Line bundles with connections on projective varieties over function fields and number fields

Klaus Künnemann (Regensburg)
Fields Institute, Toronto, October 23rd 2008

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Define $\mathcal{O}_X$-module of **principal parts** or 1**-jets of** E as

$$P^1_{X/k}(E) = q_{1*}q_2^*E = E \oplus (\Omega^1_{X/k} \otimes E)$$

with $\mathcal{O}_X$-module structure $\lambda \cdot [e, \omega] = [\lambda \cdot e, \lambda \cdot \omega - d\lambda \otimes e]$.

We obtain the **Atiyah extension**

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$$\left\{ \begin{matrix} \text{connections} \\ \nabla \colon\ E \to \Omega^1_{X/k} \otimes E \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \mathcal{O}_X\text{-linear splittings} \\ s \colon\ E \to P^1_{X/k}(E) \end{matrix} \right\}$$

mapping ∇ to splitting $s_\nabla : e \mapsto [e, -\nabla(e)]$.

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Hence the **Atiyah class**

$$at_{X/k}(E) \in \mathrm{Ext}^1_{\mathcal{O}_X}(E, \Omega^1_{X/k} \otimes E) = H^1(X, \Omega^1_{X/k} \otimes \mathrm{End}(E))$$

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If $k = \mathbb{C}$ we may apply GAGA to holomorphic connections on $E_\mathbb{C}$ over $X(\mathbb{C})$.

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$$c_1(L) := at_{X/k}(L) \in H^1(X, \Omega^1_{X/k}) = \text{'Hodge cohomology'}$$

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Observe: ∇^u_L is holomorphic and algebraizes by GAGA.

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If the monodromy of ∇ is unitary (i.e. if $\nabla_{\mathbb{C}} = \nabla_L^u$) there exists $n > 0$ such that

$$(L, \nabla)^{\otimes n} \cong (\mathcal{O}_X, d).$$

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- **unitary** (i.e. $\subseteq U(1)$) if $\lambda_1, \ldots, \lambda_n \in \mathbb{R} \cap \overline{\mathbb{Q}}$,
- but **infinite** if there is one $\lambda_j \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$.

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Let (R, Σ, F_∞) be an arithmetic ring, for example

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Group structure from 'Baer sum' or homological algebra.

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Corollary to Theorem (A):

$$\ker\hat{(}c_1^H)/\mathrm{im}\ (\pi^*)$$

is a finite group in situation a) and b).

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Theorem (A) and rational points of the universal vector extension of $\mathrm{Pic}^0_{X/k}$:

In the situation of Theorem (A) the **Picard variety** $B := \mathrm{Pic}^0_{X/k}$ classifies line bundles L on X algebraically equivalent to zero. The **universal vector extension** $B^\# = \mathrm{Pic}^{0\#}_{X/k}$ of B classifies line bundles L as above equipped with an integrable connection ∇.

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After base change to $\mathbb{C}$ this becomes ($X = X(\mathbb{C})$)

$$0 \to \Gamma(X, \Omega_{X/\mathbb{C}}) \to \frac{H^1(X,\mathbb{C})}{H^1(X, 2\pi i\mathbb{Z})} \to \frac{H^1(X,\mathcal{O}_X)}{H^1(X, 2\pi i\mathbb{Z})} \to 0$$

(use the exponential sequence).

The maximal compact subgroup

$$K := \frac{H^1(X, 2\pi i\mathbb{R})}{H^1(X, 2\pi i\mathbb{Z})} \subseteq B^{\#}(\mathbb{C}) = \frac{H^1(X, \mathbb{C})}{H^1(X, 2\pi i\mathbb{Z})} \xrightarrow{\exp} H^1(X, \mathbb{C}^*)$$

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Theorem (A') was conjectured by D. Bertrand and proved if B is defined over $\overline{\mathbb{Q}} \cap \mathbb{R}$ and admits 'real multiplication'.

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Assume $\nabla_{\mathbb{C}} = \nabla^u_L$ and fix a rigidification $\varphi : \overline{\mathbb{Q}} \xrightarrow{\sim} L_e$.

$\mathbb{G}_m$-torsor $L^\times$ associated with L defines extension of $\overline{\mathbb{Q}}$-algebraic groups (use Theorem of the square)

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Theorem (Schneider-Lang): Let G be a commutative algebraic group over $\overline{\mathbb{Q}}$ with exponential map

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Nowadays consequence of more general theorems by Bombieri and Wüstholz.

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then there exists $n > 0$ such that

$$(L, \nabla)^{\otimes n} \cong (\mathcal{O}_X, d).$$

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By construction

$$\pi^*(L, \nabla) \cong (\mathcal{O}_X, d)$$

and we can choose $n := \deg(\pi)$. □

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$$\rho_{\nabla_M}(\Delta(\gamma))$$

Proof of Theorem (A), conclusion via 'Weil restriction':
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Γ_A contains $\mathbb{R}$-basis of $\mathrm{Lie}\, A_{\mathbb{C}}$

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5 A remark on Theorem (B)

We say that abelian variety $\mathrm{Pic}^0_{X_K/K}$ has **no fixed part** if

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Consequence of the Hodge index theorem: For any $\alpha \in \mathbb{Q} \cdot c_1(\mathrm{Pic}\, X) \subseteq H^{1,1}(X)$, we have

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For any effective divisor E on C

$$F := \pi^*\mu_C = \frac{1}{\deg E} c_1(\pi^*\mathcal{O}(E)) \in H^{1,1}(X).$$

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Hence $\alpha = 0$ by Hodge index theorem and $\beta = \frac{p}{q} \cdot F$. □

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- negative (?): theory of conformal blocks.

Thank you for your attention!