{M}$. Assume that D has only simple normal crossings. Let L, L_j be a holomorphic line bundle dtermined by D and D_j respectively. Thus $\exists \sigma \in \Gamma(M, L), \exists \sigma_j$ satifies $D = (\sigma)$ and $D_j = (\sigma_j)$. $L = L_1 \otimes \cdots \otimes L_l,\ \sigma = \sigma_1 \otimes \cdots \otimes \sigma_l$.

Assume $c_1(L) > 0$. Consider three Kähler metrics on M:

1) (projective) $dd^c \log ||\sigma||^{-2}$: Imcomplete. On a nhd of D $||\sigma|| = |z_1 \cdots z_l| a(z)$ where $a(z)$ is C^∞. Thus this metric is smooth on $\overline{M}$. Hence the associated Brownian motion can be regarded as the process on $\overline{M}$. Then it is recurrent but Ass(A) does not hold.

2) (Euclidean) $dd^c ||\sigma||^{-2}$: This is complete and Ricci curvature is bounded. So Ass(R) is satisfied. (The first part of Theorem9 holds.) The associated Brownian motion is transient if $\dim M \geq 2$.

3) $w = Cdd^c \log ||\sigma||^{-2} - \sum_{j=1}^{l} dd^c \log(\log ||\sigma_j||^2)^2$:

(Cornalba-Griffiths metric)

Proposition 12 Assume $c_1(L) > 0$. There exist $C > 0$ and $||\cdot||$ s.t. Cornalba-Griffiths metric satisfies the following properties.

i) Complete.

ii) Finite volume. i.e. $\int_M w^m < \infty$.

iii) Ricci curvature is bounded.

iv) $Ric < 0$ and $-\int_M Ric \wedge w^{m-1} < \infty$.

ii) implies recurrence. iii) ensures the validity of Ass(R). Then we have

Theorem 13 Assume $c_1(L) > 0$, X : Brownian motion w.r.t CG metric. For $f \in SG(M, X)$

$$\#(\mathbf{P}^1(\mathbf{C}) \setminus f(M)) \leq 2 + \frac{2vol_{CG}(M)}{\mu_f(M)},$$

where vol_{CG} is the volume w.r.t. Cornalba-Griffiths metric, μ_f is associated with X.

[Example 2. submanifolds in C^n]

Let M be a properly immersed submanifold in C^n with $\dim M = m$.

The induced metric from C^n defines a holomorhic diffusion Y via the following Dirichlet form:

$$\mathcal{E}(u,v) = \frac{1}{2}\int_M du \wedge d^c v \wedge (dd^c||z||^2)^{m-1}, \quad u,v \in C_o^\infty(M)$$

and an admissible measure dv defined by $\int_M \phi dv = \int_M \phi (dd^c||z||^2)^m$ **where** $||z||^2 = |z_1|^2 + \cdots + |z_n|^2$.

Proposition 14 **i)** Y **is conservative.**

ii) If $m \geq 2$, Y **is transient.**

Set $w = dd^c \log(1 + ||z||^2)$.

Consider another holomorphic diffusion X defined by

$$\mathcal{E}(u, v) = \frac{1}{2} \int_M du \wedge d^c v \wedge w^{m-1}, \quad u, v \in C_o^\infty(M)$$

and an admissible measure dm defined by

$$\int_M \phi dm = \int_M \phi dd^c ||z||^2 \wedge w^{m-1}.$$

Set $V(r) = \int_{M \cap \{||z|| < r\}} (dd^c ||z||^2)^m$.

Proposition 15 (H.Kaneko) If

$$\int_1^\infty \frac{r^{2m-1}}{V(r)} dr = \infty, \qquad (*)$$

then X is recurrent.

In particular if M is algebraic, X is recurrent.
($V(r) = O(r^{2m})$ due to W.Stoll).

Hence we have

Theorem 16 Assume (R) w.r.t. the induced metric and (*). For $f \in SG(M, Y)$ w.r.t. the induced metric

$$\#(\mathrm{P}^1(\mathrm{C}) \setminus f(M)) \leq 2 + \frac{2K(M)}{e_f(M)},$$

where

$$K(M) = \limsup_{r \to \infty} \frac{-\int_{M \cap \{||z|| < r\}} R(z) dv(z)}{r^{2(m-1)}},$$

$$e_f(M) = \lim_{r \to \infty} \frac{\frac{1}{2} \int_{M \cap \{||z|| < r\}} ||df||^2 dv(z)}{r^{2(m-1)}},$$

where $dv, R(z), ||df||$ w.r.t.the induced metric.

If M is an algebraic hypersurface of degree k non-singular at infinity in $\mathbb{C}^n$, then $K(M) < \infty$ with $m = n - 1$.

[transcendental cases]

1. There exists a hypersurface M in C^n satisfying $V(r) \sim r^{2n-2} \log r$. It satisfies the Kaneko's criteria and supports a recurrent holomorphic diffusion.

Then

$$Cap(\mathrm{P}^1(\mathrm{C}) \setminus f(M)) = 0.$$

$\liminf_{t\to\infty} \tilde{T}_x(t)/(\log t)^2 > 0 \ \Rightarrow$ **#omitted values is finite.**

Rem. f: **polynomial**$|_M \Rightarrow \ \tilde{T}_x(t) = O(\log t)$.

2. $M = \{e^x + e^y = 1\} \subset \mathrm{C}^2$.

Ric is bounded. $\Rightarrow$ Ass(R).

Proposition 17 Assume that X is BM assciated with the induced metric and $f \in SG(M, X)$.

Set

$$C_x(f) := \liminf_{t \to \infty} \frac{\tilde{T}_x(t)}{\sqrt{t}}.$$

We have

$$\sum_a \tilde{\delta}_x(a, f) \leq 2 + \frac{2}{C_x(f)},$$

where $\tilde{\delta}_x(a, f) = \liminf_{t \to \infty} \frac{\tilde{m}_x(a, t)}{\tilde{T}_x(t)}$.