Deficiencies of Holomorphic Curves for Hypersurfaces and Linear Systems

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Introduction

We first give a construction of holomrphic curves f from $\mathbb{C}$ into the complex projective space $\mathbb{P}_n(\mathbb{C})$ with deficient hypersurfaces. There have been a few studies on the construction of holomorphic curves with a deficient hypersurface. We prove the existence of holomorphic curves that have a preassigned positive deficiency for a given divisor D in $\mathbb{P}_n(\mathbb{C})$. We notice that there has been a conjecture stating the estimate

$$\delta_f(D) \leq \frac{C}{d}$$

holds under a generic condition for D, where C is a positive constant independent of f and D.

We can construct many examples of singular hypersurfaces for which the estimate of the above type does not hold. Our construction is based on some properties of entire functions of order zero proved by Valiron.

Next, we consider deficiencies on linear systems on algebraic manifolds M. Let $L \to M$ an ample line bundle and $f : \mathbb{C} \to M$ a transcendental holomorphic curve. We consider the Nevanlinna deficiency of holomorphic curve f as a function of linear systems $\Lambda \subseteq |L|$. We define the deficiency for the base locus of linear system by means of the new language in the value distribution theory for coherent ideal sheaves. In particular, we construct holomoephic curves with deficiencies for Λ.

§1 Notation

Let z be the natural coordinate in $\mathbb{C}$ and $d^c = (\sqrt{-1}/4\pi)(\bar{\partial} - \partial)$. Set

$$\Delta(r) = \{z \in \mathbb{C} : \ |z| < r\} \quad \text{and} \quad C(r) = \{z \in \mathbb{C} : \ |z| = r\}.$$

For a (1,1)-current φ of order zero on $\mathbb{C}$ we set

$$N(r, \varphi) = \int_1^r \langle \varphi, \chi_{\Delta(t)} \rangle \frac{dt}{t},$$

where $\chi_{\Delta(r)}$ denotes the characteristic function of $\Delta(r)$. Let M be a compact complex manifold and $L \to M$ line bundle over M. We denote by $\Gamma(M, L)$ the space of all holomorphic sections of $L \to M$. Let $|L| = \mathbb{P}(\Gamma(M, L))$ be the complete linear system defined by L and Λ a linear system included in

$|L|$. When $\Lambda \neq \emptyset$, we define the base locus of Λ by

$$\mathrm{Bs}\ \Lambda = \bigcap_{D \in \Lambda} \mathrm{Supp}\ D.$$

Denote by $||\cdot||$ a hermitian fiber metric in L and by ω its Chern form. Let $f : \mathbb{C} \to M$ be a holomorphic curve. We set

$$T_f(r, L) = \int_1^r \frac{dt}{t} \int_{\Delta(t)} f^*\omega,$$

and call it the characteristic function of f with respect to L. We define the order ρ_f of $f : \mathbb{C} \to M$ by

$$\rho_f = \limsup_{r \to +\infty} \frac{\log T_f(r, L)}{\log r}.$$

We notice that the definition of ρ_f is independent of a choice of positive line bundles $L \to M$. We call f of finite type if

$\rho_f < +\infty$. Let $D = (\sigma) \in |L|$ with $||\sigma|| \leq 1$ on M. Assume that $f(\mathbb{C}) \not\subseteq \mathrm{Supp}\ D$. We define the proximity function of D by

$$m_f(r,\, D) = \int_{C(r)} \log\left(\frac{1}{||f^*\sigma||}\right) \frac{d\theta}{2\pi}.$$

We define Nevanlinna's deficiency $\delta_f(D)$ by

$$\delta_f(D) = \liminf_{r\to+\infty} \frac{m_f(r,\, D)}{T_f(r,\, L)}.$$

We have then defect functions δ_f defined on $|L|$. If $\delta_f(D) > 0$, then D is called a deficient divisor in the sense of Nevanlinna. Let $E = \sum_j \nu_j p_j$ be an effective divisor on $\mathbb{C}$, where $\nu_j \in \mathbb{Z}^+$ and $p_j \in \mathbb{C}$ are distinct points. For a positive integer k, we define the truncated counting function of E by

$$N_k(r,\, E) = \sum_j \min\{k,\, \nu_j\}\, N(r,\, p_j).$$

In general, for an effective divisor D on M, we write $L(D)$ for the line bundle determined by D. We now consider the case where $M = \mathbb{P}_n(\mathbb{C})$. Let $L(H) \to \mathbb{P}_n(\mathbb{C})$ be the hyperplane bundle over $\mathbb{P}_n(\mathbb{C})$ and ω_0 the Fubini-Study form on $\mathbb{P}_n(\mathbb{C})$. In the case where $M = \mathbb{P}_n(\mathbb{C})$ and $L = L(H)$, we always take ω_0 for ω and we simply write $T_f(r)$ for $T_f(r,\ L(H))$.

§2. Construction of holomorphic curves with deficient divisor.

We first consider the case of hyperplane.

Theorem 2.1. *Let α be an arbitrary positive real number less than one and let H be an arbitrary hyperplane in $\mathbb{P}_n(\mathbb{C})$. Then there exists an algebraically nondegenerate holomorphic curve $f : \mathbb{C} \to \mathbb{P}_n(\mathbb{C})$ such that $\delta_f(H) = \alpha$.*

We next deal with the case where a given divisor D is a hypersurface of degree d not less than two, that is, $D \in |L(H)^{\otimes d}|$ with $d \geq 2$. Let $P(\zeta) = P(\zeta_0, \cdots, \zeta_n)$ be a homogeneous polynomial of degree d and define a divisor D in $\mathbb{P}_n(\mathbb{C})$ by $P = 0$.

Theorem 2.2. *There exists a positive constant $\lambda(D)$ with $\lambda(D) \leq d$ depending only on D that satisfies the following property: For each positive real number α with $\alpha \leq \lambda(D)/d$, there exists an algebraically nondegenerate holomorphic curve $f : \mathbb{C} \to \mathbb{P}_n(\mathbb{C})$ such that $\delta_f(D) = \alpha$.*

Theorem (Valiron). *Let* f *be a transcendental holomorphic function on* $\mathbb{C}$*. Suppose that* $T_f(r) = O((\log r)^2)$ *as* $r \to +\infty$*. Then*

$$\lim_{r \to +\infty} \frac{\log M(r, f)}{T_f(r)} = \lim_{r \to +\infty} \frac{N(r, 0, f)}{T_f(r)} = 1.$$

Furthermore, there exists a Borel subset $\varepsilon(r)$ *of* $C(r)$ *such that*

$$\log |f(z)| = (1 + o(1)) \log M(r, f)$$

for all $z \in C(r) \setminus \varepsilon(r)$ *and* $\mu(\varepsilon(r)) \to 0$ *as* $r \to +\infty$*, where* μ *denotes the Haar measure on* $C(r)$ *normalized so that* $\mu(C(r)) = 1$*.*

$\boxed{\textbf{Remark 2.3.}}$ We give here note on the constant $\lambda(D)$ in Theorem 2.2. Let $P(\zeta)$ be a homogeneous polynomial of degree d such that $D = \{P = 0\}$. We rewrite P as follows:

$$P(\zeta) = \sum_{j=0}^{d} a_j \zeta_0^{d-j} \zeta_1^j + c_2 \zeta_2^d + \cdots + c_n \zeta_n^d + Q(\zeta),$$

where Q is a polynomial in ζ which does not contain terms $\zeta_0^{d-j}\zeta_1^j$ for $j = 0, \cdots, d$. Let d_j be the degree in ζ_j that are contained in P. Set $\tilde{d} = \min_{0 \le j \le n} d_j$. We define a polynomial $L(z)$ in z by

$$L(z) = \sum_{j=0}^{d} a_j z^j.$$

Denote by κ the largest multiplicity of roots of the equation $L(z) = 0$, where $1 \leq \kappa \leq d-1$. We now give a list of the constant $\lambda(D)$ in Theorem 2.2:

(I) If $d = \tilde{d}$, then $\lambda(D) = \kappa$.

(II) If $\tilde{d} < d$, then $\lambda(D) = d - \tilde{d}$.

If D has a bad singularity, the estimate of type

$$\delta_f(D) \leq \frac{C}{d}$$

do not hold.

§3. Examples.

We give here some examples of irreducible hypersurfaces of degree d.

$\boxed{\textbf{Example 3.1.}}$ We define an irreducible hypersurface D_d of degree d in $\mathbb{P}_n(\mathbb{C})$ by

$$\zeta_1^d + \cdots + \zeta_n^d = 0.$$

Note that D_d has just one singular point $(1, 0, \cdots, 0)$. In this case, $\lambda(D) = d$. Hence, for an arbitrary positive real number α not greater than one, there exists an algebraically nondegenerate holomorphic curve $f : \mathbb{C} \to \mathbb{P}_n(\mathbb{C})$ such that $\delta_f(D_d) = \alpha$.

Example 3.2. We next give an example of a nonsingular hypersurface. We define a nonsingular hypersurface S_d in $\mathbb{P}_n(\mathbb{C})$ of degree $d \geq 2$ by

$$\zeta_0^{d-1}\zeta_2 - \zeta_1^d + \zeta_1\zeta_2^{d-1} + \sum_{j=3}^{n} \zeta_j^d = 0.$$

In this case, we have $\lambda(D) = 1$.

Example 3.3. Let $n = 2$ and define an irreducible curve C_d by

$$\zeta_0\zeta_2^{d-1} - \zeta_1^d = 0.$$

Note that C_d also has just one singular point $(1,0,0)$, if $d \geq 3$. We also note that C_d is a rational curve. For C_d, we have $\lambda(C_d) = d-1$ by Theorem 2.2. Hence, for an arbitrary positive number $\alpha \leq (d-1)/d$, there exists an algebraically nondegenerate holomorphic curve $f : \mathbb{C} \to \mathbb{P}_2(\mathbb{C})$ such that $\delta_f(D_d) = \alpha$.

$\boxed{\textbf{Remark 3.4.}}$ We note that, for each positive integer d not less than two, there exists a holomorphic curve $f : \mathbb{C} \to \mathbb{P}_2(\mathbb{C})$ such that f omits C_d. In fact, if we define f by

$$f(z) = (\exp z + \exp(1-d)z^2,\ 1,\ \exp z^2),$$

then we easily see $f(\mathbb{C}) \cap C_d = \emptyset$. Note that f is algebraically nondegenerate.

The holomorphic curves constructed above is of order zero. In the above examples, we use exponential curves and obtaine holomorphic curves of order one with deficiencies (Ahlfors-Weyl's method). Note that this method works in the case that can be reduced to the hyperplane case. Indeed, let F_d be the Fermat surface degree d, that is,

$$F_d: \ \zeta_0^d + \cdots + \zeta_n^d = 0.$$

Then our method gives a holomorphic curve f with $\delta_f(F_d) = \alpha$ $(0 < \alpha \leq 1/d)$, but we cannot construct a holomorphic curve with positive deficiency for F_d by Ahlfors-Weyl's method. Hence it seems that our method has a wide range of applicability.

Remqrks 3.5. (1) We notice that Theorem 2.2 is also valid for meromorphic mappings $\mathbb{C}^m \to \mathbb{P}_n(\mathbb{C})$. When $m \geq n$, we get a dominant mapping. We can find many examples of singular divisors and meromorphic mappings $f : \mathbb{C}^m \to \mathbb{P}_n(\mathbb{C})$ for which Griffiths' defect relation

$$\sum_{j=1}^{q} \delta_f(D_j) \leq \frac{n+1}{d}$$

does not hold. For instance, we consider the Example 3.3. Namely, let C_d be a curve as in Example 3.3 and α a positive real number less than $(d-1)/d$. Then there exists a dominant meromorphic mapping $f : \mathbb{C}^m \to \mathbb{P}_2(\mathbb{C})$ such that

$$\delta_f(C_d) = \frac{d-2}{d}.$$

Hence we also have an example for which Griffiths' defect relation does not hold.

(2) Suppose that $d \geq 3$. We note that, if $\tilde{d} \leq d-2$, then D has a singular point. Indeed, we may assume that $\tilde{d} = d_0$. We write P as follows:

$$P(\zeta) = \zeta_0^{d-k} Q_1(\zeta) + Q_2(\zeta),$$

where $Q_2(\zeta)$ does not contain ζ_0 and ζ_0^{d-k} is the greatest common divisor in $P - Q_2$. Since $d-k \leq d-2$, we see that D has a singular point $(1, 0, \cdots, 0)$. Set $w_j = \zeta_j/\zeta_0$ for $j = 1, \cdots, n$. Define $\tilde{P}(w) = \zeta_0^{-d} P(\zeta)$, where $w = (w_1, \cdots, w_n)$. If $d - d_0 \geq n+1$, then the polynomial $\tilde{P}(w)$ has a zero at $(0, \cdots, 0)$ with multiplicity at least $n+1$. Hence D is not normal crossings at $(1, 0, \cdots, 0)$. This fact shows that the hypothesis in Griffiths' defect relation, that is, D is at most simple normal crossings, cannot be simply dropped.

Remark 3.6.

(Effect of the resolution of singularities to deficiencies)

We considered an example of the singular curve C_d defined by

$$C_d : \zeta_0\zeta_2^{d-1} - \zeta_1^d = 0.$$

This curve has only one singular point P(1, 0, 0), if $d \geq 3$. If $\pi : Q_P(\mathbb{P}_2(\mathbb{C})) \to \mathbb{P}_2(\mathbb{C})$ is a monoidal transformation with the center P, then this gives a resolution of singularity of C. Namely, let $\tilde{C}$ and $\bar{C}$ be the total transform and the proper transform of C_d, respectively. We also denote by E the exceptional curve. Then $\Sigma_1 = Q_P(\mathbb{P}_2(\mathbb{C}))$ is the Hirzeburch surface of rank one.

We see

$$\tilde{C} = (d-1)E + \bar{C},$$

where $\bar{C}$ is a nonsingular curve in Σ_1 We define a holomorphic curve $\tilde{f} : \mathbb{C} \to \Sigma_1$ by $\tilde{f} = \pi^{-1} \circ f$. We have then an estimate for $\delta_{\tilde{f}}(\bar{C})$ depending on the structure of the singularity:

$$\delta_{\tilde{f}}(\bar{C}) = \frac{\alpha}{1 + (1-\alpha)(d-1)}.$$

In particular, the estimate

$$\frac{\alpha}{d} < \delta_{\tilde{f}}(\bar{C}) < \frac{d-1}{2d-1}$$

is valid.

§4. Value distribution theory for coherent ideal sheaves

Let M be a projective algebraic manifolds and $L \to M$ an ample line bundle. Let $f : \mathbb{C} \to M$ be a transcendental holomorphic curve. We consider the Nevanlinna deficiency of f as a function of linear systems $\Lambda \subseteq |L|$. We define the deficiency for the base locus of linear system by means of the new language in the value distribution theory for acoherent ideal sheaves due to Noguchi-Winkelman-Yamnoi. In particular, we construct holomoephic curves with deficiencies for linear systems.

Let $\mathcal{I}$ be a coherent ideal sheaf of the structure sheaf $\mathcal{O}_M$ of M. Let $\mathcal{U} = \{U_j\}$ be a finite open covering of M with a partition of unity $\{\eta_j\}$ subordinate to $\mathcal{U}$. We can assume that there exists a finitely many sections $\sigma_{jk} \in \Gamma(U_j, \mathcal{I})$ such that every stalk $\mathcal{I}_p$ over $p \in U_j$ is generated by germs $\underline{\sigma_{j1}}_p, \cdots, \underline{\sigma_{jl_j}}_p$. Set

$$\rho_{\mathcal{I}}(p) = \left(\sum_j \eta_j(p) \sum_{k=1}^{l_j} \left| \sigma_{jk}(p) \right|^2 \right)^{1/2}$$

We take a positive constant C such that $C\rho_{\mathcal{I}}(p) \leq 1$ for all $p \in M$. Set

$$\phi_{\mathcal{I}}(p) = -\log \rho_{\mathcal{I}}(p).$$

We call it the proximity potential for $\mathcal{I}$. It is easy to verify that $\phi_{\mathcal{I}}$ is well-defined up to addition by a bounded continuous function on M. We now define the proximity function $m_f(r, \mathcal{I})$ of f for $\mathcal{I}$, or equivalently, for the complex subspace (may be non-reduced)

$$Y = (\mathrm{Supp}\,(\mathcal{O}_M/\mathcal{I}),\ \mathcal{O}_M/\mathcal{I}),$$

by

$$\boxed{m_f(r, \mathcal{I}) = \int_{C(r)} f^* \phi_{\mathcal{I}}(z) \frac{d\theta}{2\pi}}$$

provided that $f(\mathbb{C}) \not\subseteq \mathrm{Supp}\, Y$.

For $z_0 \in f^{-1}(\mathrm{Supp}\ Y)$, we can choose an open neighborhood U of z_0 and a positive integer ν such that

$$f^*\mathcal{I} = ((z - z_0)^\nu) \quad \text{on} \quad U.$$

Then we see

$$\log \rho_{\mathcal{I}}(f(z)) = \nu \log |z - z_0| + h_U(z) \quad \text{for} \quad z \in U,$$

where h_U is a C^∞-function on U. Thus we have the counting functions $N(r, f^*\mathcal{I})$ and $N_l(r, f^*\mathcal{I})$ as in §1. Moreover, we set

$$\omega_{\mathcal{I},f} = -dd^c h_U \quad \text{on} \quad U$$

and thus obtain a well-defined smooth (1, 1)-form on $\mathbb{C}$. Define the characteristic function $T_f(r, \mathcal{I})$ of f for $\mathcal{I}$ by

$$\boxed{T_f(r, \mathcal{I}) = \int_1^r \frac{dt}{t} \int_{\Delta(t)} \omega_{\mathcal{I},f}.}$$

We summarize the basic facts on value distribution theory for coherent ideal sheaves due to Noguchi-Winkelman-Yamanoi as follows (Forumn Math. **20** (2008)):

$\boxed{\textbf{Theorem.}}$ *Let* $f : \mathbb{C} \to M$ *and* $\mathcal{I}$ *be as above. Then the following hold*:

(i) (First Main Theorem) $T_f(r, \mathcal{I}) = N(r, f^*\mathcal{I}) + m_f(r, \mathcal{I}) + O(1)$.

(ii) *If* $L \to M$ *be an ample line bundle, then* $T_f(r, \mathcal{I}) = O(T_f(r, L))$.

(iii) *Let* $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *be coherent ideal sheaves. Suppose that* $f(\mathbb{C}) \not\subseteq \mathrm{Supp}\ (\mathcal{O}_M/\mathcal{I}_1 \otimes \mathcal{I}_2)$.
Then

$$T_f(r, \mathcal{I}_1 \otimes \mathcal{I}_2) = T_f(r, \mathcal{I}_1) + T_f(r, \mathcal{I}_2) + O(1)$$

and

$$m_f(r, \mathcal{I}_1 \otimes \mathcal{I}_2) = m_f(r, \mathcal{I}_1) + m_f(r, \mathcal{I}_2) + O(1).$$

(iv) *Let* $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *and* f *be as in* (iii). *If* $\mathcal{I}_1 \subset \mathcal{I}_2$, *then*

$$m_f(r, \mathcal{I}_2) \leq m_f(r, \mathcal{I}_1) + O(1).$$

Remark. When $\mathcal{I}$ defines an effective divisor D on M, we easily see

$$T_f(r, \mathcal{I}) = T_f(r, L(D)) + O(1)$$

and

$$m_f(r, \mathcal{I}) = m_f(r, D) + O(1).$$

Let $L \to M$ be an ample line bundle and $W \subseteq \Gamma(M, L)$ a subspace with $\dim W \geq 2$. Let $\Lambda = \mathbb{P}(W)$. We define a coherent ideal sheaf $\mathcal{I}_0$ in the following way: For each $p \in M$, the stalk $\mathcal{I}_{0,p}$ is generated by all germs $\underline{\sigma}_p$ for $\sigma \in W$. Then $\mathcal{I}_0$ defines the base locus of Λ as a complex subspace B_Λ, that is,

$$B_\Lambda = (\mathrm{Supp}\ (\mathcal{O}_M/\mathcal{I}_0), \mathcal{O}_M/\mathcal{I}_0).$$

Hence $\mathrm{Bs}\,\Lambda = \mathrm{Supp}\ (\mathcal{O}_M/\mathcal{I}_0)$. We notice that $\mathcal{I}_0$ can be written as

$$\mathcal{I}_0 = \mathcal{I}_1 \otimes \mathcal{I}_2, \ \ \mathrm{codim}\,\mathrm{Supp}\,(\mathcal{O}_M/\mathcal{I}_1) = 1, \ \ \mathrm{codim}\,\mathrm{Supp}\,(\mathcal{O}_M/\mathcal{I}_2) \geq 2.$$

§5. S.M.T.

We let $\Gamma(M, L)$ denote the space of all holomorphic sections of $L \to M$ and $|L|$ the complete linear system defined by L. Let $W \subseteq \Gamma(M, L)$ be a linear subspace with $l+1 = \dim W \geq 2$. Denote by Λ the linear system determined by W, that is, $\Lambda = \mathbb{P}(W)$. The linear system Λ may have the non-empty base locus. We now give SMT that gives a generalization of Ochiai's (T. Ochiai, Osaka J. Math. **11** (1974)).

Let $D_1, \cdots, D_q$ be divisors in Λ such that $D_j = (\sigma_j)$ for $\sigma_j \in W$.

We first give a definition of *subgeneral position*. Set $Q = \{1, \cdots, q\}$ and take a basis $\psi_0, \cdots, \psi_l$ of W. We write

$$\sigma_j = \sum_{k=0}^{l} c_{jk} \psi_k$$

for each $j \in Q$. For a subset $R \subseteq Q$, we define a matrix A_R by $A_R = (c_{jk})_{j \in R, 0 \leq k \leq l}$.

Definition 5.1. Let $N \geq l$ and $q \geq N+1$. We say that $D_1, \cdots, D_q$ are in N-subgeneral position in Λ if

$$\operatorname{rank} A_R = l+1 \quad \text{for every subset } R \subseteq Q \text{ with } \sharp R = N+1.$$

If they are in l-subgeneral position, we simply say that they are in general position.

Remark The above definition is different from the usual one. In fact, the divisors $D_1, \cdots, D_q$ are usually said to be in N-subgeneral position in Λ provided that

$$\bigcap_{j \in R} D_j = \emptyset \qquad \text{for every subset} \quad R \subseteq Q \quad \text{with} \quad \sharp R = N + 1.$$

However, the divisors $D_1, \cdots, D_q$ may have a common point when they are in N-subgeneral position in the sense of Definition 5.1. Thus our definition is weaker than the usual one. We always use " N-subgeneral position" in the sense of Definition 5.1.

Let $f : \mathbb{C} \to M$ be a transcendental holomorphic curve that is nondegenerate with respect to Λ, namely, the image of f is not contained in the support of any divisor in Λ. We have then the following generalized Crofton type formula due to R. Kobayashi:

Proposition 5.2. *Suppose that* $\mathrm{Bs}\,\Lambda \neq \emptyset$ *and* $f(\mathbb{C}) \not\subseteq \mathrm{Bs}\,\Lambda$. *Then*

$$\int_{D\in\Lambda} m_f(r, D)d\mu(D) = m_f(r, \mathcal{I}_0) + O(1).$$

Theorem 5.3. Let $D_1, \cdots, D_q \in \Lambda$ be divisors in N-subgeneral position. Then

$$(q-2N+\dim\Lambda-1)(T_f(r,L)-T_f(r,\mathcal{I}_0)) \leq \sum_{j=1}^{q} N(r,f^*D_j)+S_f(r),$$

where

$$S_f(r) = O(\log T_f(r,L)) + O(\log r)$$

as $r \to +\infty$ except on a Borel subset $E \subseteq [1,+\infty)$ with finite measure.

§6. Deficiency for linear systems.

Let $f : \mathbb{C} \to M$ be a transcendental holomorphic curve. Suppose that f is nondegenerate with respect to Λ. We define a deficiency $\delta_f(B_\Lambda)$ for B_Λ by

$$\boxed{\delta_f(B_\Lambda) = \liminf_{r\to+\infty} \frac{m_f(r, \mathcal{I}_0)}{T_f(r, L)}.}$$

Then we have the following:

Proposition 6.1. *Let* $f : \mathbb{C} \to M$ *be a transcendental holomorphic curve. Suppose that* f *is nondegenerate with respect to* Λ. *Then*

$$\int_{D \in \Lambda} \delta_f(D) d\mu(D) = \delta_f(B_\Lambda).$$

Theorem 6.2. *Let* $f : \mathbb{C} \to M$ *be as above. Then*

$$\delta_f(D) \geq \delta_f(B_\Lambda)$$

for all $D \in \Lambda$ *and*

$$\delta_f(D) = \delta_f(B_\Lambda)$$

for almost all $D \in \Lambda$ *in the sense of Lebesgue measure in* Λ*.*

Hence we have a defect function $\delta_f : \Lambda \to [\delta_f(B_\Lambda), 1]$. We consider $\delta_f(B_\Lambda)$ as a deficiency of f for Λ.

Next, we deduce the defect relation from Theorem 5.3. We let $E(f;N)$ denote the set of all $r \in [1, +\infty)$ satisfying

$$T_f(r,\ L) + N \leq T_f\left(r + \frac{1}{(T_f(r,\ L)+N)^2},\ L\right)$$

where N is a positive integer. Then the Lebesgue measure $|E(f;N)|$ is finite, and $E(f;N_2) \subseteq E(f;N_1)$ if $N_1 < N_2$. Set

$$E(f) = \bigcap_{N \in \mathbb{Z}^+} E(f;N).$$

We call $E(f)$ the exceptional growth set for f. The existence of non-empty $E(f)$ affects on deficiency.

In the case where f is of finite type, we set $E(f) = \emptyset$.

After Nochka , we now define modified deficiency in the sense of Nochka. We define N-th Nevanlinna's deficiency $\delta_f(D;N)$ by

$$\delta_f(D;N) = \liminf_{\substack{r\to+\infty \\ r\notin E(f;N)}} \frac{m_f(r,\ D)}{T_f(r,\ L)}.$$

It is clear that $\delta_f(D;N_2) \leq \delta_f(D;N_1)$ if $N_1 < N_2$.

We define the modified deficiency of f in the sense of Nochka by

$$\boxed{\tilde{\delta}_f(D) = \lim_{N\to+\infty} \delta_f(D;N).}$$

Then $\delta_f(D) \leq \tilde{\delta}_f(D)$ and $\delta_f(D) = \tilde{\delta}_f(D)$ if f is of finite type. bigskip

We define $\tilde{\delta}_f(B_\Lambda; N)$ and $\tilde{\delta}_f(B_\Lambda)$ by the same way. Furthermore, we also define

$$\gamma_f(\Lambda) = \liminf_{r \to +\infty} \frac{T_f(r,\ \mathcal{I}_0)}{T_f(r,\ L)}.$$

Then $0 \leq \gamma_f(\Lambda) \leq 1$ and $\delta_f(B_\Lambda) \leq \gamma_f(\Lambda)$. We also define $\tilde{\gamma}_f(\Lambda; N)$ and $\tilde{\gamma}_f(\Lambda)$.

Remark 6.3. We also have the following:

$$\int_{D \in \Lambda} \tilde{\delta}_f(D) d\mu(D) = \tilde{\delta}_f(B_\Lambda).$$

$$\tilde{\delta}_f(D) \geq \tilde{\delta}_f(B_\Lambda)$$

for all $D \in \Lambda$.

$$\tilde{\delta}_f(D) = \tilde{\delta}_f(B_\Lambda)$$

for almost all $D \in \Lambda$.

By Theorem 5.3, we get the following defect relation:

Theorem 6.4. *Let* Λ, f *and* $D_1, \cdots, D_q$ *be as in Theorem* 5.3. *Then*

$$\sum_{j=1}^{q} (\tilde{\delta}_f(D_j) - \tilde{\delta}_f(B_\Lambda)) \leq (1 - \tilde{\gamma}_f(\Lambda))(2N - \dim \Lambda + 1).$$

§7. The existence of holomorphic curves with deficiencies

We can show the existence of holomorphic curves with

$$0 < \delta_f(B_\Lambda) < 1$$

in the case where $M = \mathbb{P}_n(\mathbb{C})$ and $L = L(H)^{\otimes d}$.

Proposition 7.1. *Let $0 < e_0 \leq 1$. Then there exist an algebraically nondegenerate transcendental holomorphic curve $f : \mathbb{C} \to \mathbb{P}_n(\mathbb{C})$ and a linear system Λ included in $|L(H)^{\otimes d}|$ such that $\delta_f(B_\Lambda) = e_0$.*

Let $\Lambda \subseteq |L(H)^{\otimes d}|$ be a linear system with the non-empty base locus. We will show the existence of holomorphic curves with $0 < \delta_f(B_\Lambda) < 1$. We will give a proof by constructing a holomorphic curve by using exponential cueves. We recall some known facts on exponential curves. Let $f : \mathbb{C} \to \mathbb{P}_n(\mathbb{C})$ be a nonconstant holomorphic curve defined by

$$f(z) = (\exp a_0 z, \cdots, \exp a_n z),$$

where $a_0, \cdots, a_n \in \mathbb{C}$.

We denote by $\mathcal{C}_f$ the circumference of the convex polygon spanned by the set $\{a_0, \cdots, a_n\}$. Let H be a hyperplane in $\mathbb{P}_n(\mathbb{C})$ defined by

$$H: \; L(z) = \sum_{j=0}^{n} \alpha_j \zeta_j = 0 \quad (\alpha_0, \cdots, \alpha_n \in \mathbb{C}),$$

where $\zeta = (\zeta_0, \cdots, \zeta_n)$ is a homogeneous coordinate system in $\mathbb{P}_n(\mathbb{C})$. We define the set J_H of index by $J_H = \{j : \; \alpha_j \neq 0\}$. Let $\mathcal{C}_{f,H}$ be the circumference of the convex polygon spanned by the set $\{a_j : \; j \in J_H\}$. According to H. and J. Weyl,

$$\boxed{T_f(r) = \frac{\mathcal{C}_f}{2\pi} r + O(1).}$$

Then we have the following lemma:

Lemma 7.2. *Let* f *and* H *be as in the above. Then the deficiency of* f *for* H *is given by*

$$\delta_f(H) = 1 - \frac{\mathcal{C}_{f,H}}{\mathcal{C}_f}.$$

Furthermore, the constant $\mathcal{C}_{f,H}$ *is depending only on* f *and on* J_H.

We first consider the case where $d = 1$. By making use of Lemma 7.2, we have the following:

Theorem 7.3. *Let* $\Lambda \subseteq |L|$ *and* e_0 *an arbitrary positive number less than one. Suppose* $\mathrm{Bs}\,\Lambda \neq \emptyset$*. Then there exists a holomorphic curve* $f : \mathbb{C} \to \mathbb{P}_n(\mathbb{C})$ *such that* $e_0 = \delta_f(B_\Lambda)$*.*

Example 7.4. Let $(\zeta_0, \zeta_1, \zeta_2)$ be a homogeneous coordinate system in $\mathbb{P}_2(\mathbb{C})$ and W a subspace of $\Gamma(\mathbb{P}_2(\mathbb{C}), L(H))$ generated by ζ_1 and ζ_2. Then $\mathrm{Bs}\,\Lambda = \{(1,0,0)\}$. We define an algebraically nondegenerate holomorphic curve $f : \mathbb{C} \to \mathbb{P}_2(\mathbb{C})$ by

$$f(z) = (1, e^z, e^{cz}),$$

where c is a positive number greater than one. In this case, we have

$$\phi_{\mathcal{I}_0} = \frac{1}{2} \log \left(\frac{|\zeta_0|^2 + |\zeta_1|^2 + |\zeta_2|^2}{|\zeta_1|^2 + |\zeta_2|^2} \right).$$

Then, a direct calculation gives us the following:

$$T_f(r) = \frac{c}{\pi} r + O(1) \qquad \text{and} \qquad m_f(r, \mathcal{I}_0) = \frac{1}{\pi} r + O(1).$$

Hence we have $\delta_f(B(\Lambda)) = 1/c$. We notice that f does not hit $\mathrm{Bs}\,\Lambda$.

Theorem 7.5. *There exist a linearly nondegenerate transcendental holomorphic curve $f : \mathbb{C} \to \mathbb{P}_n(\mathbb{C})$ of finite type and finitely many linear systems $\{\Lambda_j\}$ included in $|L(H)|$ that have the following properties: The set of values of δ_f is a finite set (say $\{e_j\}$) with $e_j = \delta_f(B_\Lambda)$. Furthemore,*

$$\delta_f(D) = e_j \quad \text{for all} \quad D \in \Lambda_j \setminus \bigcup_k \Lambda_{j_k},$$

where $\{\Lambda_{j_k}\} \subset \{\Lambda_j\}$ and $0 < e_j < 1$ for at least one j. For a sufficiently small positive number δ, there exist $\Lambda_{j_1}, \cdots, \Lambda_{j_t}$ in $\{\Lambda_j\}$ such that

$$\{H \in |L(H)| : \ \delta_f(H) \geq \delta\} = \bigcup_{k=1}^{t} \Lambda_{j_k}.$$

Next we consider the case where $d \geq 2$. In this case, by using Veronese map and Lemma 7.2, we have the following theorem:

Theorem 7.6. *Let* $\Lambda \subseteq |L(H)^{\otimes d}|$. *Suppose that* $\mathrm{Bs}\,\Lambda \neq \emptyset$. *Then there exists a holomorphic curve* $f : \mathbb{C} \to \mathbb{P}_n(\mathbb{C})$, *nondegenerate with respect to* Λ, *such that*

$$0 < \delta_f(B_\Lambda) < 1.$$