

S-unit equations in number fields:  
effective results, generalizations,  
applications, abc conjecture

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I. S-unit equations

$K$  algebraic number field,  $M_K$  set of places

$S$  finite subset of  $M_K$ ,  $S \supseteq S_\infty$

$\mathfrak{p}_1, \dots, \mathfrak{p}_t$  prim ideals associated to  $S \setminus S_\infty$

$\alpha \in K^*$ :

$S$ -integer:  $\text{ord}_{\mathfrak{p}}(\alpha) \geq 0$  for  $\mathfrak{p} \notin \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$

$S$ -unit:  $\text{ord}_{\mathfrak{p}}(\alpha) = 0$  - " -

$\mathcal{O}_S$  ring of  $S$ -integers,  $\mathcal{O}_S^*$  group of S-units  
for  $S = S_\infty$  ( $t=0$ ),  $\mathcal{O}_S = \mathcal{O}_K$ ,  $\mathcal{O}_S^* = \mathcal{O}_K^*$

many diophantine problems  $\Rightarrow$

S-unit equation

$$(1.1) \quad \alpha x + \beta y = 1 \quad \text{in } x, y \in \mathcal{O}_S^*$$

$$\alpha, \beta \in K^*; \quad \alpha = \beta = 1 \Rightarrow abc$$

(1.1) exponential diophantine equ.

$$x = \zeta \varepsilon_1^{a_1} \dots \varepsilon_{s-1}^{a_{s-1}}, \quad y = \delta \varepsilon_1^{b_1} \dots \varepsilon_{s-1}^{b_{s-1}}$$

$\begin{array}{c} | \\ |s| \end{array}$

$\{\varepsilon_1, \dots, \varepsilon_{s-1}\}$  fundamental system of  $S$ -units

unknowns:  $\zeta, \delta$  roots of unity,  $a_i, b_i \in \mathbb{Z}$

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## ineffective finiteness results

results proved in an implicit way

Siegel, Mahler, Parry

Lang (1960)

several bounds for the number of solutions

generalizations

- several unknowns
- systems of polynomial equations
- subgroups of finite rank of  $\mathbb{C}^n$
- ⋮

analogues in function fields

great number of applications

## effective finiteness theorems

- a) theory of logarithmic forms  
results proved in an implicit way  
 (for special equations):

Baker (1968), Coates (1969)

bounds for the solutions (general case):

Gy (1974, '79), Kotov-Trelina,

Sprindzuk, Schmidt (1992),

Bugeaud-Gy (1996), Haristoy,

Gy-K.Yu (2006)

- b) alternative effective method,  
 extension of the Thue-Siegel method,  
Dyson lemma, some geometry of  
 numbers

Bombieri (1993), Bombieri-Cohen

(1997, 2003)

- c) combination of b) and a),

Bugeaud (1998)

## applications

- Thue eqs, decomposable form eqs
- superelliptic eqs
- recurrence sequences
- polynomials, binary forms with given discriminant, power integral bases
- irreducible polynomials
- 
-

# bounds for the solutions

$$(1.1) \quad \alpha x + \beta y = 1 \quad \text{in } x, y \in \mathcal{O}_S^*$$

$$n = [K : \mathbb{Q}], \quad s = |S|, \quad p_1, \dots, p_t$$

$$P = \max_i N(p_i), \quad t \leq 2n P / \log P$$

S-regulator

$$R_S = i_S R \prod_i \log N(p_i), \quad i_S | h$$

$h(y)$  absolute height,  $y \in \bar{\mathbb{Q}}$

$h(y)$  - " - log - " - ,  $\log H(y)$

$$H = \max(1, h(\alpha), h(\beta))$$

$$\log^* a = \max(\log a, 1)$$

Bugeaud - Gy (1996)

$$(1.2) \quad \max(h(x), h(y)) \leq C_1 P R_S (\log^* R_S)^2 H$$

$$C_1 = (c_1 n s)^{c_2 s}, \quad \underbrace{c_1, c_2}_{\text{explicit}} > 0 \text{ absolute constants}$$

Bombieri, Bombieri - Cohen, Bugeaud (1998)

$$(1.3) \quad \max(h(x), h(y)) \leq C_2 P (\log^* P) R_S \times$$

$$\times \max(C_3 P (\log^* P) R_S, H)$$

$C_2, C_3$  of the same form as  $C_1$

(1.2), (1.3) best possible in terms of  $H$

in terms of  $S$ ,  $s^s$  dominating factor

in (1.2), (1.3) whenever  $t > \log P$

Thm 1 (Gy-Yu, 2006). All solutions of (1.1) satisfy

$$(1.4) \max(h(x), h(y)) \leq C_4 P R_S (\log^* R_S) H$$

$C_4$  explicit, of the same form as  $C_1, C_2, C_3$

Thm 2. Every solution of (1.1) satisfies

$$(1.5) \max(h(x), h(y)) \leq C_5 (P/\log^* P) R_S H$$

$$C_5 = c_3^S, \quad c_3 = c_3(n, h, R) \text{ explicit}$$

Remarks:

- (1.5) modified, more precise version of a thm of Yu and Gy (2006); first bounds without  $S^S$   $\Rightarrow$  important applications
- $\alpha = \beta = 1, H = 1$ , (1.5)  $\Rightarrow$  results in the direction of the abc conjecture
- $C_4$  much smaller than  $C_1, C_2, C_3$

Proofs:

- improved estimates for  $S$ -units
- recent thm of Loher-Masser on mult. independent alg. numbers
- refinements of arguments of Gy (1979) for (1.5) and Bugeaud-Gy (1996) for (1.4)
- recent estimates of Matveev (2000) and Yu (2007) on logarithmic forms

$S^S$  in (1.4): Minkowski on successive minima

$$\underline{K = \mathbb{Q}}, (1.1) \Rightarrow$$

$A, B, C, a, b, c$  non-zero integers

$$(A, B, C) = 1, (a, b, c) = 1$$

$$\max(|A|, |B|, |C|) = H, |abc| > 1$$

$$(1.6) \quad Aa + Bb + Cc = 0$$

$$\underline{P = P(abc)}, \underline{t = w(abc)}$$

explicit version of (1.5)  $\Rightarrow$

Corollary 1 of Thm 2. If (1.6) holds then

$$(1.7) \quad \log \max(|a|, |b|, |c|) <$$

$$2^{10t+22} t^4 (P/\log P) \left( \prod_{p|abc} \log p \right) \log^* H$$

radical of  $(a, b, c)$

$$N = N(a, b, c) = \prod_{p|abc} p$$

$$(1.8) \quad \left\{ \begin{array}{l} P \leq N, \quad \prod_{p|abc} \log p \leq (\log N/t)^t \\ t < 1.5 \log N / \log_2 N^* \\ N^* = \max(N, 16), \log_i \text{ } i\text{th iterate of } \log \end{array} \right.$$

Corollary 1  $\Rightarrow$  upper bound in terms of  $H$  and  $N$  only

Question: best possible upper bound?

## II. abc conjecture in $\mathbb{Z}$

For  $A=B=C=1$ , in 1985:

abc conjecture (Oesterlé, Masser)

For any given  $\varepsilon > 0$

$$(2.1) \quad a + b = c$$

with coprime positive integers  $a, b, c$   
implies

$$c \ll_{\varepsilon} N^{1+\varepsilon}, \quad N = N(a, b, c) \text{ radical}$$

best possible in  $\varepsilon$

refinements, more explicit versions

Baker (1998), Granville (1998)

$$c \ll N (\log N)^t / t!, \quad t = w(abc)$$

implied constant absolute, Baker (2004):  $\frac{6}{5}$

very extensive literature, extraordinary

consequences (asymptotic Fermat,  
generalized Fermat eqn, Pillai eqn, ...)

abc: unifies and motivates a number  
of results and problems in  
number theory

abc conjecture home page

created and maintained by A. Nitaj

<http://www.math.unicaen.fr/~nitaj/abc.html>

The abc conjecture seems out of reach since 1986, several upper bounds in terms of  $N = N(a, b, c)$  for

(2.1)  $a + b = c$ ,  $a, b, c > 0$  coprime

theory of logarithmic forms:

Stewart - Tijdeman (1986), Stewart - Yu (1991, 2001), Chi (2005)

Stewart - Yu (2001):

$$p' = \min(P(a), P(b), P(c))$$

(2.1)  $\Rightarrow$

(2.2)  $\log c < p' N^{c_4 \log_3 N^* / \log_2 N}$

and

(2.3)  $\log c < c_5 N^{1/3} (\log N)^3$

$c_4, c_5 > 0$  effective absolute constants

Chi (2005):  $c_4 = 710$

Corollary 1 with  $A=B=C=1$  and

$$(1.8) \left\{ \begin{array}{l} P \leq N, \prod_{p|abc} \log p \leq (\log N / t)^t \\ t < 1.5 \log N / \log_2 N \end{array} \right.$$

→

Corollary 2 of Thm 2. If  $a+b=c$  with coprime positive integers  $a, b, c$  then

$$(2.4) \log c < 2^{10t+22} t^4 (P/\log P) \left( \prod_{p|abc} \log p \right)$$

$$(2.5) \text{ -- " -- } < (2^{10t+22} / t^{t-4}) N (\log N)^t$$

and

$$(2.6) \text{ -- " -- } < 2^{23} \underbrace{(P/\log P)}_{< N} N^{\underbrace{653 \log_3 N^* / \log_2 N^*}_{< \epsilon \text{ if } N \text{ large}}}$$

(2.6): in the direction of Oesterle, Masser

(2.5) -- " -- of Baker, Granville

Comparison of Cor. 2 and Stewart-Yu (2000)

a) (2.2), (2.3) slightly better than (2.6)

reason:

direct proof for (2.2), (2.3), use of specific properties of  $\mathbb{Z}$  (e.g.  $a+b=c$ ,  $b > a \Rightarrow 2b > c > b$ )

(2.4) - (2.6) special cases of more general results

b) in general (2.4) better than (2.2),  $C_4 = 710$

ex. (de Weger)  $11^2 + 3^2 \cdot 5^6 \cdot 7^3 = 2^{21} \cdot 2^3$ ,  $\log c > 2^4$

bound in (2.2):  $> 2^{4950}$

bound in (2.4):  $< 2^{100}$

### III. abc conjecture in number fields

$K$  number field,  $n = [K:\mathbb{Q}]$ ,  $M_K$  set of places

$v \in M_K$ ,  $| \cdot |_v$  normalized :  $\alpha \in K^*$

$v$  infinite

$$|\alpha|_v = |\sigma(\alpha)|^{n_v}, \quad n_v = \begin{cases} 1, & \sigma(K) \subseteq \mathbb{R} \\ 2 & \text{otherwise} \end{cases}$$

$v$  finite

$$\mathfrak{p} \text{ prime ideal}, \quad |\alpha|_v = N_{K/\mathbb{Q}}(\mathfrak{p})^{-\text{ord}_{\mathfrak{p}}(\alpha)}$$

height of  $(a, b, c) \in (K^*)^3$

$$H_K(a, b, c) = \prod_{v \in M_K} \max(|a|_v, |b|_v, |c|_v)$$

radical of  $(a, b, c)$

$$N_K(a, b, c) = \prod_v N_{K/\mathbb{Q}}(\mathfrak{p})^{\text{ord}_{\mathfrak{p}}(\mathfrak{p})}$$

$\mathfrak{p}$  prime  $\mathfrak{p} | p$ ,  $\prod$  over all finite  $v$

s.t.  $|a|_v, |b|_v, |c|_v$  not all equal

$\Delta_K$  absolute value of the discriminant of  $K$

Voita ('87), Elkies ('91), Broberg ('99),  
Granville-Stark (2000), Browkin (2000),  
Masser (2002)

abc conjecture in  $K$  : For every  $\varepsilon > 0$   
there exists  $C_\varepsilon$ , depending only on  $\varepsilon$ ,  
s.t.

$$H_K(a, b, c) < C_\varepsilon^n (\Delta_K N_K(a, b, c))^{1+\varepsilon}$$

for all  $a, b, c \in K^*$  with  $a+b+c=0$

best possible in  $E$

uniform: good behaviour under field extension

$K = \mathbb{Q}$ : Oesterle, Masser conjecture

very rich literature, profound implication

- Faltings' thm on rational points (Elkies '91)
- effective version of abc  $\Leftrightarrow$  effective Siegel's thm (Surroca, 2004) with
- no Siegel's zeros for L-functions of char. with negative discriminant (Granville-Stark, 2000)

uniform bound for the sols

weaker but unconditional and effective bounds for  $H_K(a, b, c)$

(3.1)  $a + b + c = 0, a, b, c \in K^*$

$S = S_\infty \cup \{ \text{finite } v \in M_K \text{ s.t. } |a|_v, |b|_v, |c|_v \text{ not all equal} \}$

$\Rightarrow x = -a/c, y = -b/c$  solution of

(3.2)  $x + y = 1$  in  $O_S^*$

bound for  $h(x), h(y) \Rightarrow$  bound for  $H_K(a, b, c)$

Surroca (2007): bound of Bugeaud-Gy (196)  
for the solutions of (3.2)  $\Rightarrow$

(3.3)  $\log H_K(a, b, c) < ((C_6 \sim \Delta_K)^{C_7} N_K(a, b, c)^{C_8})^n$ ,

where  $C_6 - C_8 > 0$  eff. absolute constants

Considerable improvement of (3.3)

Theorem 3. Let  $\varepsilon > 0$ . Then

$$(3.1) \quad a + b + c = 0, \quad a, b, c \in K^*$$

implies

$$(3.4) \quad \log H_K(a, b, c) < \underbrace{C_9(n, \Delta_K, \varepsilon)}_{\text{eff.}} N^{1+\varepsilon},$$

where  $N = N_K(a, b, c)$ . Further, if

$$N > \max \{ \exp \exp(\max(\Delta_K, \varepsilon)), \Delta_K^{2/\varepsilon} \}$$

then

$$(3.5) \quad \log H_K(a, b, c) < \underbrace{C_{10}(n, \varepsilon)}_{\text{eff.}} (\Delta_K N)^{1+\varepsilon}$$

main steps in the proof:

$$P := \max N_{K/\mathbb{Q}}(\rho) \text{ in } N_K(a, b, c) = \prod_{\rho} N_{K/\mathbb{Q}}(\rho)^{\text{ord}_{\rho}(P)}$$

for appropriate choice of  $S$ , Theorem 2

$\Rightarrow$

$$(3.6) \quad \log H_K(a, b, c) < \underbrace{C_3(n, h, R)}_{\text{explicit}}^s P R_S;$$

estimating  $h, R, s, P$  and  $R_S$  in terms  
of  $\Delta_K$  and  $N$

$$(3.6) \Rightarrow (3.5) \Rightarrow (3.4)$$

## IV. Common generalization of binomial Thue equations and S-unit equations (over $\mathbb{Q}$ ) (joint work with A. Pinter)

### Binomial Thue equations

$$a, b, c \in \mathbb{Z} \setminus \{0\}, n \geq 3$$

$$(4.1) \quad ax^n - by^n = c, \text{ unknowns : } x, y \in \mathbb{Z}$$

Thue (1909) general thm  $\Rightarrow$  finitely many sols

$\mathcal{P}$ : set of integers composed of primes  $p_1, \dots, p_s$

Mahler (1933):  $c \in \mathcal{P}, (x, y) = 1$

Baker (1968):  $\max(|x|, |y|) < C_1^{\text{eff}}(a, b, c, n)$

Coates (1969):  $c \in \mathcal{P}, (x, y) = 1$

several improvements and generalizations

$n$  also unknown

Tijdeman (1976):  $n \leq C_2^{\text{eff}}(a, b, c) \Rightarrow |x|, |y| \leq C_3^{\text{eff}}(a, b, c)$

van der Poorten (1977):  $c \in \mathcal{P}, (x, y) = 1$

Bugeaud - Gy (2004):  $c \in \mathcal{P}$ , explicit bound for  $n$

### Common generalization of (4.1) and S-unit eqns

$$(4.2) \quad ax^n - by^n = c, \text{ unknowns; } x, y, a, b, c, n \in \mathbb{Z}$$

with  $|xy| \geq 1, a, b, c \in \mathcal{P}, n \geq 3$ , and

$$(4.3) \quad (ax, by, c) = 1, a, b, c \text{ } n\text{th powerfree}$$

Theorem 4 (Gy - Pinter, 2008). All sols of (4.2) with (4.3) satisfy

$$\max(|ax^n|, |by^n|, |c|) \leq C_4^{\text{eff}}(\mathbb{Q}) \leq C_5^{\text{eff}}(\mathcal{P})$$

$$\mathbb{Q} = p_1 \dots p_s, \quad P = \max p_i$$

if  $|x y| \geq 1 \Rightarrow$   
 $n \leq c_6^{\text{eff}} Q^3$ ,  $c_6 > 0$  absolute constant

Thm 4: proved in the number field case  
in  $C_4, C_5$  dependence on  $Q, P$  necessary  
-  $a, b$  fixed  $\Rightarrow$  thms of Tijdeman and van der Poorten  
-  $|x y| = 1 \Rightarrow$  effective results for  $S$ -unit equations

method:

$N_{\mathcal{O}}(a)$ :  $\mathcal{O}$ -free part of  $a \in \mathbb{Z} \setminus \{0\}$  in absolute value

$\max(N_{\mathcal{O}}(x), N_{\mathcal{O}}(y))$  large  
theory of logarithmic forms, complex and  $p$ -adic

$\max(N_{\mathcal{O}}(x), N_{\mathcal{O}}(y))$  not large  
 $\Rightarrow S'$ -unit eqn with appropriate  $S' \supseteq S$

abc conjecture  $\Rightarrow n < \underbrace{c}_{\text{absolute}} \log Q$

equation (4.2): 3-parameter family of  $S$ -unit equations

(4.4)  $t^n a - w^n b = c$  in  $a, b, c \in \mathcal{O}$   
with parameters  $t, w$  and  $n$

Corollary to Theorem 4. There is an effective constant  $C_2 = C_2(P) > 0$  s.t. equation (4.4) has no solution for integer parameters  $t, w, n$  with  $n \geq 3$ ,  $(tw, p_1 \dots p_s) = 1$ ,  $|tw| > 1$  and  $\max(|t|, |w|, n) > C$ .

other results on parametric families

1) as a consequence of more general results

Corvaja and Zannier (2006) } finiteness  
Levin (2006)

results on one-parameter families of S-unit equations of the form

$$(4.5) \quad f(t)a + g(t)b = h(t)c \quad \text{int } t \in \mathbb{Q}, a, b, c \in \mathcal{O}$$

with  $f(t)g(t)h(t) \neq 0$

$f, g, h$  certain polynomials in  $\mathbb{Q}[t]$ ,

e.g.  $\deg f + \deg g = \deg h > 2$  or  $f, g, h$

linear

2) each S-unit <sup>(equation)</sup> over  $\mathbb{Q}$  :

$$(4.6) \quad \left\{ \begin{array}{l} Aa + Bb + Cc = 0, \text{ unknowns: } a, b, c \in \mathcal{O} \\ \text{coefficients: } A, B, C \text{ relatively prime} \\ \text{integers with } (ABC, p_1 \dots p_s) = 1 \end{array} \right.$$

Evertse, Gy, Stewart, Tijdeman (1988):

(i) there are only finitely many equations of the form (4.6) with more than 2 non-proportional solutions  $a, b, c$

(ii) there are infinitely many eqns of the form (4.6) with exactly 2 non-proportional solutions  $a, b, c$

1) and 2)  $\left\{ \begin{array}{l} \text{ineffective} \\ \text{over number fields} \end{array} \right.$

above Corollary (effective), Corvaja and Zannier, Levin:

(iii) there are infinitely many eqns of the form (4.6) having no solution

V. Polynomial equations in two unknowns  
from a multiplicative division group  
(joint work with A. Bérczes, J.H. Evertse  
and C. Poonen)

$P(x, y) \in \bar{\mathbb{Q}}[x, y]$  absolute irreducible

$\Gamma$  finitely generated subgroup of  $(\bar{\mathbb{Q}}^*)^2$

(5.1)  $P(x, y) = 0$  in  $(x, y) \in \Gamma$

assume:  $P$  has  $\geq 3$  terms (otherwise (5.1)  
may have infinitely many trivial solutions)

another interpretation

curve  $\mathcal{C} : P(x, y) = 0$  in  $(\bar{\mathbb{Q}}^*)^2$

(5.2)  $\mathcal{C} \cap \Gamma$

solutions (points) from larger sets:

division group of  $\Gamma$

$\bar{\Gamma} := \{ \underline{x} \in (\bar{\mathbb{Q}}^*)^2 \mid \exists k \in \mathbb{Z}_{>0} \text{ with } \underline{x}^k \in \Gamma \}$

$(\zeta, \zeta') \in \bar{\Gamma}$  for any roots of unity  $\zeta, \zeta'$

for  $\varepsilon > 0$

$\bar{\Gamma}_\varepsilon := \{ \underline{x} \in (\bar{\mathbb{Q}}^*)^2 \mid \exists \underline{y}, \underline{z} \text{ with } \underline{x} = \underline{y} \cdot \underline{z}, \underline{y} \in \bar{\Gamma},$

$\underline{z} \in (\bar{\mathbb{Q}}^*)^2, h(\underline{z}) = h(z_1) + h(z_2) < \varepsilon \}$

$\underline{z} = (z_1, z_2)$

"cylinder" around  $\bar{\Gamma}$

$C(\bar{\Gamma}, \varepsilon) := \{ \underline{x} \in (\bar{\mathbb{Q}}^*)^2 \mid \exists \underline{y}, \underline{z} \text{ with } \underline{x} = \underline{y} \cdot \underline{z},$

$\underline{y} \in \bar{\Gamma}, \underline{z} \in (\bar{\mathbb{Q}}^*)^2, h(\underline{z}) < \varepsilon(1 + h(\underline{y})) \}$

"truncated cone" around  $\bar{\Gamma}$

$\bar{\Gamma}_\varepsilon$ : was introduced by Poonen (1999) } in more  
 $C(\bar{\Gamma}, \varepsilon)$ : - " - Evertse (2002) } general  
 context

$$\Gamma \subset \bar{\Gamma} \subset \bar{\Gamma}_\varepsilon \subseteq C(\bar{\Gamma}, \varepsilon)$$

not groups

points  $\underline{x} \in \bar{\Gamma}_\varepsilon$  are "close" to  $\bar{\Gamma}$  if  $\varepsilon > 0$  small  
 in general the coordinates of  $\underline{x}$  in  $\bar{\Gamma}, \bar{\Gamma}_\varepsilon$   
 or  $C(\bar{\Gamma}, \varepsilon)$  are not contained in a  
prescribed number field

### ineffective results

Liardet (1974):  $\mathcal{C} \cap \bar{\Gamma}$  finite

Poonen (1999):  $\mathcal{C} \cap \bar{\Gamma}_\varepsilon$  - " - for small  $\varepsilon > 0$

Evertse (2002): bound for  $\#(\mathcal{C} \cap \bar{\Gamma}_\varepsilon)$ ,

$\mathcal{C} \cap C(\bar{\Gamma}, \varepsilon)$  finite for small  $\varepsilon > 0$

Rémond (2002): bound for  $\#(\mathcal{C} \cap C(\bar{\Gamma}, \varepsilon))$

Pontreau (200?): improved bound for - " -

} in  
 more  
 genera  
 form

### multivariate (higher dimensional)

generalizations, description of the  
 (infinite) set of solutions (points)

Laurent, Gy, Bombieri, Masser,  
 Zannier, Faltings, Vojta, McQuillan,  
 Zhang, Szpiro, Ullmo, Poonen,  
 David, Philippon, Chambert-Loir,  
 Evertse, Schlickewei, Schmidt,  
 Rémond, ...

effective results

(5.1)  $P(x, y) = 0$  in  $(x, y) \in \Gamma, \bar{\Gamma}, \bar{\Gamma}_\varepsilon$  resp.  $C(\bar{\Gamma}, \varepsilon)$

$P(x, y) \in \bar{\mathbb{Q}}[x, y]$  absolute irreducible, not  
of the form

$$\alpha x^m + \beta y^n \text{ or } \gamma x^m y^n + \delta$$

$\Rightarrow \mathcal{C}$  not a translate of a proper  
algebraic subgroup of  $(\bar{\mathbb{Q}}^*)^2$

(5.2)  $\mathcal{C} \cap \Gamma, \mathcal{C} \cap \bar{\Gamma}, \mathcal{C} \cap \bar{\Gamma}_\varepsilon, \mathcal{C} \cap C(\bar{\Gamma}, \varepsilon)$

- $P(x, y)$  linear,  $\Gamma = (\mathcal{O}_S^*)^2$  in a number field (S-unit eqns): several explicit results for  $\mathcal{C} \cap \Gamma$
- Bombieri (1983), Bombieri - Cohen (1987, 2003)  
 $P(x, y)$  linear,  $\mathcal{C} \cap \Gamma$  finite + effective
- Bombieri and Gubler (2006): general case,  $\mathcal{C} \cap \Gamma$  finite + effective

new results

- Bérczes, Evertse, Gy (200?):  $P(x, y)$  linear,  
each set in (5.2) finite (for small  $\varepsilon > 0$ )  
+ explicit bounds for the heights and degrees  
generalization (with weaker bounds)
- Bérczes, Evertse, Gy, Pontreau (200?):  
general case, each set in (5.2) finite  
(for small  $\varepsilon > 0$ ) + explicit bounds for  
the heights and degrees

⇒ for these special class<sup>es</sup> of varieties, effective versions of some (very general but ineffective) thms of Laurent, Poonen, Evertse and Reimond, respectively

Quantitative versions, bounds for the solutions of eqn. (5.1)

$\Gamma$  finitely generated subgroup of  $(\mathbb{Q}^x)^2$  of rank  $r$ ,  $\{\underline{w}_1, \dots, \underline{w}_r\}$  basis of  $\Gamma/\Gamma_{\text{tors}}$   
 $h_0 = \max\{1, h(\underline{w}_1), \dots, h(\underline{w}_r)\}$

$K$  number field s.t.  $\Gamma \subset (K^x)^2$ ,  $n = [K:\mathbb{Q}]$

$S$  finite subset of  $M_K$  with  $S \supseteq S_\infty$  s.t.

$$\Gamma \subset (\mathcal{O}_S^x)^2, \quad s = |S|$$

$N = \max\{2, \max_{\mathfrak{p}_v \in S \setminus S_\infty} N(\mathfrak{p}_v)\}$ ,  $\mathfrak{p}_v$  prime ideal

$\delta = \deg P$ ,  $H = \max\{1, \underbrace{h(P)}\}$

$$\sum_{v \in M_K} \log \max_i |a_i|_v \quad \text{coeff. of } P$$

$$A = (e^{13} \delta^7 n^3)^{r+3} s \frac{N^{2\delta^2}}{\log N} \log(\max(\delta n s N, \delta h_0)) h_0^r$$

$L$ : extension of  $K$  generated by coeffs of  $P$

Thm 5 (BEGP, 2003). Let

$$\varepsilon = (2^{48} \delta (\log \delta)^5)^{-1}$$

Then for every  $\underline{x} \in \mathcal{C} \cap \bar{\Gamma}_\varepsilon$ ,

$$(5.3) \quad h(\underline{x}) \leq r h_0 \delta A + AH, \quad [L(\underline{x}):L] \leq 2^{50} \delta (\log \delta)^6$$

generated by  $x = (x_1, x_2)$

good dependence on  $H, h_0, n, s, N$   
bounds independent of  $L$

Corollary (BEGyP, 200?). For every  
 $\underline{x} \in \mathcal{C} \cap \bar{\Gamma}$ , (5.3) holds.

generalization in quantitative form  
of thm of Bombieri and Gubler (2006)

with a smaller  $\varepsilon$ , almost the same  
holds for  $\mathcal{C} \cap C(\bar{\Gamma}, \varepsilon)$

Thm 6 (BEGyP, 200?). Let

$$\varepsilon = (2^{50} \delta (\log \delta)^5)^{-1} (h_0 \delta A + AH)^{-1}.$$

Then for every  $\underline{x} \in \mathcal{C} \cap C(\bar{\Gamma}, \varepsilon)$ ,

$$h(\underline{x}) \leq 2 + h_0 \delta A + 2AH, [L(\underline{x}):L] \leq 2^{50} \delta (\log \delta)^5$$

For  $P(x, y) = \alpha x + \beta y - 1$  much better  
bounds in an earlier paper of  
Bérczes, Evertse, Gy (200?)

For example, in Thm 5:  $\varepsilon < 0.0225$ ,  
 $[L(\underline{x}):L] \leq 2$ , smaller bound for  $h(\underline{x})$

main tools in the proofs of Thms 5, 6  
bounds for  $h(x)$

based on a recent thm of BEGy (200?)

$|\alpha - \beta|_v >$  explicit lower bound,  
 depending on  $h(\beta)$

$\alpha \in K^*$ ,  $\beta \in \underbrace{G} \subset K^*$ ,  $v \in M_K$   
 finitely generated

used

logarithmic forms estimates  
 geometry of numbers

bounds for  $[L(x):L]$

estimates for the number of points  
 with small height

many applications, especially in the  
 case  $P(x, y) = \alpha x + \beta y - 1$ ,  $\Gamma, \bar{\Gamma}$

using  $\Gamma$  in place of  $O_S^*$  (or  $(O_S^*)^2$ ),  
much better quantitative results

- purely exponential dioph. eqns
- discriminant eqns and related eqns
- decomposable form eqns
- linear recurrences

⋮