

ON THE NOTION OF GEOMETRY OVER \mathbb{F}_1

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- Chevalley groups
- The geometries of J. Tits
- Chevalley groups over finite fields
- Chevalley groups over \mathbb{F}_1 (after Tits)

- Graded gadgets and affine varieties over \mathbb{F}_1
- Chevalley schemes as graded varieties over \mathbb{F}_1
- Zeta-function of a Chevalley scheme over \mathbb{F}_1

The geometries of J. Tits

Reference: J. Tits (1956) “Sur les analogues algébriques des groupes semi-simples complexes”

Motivations: $\begin{cases} \text{geometric axiomatic} \\ \text{geometric interpretation of the algebraic theory of Chevalley groups} \end{cases}$

Simple complex Lie groups: $\begin{cases} \text{classical: } A_n, B_n, C_n, D_n \\ \text{exceptional: } G_2, F_4, E_6, E_7, E_8 \end{cases}$

Over a field K : Jordan(1870), Dickson(1901), Dieudonné(1948)
but partial and ‘ad hoc’ constructions

C. Chevalley (1955): algebraic structure theorems for semi-simple complex Lie algebras \mathfrak{g} and Lie groups to transfer simultaneously their definition over any field K

Reference: C. Chevalley (1955) “Sur certains groupes simples”

Main tool: definition of an integral model $\mathfrak{g}_{\mathbb{Z}}$
(integral version of the structure theorem)

\mathfrak{g} complex, (semi)simple Lie algebra, $\boxed{\Phi = \text{roots set}}$

$\mathfrak{h} \subset \mathfrak{g}$ Cartan algebra

$\Phi \ni r : \mathfrak{h} \rightarrow \mathbb{C}; \quad X_r \in \mathfrak{g}$ root elt $([H, X_r] = r(H)X_r, \ H \in \mathfrak{h})$

Theorem (Chevalley) If $r, s, r+s \in \Phi$, the roots elements can be chosen so that

$$[X_r, X_{-r}] = n_r \in \mathfrak{h}; \quad [X_r, X_s] = N_{r,s}X_{r+s}, \quad N_{r,s} \in \mathbb{Z}$$

\mathfrak{g} has an integral basis: $\underbrace{n_{r_1}, n_{r_2}, \dots}_{\text{co-weights } \in \mathfrak{h}} \quad X_{r_1}, X_{r_2}, \dots$

- the structure of $\mathfrak{g}_{\mathbb{Z}}$ as Lie algebra depends only on \mathfrak{g}

$$\mathcal{H} = \langle n_{r_i} \in \mathfrak{h} : r_i \in \Phi \rangle \subset \mathfrak{h}, \quad r(n_r) = 2$$

choose a basis: $n_1, \dots, n_{\ell}; \quad X_1, \dots, X_{\nu} \in \mathfrak{g}_{\mathbb{Z}}$

$$\mathfrak{g}_K := K \otimes_{\mathbb{Z}} \mathfrak{g}; \quad \mathfrak{h}_K = K \otimes \mathcal{H} = \langle n_i^* = 1 \otimes n_i \rangle$$

$$\mathfrak{g}_K = \mathfrak{h}_K \oplus \langle X_j^* = 1 \otimes X_j; j = 1, \dots, \nu \rangle$$

$$\boxed{L = \mathbb{Z} \cdot \Phi \text{ lattice}}, \quad \text{rk } L = \ell = \text{rk } \mathfrak{g}$$

$$\boxed{\text{Hom}(L, K^*) \xrightarrow{\sim} \mathfrak{H} \subset \text{Aut}(\mathfrak{g}_K)} \quad \chi \mapsto h(\chi)$$

$$h(\chi)(n_r^*) = n_r^*; \quad h(\chi)(X_r^*) = \chi(r)X_r^*$$

$$r\in \Phi\qquad \phi_r:\mathsf{SL}(2,K)\rightarrow \mathsf{Aut}(\mathfrak{g}_K)$$

$$\phi_r\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}=h(\chi_r),\quad \chi_r(s)=t^{s(n_r)};\quad \phi_r\begin{pmatrix}0&1\\-1&0\end{pmatrix}=\omega_r$$

$$\phi_r\begin{pmatrix}1&0\\t&1\end{pmatrix}=x_{-r}^*(t),\quad \phi_r\begin{pmatrix}1&t\\0&1\end{pmatrix}=x_r^*(t)$$

$$(x_r(t) = \exp t (\mathrm{ad} {\sf X_r})), \quad t \in \mathbb{C}$$

$$\boxed{\mathfrak{X}_r=\{x_r^*(t):t\in K\}\subset \mathsf{Aut}(\mathfrak{g}_K)}$$

$$G_K:=<\mathfrak{H},\mathfrak{X}_r:r\in\Phi>\quad\textbf{CHEVALLEY GROUP}$$

$$\Phi^o=\{a_1,\ldots,a_\ell\}\subset\Phi\quad \underline{\text{fundamental}}\text{ (simple) roots}$$

$$(\text{Chevalley})\qquad G_K=<\mathfrak{H},\mathfrak{X}_{\pm a}:a\in\Phi^o>$$

$$4 \\$$

Abstract Root System: $(L, \Phi, n_r; r \in \Phi)$

L = lattice (group of weights)

$\Phi \subset L$ finite set (roots); $n_r : L \rightarrow \mathbb{Z}$ (r co-root)

ax1. $L \otimes \mathbb{Q}$ is generated by Φ and $\cap_{r \in \Phi} \text{Ker}(n_r)$

ax2. $n_r(r) = 2, \forall r \in \Phi$

ax3. $r \in \Phi, ar \in \Phi, a \in \mathbb{Q}, \Rightarrow a = \pm 1$

ax4. $r, s \in \Phi, \Rightarrow w_s(r) := r - n_s(r)s \in \Phi$

$w_s : L \xrightarrow{\sim} L, w_s(x) = x - n_s(x)s$ reflection w.r.t s

$$\phi_s \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (r) = \omega_s(r) = w_s(r)$$

Theorem (Chevalley, Grothendieck/Demazure)

$(L, \Phi, n_r) \rightsquigarrow \mathfrak{G} = \mathfrak{G}(L, \Phi, n_r)$ reductive group scheme $_{/\mathbb{Z}}$

$$\mathfrak{G}(K) = G_K$$

- $\mathcal{T} \subset \mathfrak{G}$ maximal split torus, $\mathcal{T}(K) = \mathfrak{H}$
- $\mathcal{N} = \mathcal{N}_{\mathfrak{G}}(\mathcal{T})$, normalizer of \mathcal{T} in \mathfrak{G}

$\mathcal{N}/\mathcal{T} \simeq W(M)$	Coxeter/Weyl group
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$$W(M) := \langle r_i \in \Phi^o; (r_i r_j)^{m_{ij}} = 1 \rangle \simeq \langle w_{r_i} : L \xrightarrow{\sim} L \rangle$$

$$M = (m_{ij}) = (m_{ji}) \quad \textbf{Coxeter matrix}$$

$$2m_{ij} = \#\{r \in \Phi : r = c_i r_i + c_j r_j, r_i, r_j \in \Phi^o; c_i, c_j \in \mathbb{Z}\}$$

- $\mathcal{U} := \langle \mathfrak{X}_r : r \in \Phi^+ \rangle \subset \mathfrak{G}$, $\mathfrak{X}_r := \text{Im}(x_r : \mathbb{G}_{a/\mathbb{Z}} \rightarrow \mathfrak{G})$
- $\mathcal{U}_w := \langle \mathfrak{X}_r : r \in \Phi_w \rangle$; $\Phi_w = \{r \in \Phi^+ : w(r) \in \Phi^-\}$

Theorem (Chevalley) If K is a field

$$\mathfrak{G}(K) = \coprod_{w \in W} \mathcal{U}(K) \mathcal{T}(K) n_w \mathcal{U}_w(K)$$

$$\langle \mathfrak{H}, \omega_r : r \in \Phi \rangle \ni n_w \in \mathcal{N}(K) \rightarrow N(K)/\mathcal{T}(K) \quad \omega_r \mapsto w_r$$

Geometries of the Chevalley groups (Tits)

\mathfrak{G} Chevalley group scheme/ \mathbb{Z}

$$\Phi^o = \{a_1, \dots, a_\ell\}, \quad \mathcal{A}_i := \{r \in \Phi : r = \sum_{\substack{j=1 \\ j \neq i}}^{\ell} c_j a_j, \quad c_j \in \mathbb{Z}\}$$

$\mathfrak{G}(K) \rightsquigarrow G(G_1, \dots, G_\ell) = \Gamma(G; G_i)$ **collection of index** ℓ

$G_i := <\mathcal{U}(K), \mathfrak{H}, \mathfrak{X}_{-r}, r \in \mathcal{A}_i>; \quad \Gamma(G; G_i) = \Gamma(\mathcal{E}; \mathcal{F}_i; \iota; A)$

$$\mathcal{E} = \cup_i \mathcal{F}_i, \quad \mathcal{F}_i = G/G_i, \quad A \simeq G/\bigcap_{g \in G, i} g^{-1}G_i g$$

$\Gamma(G; G_i) \rightsquigarrow \Sigma = \{\Gamma^{(j)} = \Gamma(G_j; G_i \cap G_j, i \neq j), j = 1 \dots \ell\}$

complete system of geometries of type $\Gamma(G; G_i)$

$\Sigma(G_K) = \{\Gamma^{(j)}, j = 1 \dots \ell\} \leftrightarrow S(G_K)$ “**scheme**”

Example: $\mathfrak{G}(K) = G_K = PGL_{\ell+1}(K)$

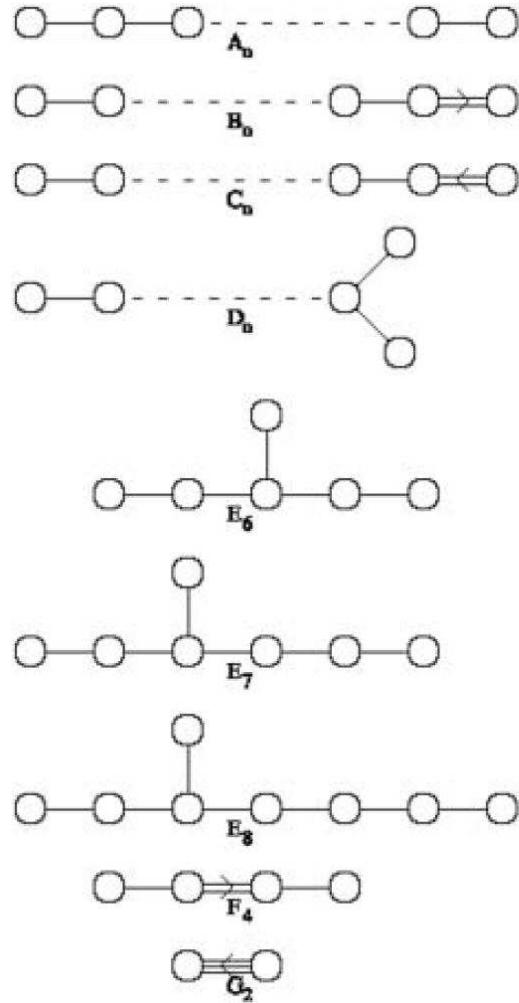
$\Sigma(G_K) = \ell\text{-dim. projective geometry over } K$

$S(G_K)$ **Coxeter** (Witt-Dynkin) **diagram** A_ℓ

The definition of $\Gamma(G; G_i)$ depends on K

(Tits) **The scheme** $S(G_K)$ **is universal!!**

Witt-Dynkin schemes



(Tits) **There are 4 elementary types: geometries over K are entirely characterized by those corresponding to:**

$$A_1 \times A_1 \text{ (no link)}, \quad A_2 \text{ } (\circ - \circ), \quad B_2 \text{ } (\circ = \circ), \quad G_2 \text{ } (\circ \equiv \circ)$$

Chevalley groups over \mathbb{F}_q

$\mathfrak{G} = \mathfrak{G}(L, \Phi, n_r)$ Chevalley group scheme

$$\mathfrak{G}(\mathbb{C}) = G$$

K any field:

$$\begin{aligned} |\mathfrak{G}(K)| &= \sum_{w \in W} |\mathcal{U}(K)\mathcal{T}(K)n_w\mathcal{U}_w(K)| \\ &= |\mathcal{U}(K)||\mathcal{T}(K)| \sum_{w \in W} |\mathcal{U}_w(K)| \\ &= |\mathbb{A}^N(K)| |\mathbb{G}_m(K)^\ell| \sum_{w \in W} |\mathbb{A}^{N_w}(K)| \end{aligned}$$

$$\ell = \text{rk } \mathfrak{g}, \quad N = \#\Phi^+, \quad N_w = \#\Phi_w; \quad 2N + \ell = \dim \mathfrak{g}$$

If $K = \mathbb{F}_q$: $|\mathfrak{G}(\mathbb{F}_q)| = (q-1)^\ell q^N \sum_{w \in W} q^{N_w}$

(R. Bott) $\sum_{w \in W} q^{N_w} = \prod_{i=1}^{\ell} \frac{q^{d_i} - 1}{q - 1}$

d_i **exponents of the Weyl group**

$$|\mathfrak{G}(\mathbb{F}_q)| = q^N \prod_{i=1}^{\ell} (q^{d_i} - 1)$$

$$P_{\mathfrak{g}}(q) := \prod_{i=1}^{\ell} (q^{d_i} - 1) \iff G_K \iff E(\mathfrak{g}) = \{d_i\}$$

determined by the action of $W(M)$ on L

$$\mathfrak{G}(K) = G_K \rightsquigarrow G(G_1, \dots, G_n) = \Gamma(G; G_i)$$

$\Gamma(G; G_i) = \Gamma(\mathcal{E}, \mathcal{F}_i, \iota, G_K)$ associated geometry

Fact $|\mathcal{F}_i| = |G/G_i| = \frac{P_{\mathfrak{g}}(q)}{(q-1)P_{\mathfrak{g}_i}(q)} = Q_{\mathfrak{g},i}(q)$

$$(q-1) \nmid Q_{\mathfrak{g},i}(q)$$

$$\mathfrak{g}_i \iff S(G_K) \setminus \{i\} \iff E(\mathfrak{g}_i) = \{d_{i_t}\} \quad (i \leftrightarrow \mathcal{F}_i)$$

Chevalley groups over \mathbb{F}_1

Main Fact $Q_{\mathfrak{g},i}(1) = \frac{|W(M)|}{\prod_{d_{i_t} \in E(\mathfrak{g}_i)} d_{i_t}} \in \mathbb{Z}_{\geq 0}$

Tits interprets $Q_{\mathfrak{g},i}(1)$ as the number of elements of the family \mathcal{F}_i^* ($\mathcal{F}_i \xrightarrow{q \rightarrow 1} \mathcal{F}_i^*$) belonging to a limiting geometric structure over \mathbb{F}_1

$$\Gamma_{\mathbb{F}_q}(\mathcal{E}, \mathcal{F}_i, \iota, G_K) \xrightarrow{q \rightarrow 1} \Gamma_{\mathbb{F}_1}(\mathcal{E}^*, \mathcal{F}_i^*, \iota^*, W(M))$$

- $W(M)$ is the group of symmetries of the limiting geometry
- $W(M)$ is the skeleton of G

$\Gamma_{\mathbb{F}_1}(\mathcal{E}^*, \mathcal{F}_i^*, \iota^*, W)$ “compositum” of the limiting version of the 4 “elementary” geometries:

i.e. union of polygons with 2,3,4 and 6 sides

Graded gadgets over \mathbb{F}_1

Tits' construction produces a notion of a “geometry” over \mathbb{F}_1

$$\Gamma_{\mathbb{F}_1}(\mathcal{E}^*, \mathcal{F}_i^*, \iota^*, W(M))$$

associated to a Chevalley group scheme $\mathfrak{G}_{/\mathbb{Z}}$

Example: $G = PGL_{\ell+1}(\mathbb{C})$, $W(M) = S_{\ell+1}$

$\mathcal{E}^* = \mathcal{P}_\ell$ finite set of $\ell + 1$ points

$\mathcal{F}_i^* \subset \mathcal{E}^*$ set of $i + 1$ points

Question: Can $\mathfrak{G}_{/\mathbb{Z}}$ be obtained by base-change from a variety G defined over \mathbb{F}_1

$$\mathfrak{G}_{/\mathbb{Z}} = G \times_{\text{Spec}(\mathbb{F}_1)} \text{Spec}(\mathbb{Z}) ?$$

Goal: Definition of G (over \mathbb{F}_1) compatible with Tits' geometry $\Gamma_{\mathbb{F}_1}(\mathcal{E}^*, \mathcal{F}_i^*, \iota^*, W(M))$

$X = (\underline{X}, X_{\mathbb{C}}, e_X)$ **(graded) gadget over \mathbb{F}_1**

- $\underline{X} = (\coprod_{k \geq 0} \underline{X}^{(k)}) : \mathcal{F}_{ab} \rightarrow \mathcal{S}ets$ covariant funct
 $\mathcal{F}_{ab} =$ (finite) abelian groups
- $X_{\mathbb{C}}$ algebraic variety over \mathbb{C}
- $e_X : \underline{X} \rightarrow \text{Hom}(\text{Spec}(\mathbb{C}[-]), X_{\mathbb{C}})$ natural transf

$\phi = (\underline{\phi}, \phi_{\mathbb{C}}) : X \rightarrow Y$ **morphism of gadgets**

- $\underline{\phi} : \underline{X} \rightarrow \underline{Y}$ natural transformation
- $\phi : X_{\mathbb{C}} \rightarrow Y_{\mathbb{C}}$ morphism of algebraic varieties
- the following diagram commutes $\forall D \in \mathcal{F}_{ab}$

$$\begin{array}{ccc}
 \underline{X}(D) & \xrightarrow{\phi(D)} & \underline{Y}(D) \\
 \downarrow e_X(D) & & \downarrow e_Y(D) \\
 \text{Hom}(\text{Spec}(\mathbb{C}[D]), X_{\mathbb{C}}) & \xrightarrow{\phi_{\mathbb{C}}} & \text{Hom}(\text{Spec}(\mathbb{C}[D]), Y_{\mathbb{C}})
 \end{array}$$

$$\phi : X \hookrightarrow Y$$

immersion of gadgets

- $\phi(D) : \underline{X}(D) \hookrightarrow \underline{Y}(D)$ injective $\forall D \in \mathcal{F}_{ab}$
- $\phi_{\mathbb{C}}$ embedding

Example (gadget) $V_{\mathbb{Z}} = \text{Spec}(A)$ defines a gadget:

$$X = \mathcal{G}(V) = (\underline{X}, X_{\mathbb{C}}, e_X)$$

$$\underline{X}(D) = \text{Hom}(A, \mathbb{Z}[D]), \quad \forall D \in \mathcal{F}_{ab}$$

$$X_{\mathbb{C}} = V_{\mathbb{C}} = V \otimes \mathbb{C}$$

$$e_X(D) : \text{Hom}(A, \mathbb{Z}[D]) \rightarrow \text{Hom}(\text{Spec}(\mathbb{C}[D]), V_{\mathbb{C}})$$

$$e_X(D)(f : A \rightarrow \mathbb{Z}[D]) = \text{Spec}(f \otimes 1_{\mathbb{C}})$$

Affine varieties over \mathbb{F}_1

$X = (\underline{X}, X_{\mathbb{C}}, e_X)$ **finite, graded gadget**

- $\exists X_{\mathbb{Z}}$ affine variety
- $\exists i : X \hookrightarrow \mathcal{G}(X_{\mathbb{Z}})$ immersion of gadgets such that:

$\forall V_{\mathbb{Z}} = \text{Spec}(A)$ and

$\forall \varphi : X \rightarrow \mathcal{G}(V_{\mathbb{Z}})$ morphism of gadgets

$\exists! \varphi_{\mathbb{Z}} : X_{\mathbb{Z}} \rightarrow V_{\mathbb{Z}}$ s.t.

$$X \xrightarrow{i} \mathcal{G}(X_{\mathbb{Z}}) \xrightarrow{\mathcal{G}(\varphi_{\mathbb{Z}})} \mathcal{G}(V_{\mathbb{Z}}) \quad \varphi = \mathcal{G}(\varphi_{\mathbb{Z}}) \circ i$$

Guiding principles

(for a meaningful definition of a graded gadget)

- $\underline{X} = \coprod_{k \geq 0} \underline{X}^{(k)} : \mathcal{F}_{ab} \rightarrow \mathcal{S}ets$
ought to contain enough points, so that
together with $X_{\mathbb{C}}$, it characterizes X
- $|\underline{X}(\mathbb{F}_{1^n})| = N(n)$ (polynomial) function s.t.
$$N(q) = |X_{\mathbb{Z}}(\mathbb{F}_q)|, \quad \text{if } n = q - 1$$

$$N(q) = \sum_k a_k (q - 1)^k, \quad a_k \in \mathbb{Z}$$

$$a_k (q - 1)^k = |\underline{X}^{(k)}(D)|, \quad q - 1 = |D|$$

Examples

$$|\mathbb{G}_m(\mathbb{F}_q)| = N(q) = q - 1$$

$$|\mathbb{A}^d(\mathbb{F}_q)| = N(q) = q^d$$

$$|\mathfrak{G}(\mathbb{F}_q)| = N(q) = q^N \prod_{i=1}^{\ell} (q^{d_i} - 1)$$

$$|\mathbb{P}^d(\mathbb{F}_q)| = N(q) = 1 + q + \cdots + q^d$$

The multiplicative group \mathbb{G}_m

$$N(q) = q - 1 = 0 + (q - 1), \quad q = p^r$$

$$\mathbb{G}_m = (\underline{\mathbb{G}_m}, \mathbb{C}^*, e_m)$$

$$\underline{\mathbb{G}_m} = \coprod_{k \geq 0} \underline{\mathbb{G}_m}^{(k)} : \mathcal{F}_{ab} \longrightarrow \mathcal{S}ets$$

$$\mathbb{G}_m(D)^{(k)} = \begin{cases} \emptyset & \text{if } k \in \mathbb{Z}_{\geq 0} \setminus \{1\} \\ D & \text{if } k = 1. \end{cases}$$

Example:

$$\mathbb{G}_m(\mathbb{F}_{1^n})^{(k)} = \begin{cases} \emptyset & \text{if } k \in \mathbb{Z}_{\geq 0} \setminus \{1\} \\ \mathbb{Z}/n\mathbb{Z} & \text{if } k = 1. \end{cases}$$

$$e_m(D) : \underline{\mathbb{G}_m}(D) \rightarrow \text{Hom}(\text{Spec } \mathbb{C}[D], \mathbb{C}^*)$$

$$e_m(D)(g) = \chi(g), \quad \chi : \mathbb{C}[D] \rightarrow \mathbb{C}, \quad g \in D$$

$$(\mathbb{G}_m)_{\mathbb{Z}} = \text{Spec}(\mathbb{Z}[T^{\pm 1}])$$

The affine space $\underline{\mathbb{A}}^d$

$$N(q) = q^d = (q-1)^d + d(q-1)^{d-1} + \cdots + d(q-1) + 1$$

$$\underline{\mathbb{A}}^d = (\underline{\mathbb{A}}^d, \mathbb{C}^d, e_d)$$

$$\underline{\mathbb{A}}^d = \coprod_{k \geq 0} (\underline{\mathbb{A}}^d)^{(k)} : \mathcal{F}_{ab} \longrightarrow \mathcal{S}ets$$

$$\underline{\mathbb{A}}^d(D)^{(k)} = \coprod_{\substack{Y \subset \{1, \dots, d\} \\ |Y|=k}} D^Y$$

Example:

$$\underline{\mathbb{A}}^2(\mathbb{F}_{1^n})^{(k)} = \begin{cases} \{0\} & \text{if } k=0 \\ \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} & \text{if } k=1 \\ (\mathbb{Z}/n\mathbb{Z})^{\{1,2\}} & \text{if } k=2 \\ \emptyset & \text{if } k \geq 3 \end{cases}$$

$$|\underline{\mathbb{A}}^2(\mathbb{F}_{1^n})| = n^2 + 2n + 1$$

$$e_d(D) : \underline{\mathbb{A}}^d(D) \rightarrow \text{Hom}(\text{Spec } \mathbb{C}[D], \mathbb{C}^d)$$

$$e_d(D)((g_j)_{j \in Y}) = (\xi_j)_{j \in \{1, \dots, d\}}, \quad \xi_j = \begin{cases} \chi(g_j) & \text{if } j \in Y; \\ 0 & \text{if } j \notin Y. \end{cases}$$

$$\mathbb{A}_{\mathbb{Z}}^d = \text{Spec}(\mathbb{Z}[T])$$

The projective space $\underline{\mathbb{P}}^d$ (as graded functor)

$$\underline{\mathbb{P}}^d = \coprod_{k \geq 0} (\underline{\mathbb{P}}^d)^{(k)} : \mathcal{F}_{ab} \longrightarrow \mathcal{S}ets$$

$$\underline{\mathbb{P}}^d(D)^{(k)} = \coprod_{\substack{Y \subset \{1, 2, \dots, d+1\} \\ |Y|=k+1}} D^Y / D$$

the right action of D is the diagonal action

$$\underline{\mathbb{P}}^d(\mathbb{F}_{1^n})^{(0)} = \{1, 2, \dots, d+1\}$$

$$|\underline{\mathbb{P}}^d(\mathbb{F}_{1^n})^{(0)}| = d+1$$

$\underline{\mathbb{P}}^d(\mathbb{F}_{1^n})$ coincides in degree zero with the $d+1$ points of the set \mathcal{P}_d on which Tits' defines a projective geometry of dimension d over \mathbb{F}_1

$$|\underline{\mathbb{P}}^d(\mathbb{F}_q)| = N(q) = 1 + q + \cdots + q^d \xrightarrow{q \rightarrow 1} |\underline{\mathbb{P}}^d(\mathbb{F}_{1^n})^{(0)}|$$

Extended Coxeter & Weyl groups

Abstract Root System: $(L, \Phi, n_r; r \in \Phi)$

$\Phi^o \subset \Phi^+$ fundamental roots

$\Pi = \{1, \dots, \ell\}$, $|\Pi| = |\Phi^o|$

$M = (m_{ij})$ Coxeter matrix $m_{ij} = m_{ji}$

$W(M) \simeq \langle r_i \in \Phi^o; (r_i r_j)^{m_{ij}} = 1 \rangle$
Coxeter group

$B(M) = \langle q_i : (q_i q_j)^{m_{ij}} = (q_j q_i)^{m_{ij}}, i, j \in \Pi \rangle$

Braid group of M

$X(M) = \text{Ker}(B(M) \twoheadrightarrow W(M))$ $q_i \mapsto r_i$

$V(M) = B(M)/[X(M), X(M)]$
extended Coxeter group

Theorem (Tits) $V(M) = B(M)/[X(M), X(M)]$ is the ‘universal extension’ of $W(M)$

$$1 \rightarrow U(M) \rightarrow V(M) \xrightarrow{f} W(M) \rightarrow 1$$

by

$$U(M) = X(M)/[X(M), X(M)]$$

free abelian group generated by a set of elements $\{g(s), s \in S\}$ in bijective correspondence with the set $S \subset W(M)$ of reflections
($S \ni s \iff r_s \in \Phi^+$ is conjugate to a $r \in \Phi^o$)

The definition of the extended Weyl group is implemented ‘over’ the construction of $V(M)$

root system (L, Φ, n_r) \longleftrightarrow $\mathfrak{G} = \mathfrak{G}(L, \Phi, n_r)$

$\mathcal{T} \subset \mathfrak{G}$ maximal, split torus

$\mathcal{N} = \mathcal{N}_{\mathfrak{G}}(\mathcal{T})$ normalizer of \mathcal{T} in \mathfrak{G}

$\mathcal{N}/\mathcal{T} \simeq W(M) = W$ **Weyl** (Coxeter) **group**

(D, ϵ) D = abelian group, $\epsilon \in D$, $\epsilon^2 = 1$

Proposition (Tits) $(D, \epsilon) \rightarrow \mathcal{N}_{D, \epsilon} = \mathcal{N}_{D, \epsilon}(L, \Phi)$

$$1 \rightarrow T \rightarrow \mathcal{N}_{D, \epsilon}(L, \Phi) \xrightarrow{p} W \rightarrow 1$$

canonical extension of W by $T = \text{Hom}(L, D)$,
functorial in (D, ϵ)

$$\begin{aligned} \mathcal{N}_{D, \epsilon} &= (V(M) \times T)/\text{Graph}(U(M) \xrightarrow{g(s_r) \rightarrow h_s^{-1}} T) \\ h_s(x) &= \epsilon^{n_r(x)} \end{aligned}$$

extended Weyl group

A commutative ring with 1

Theorem (Tits) The group extension

$$1 \rightarrow \mathcal{T}(A) \rightarrow \mathcal{N}(A) \xrightarrow{p} W \rightarrow 1$$

is canonically isomorphic to the group extension

$$1 \rightarrow \text{Hom}(L, A^*) \rightarrow \mathcal{N}_{A^*, -1}(L, \Phi) \xrightarrow{p} W \rightarrow 1$$

$$\mathcal{U}(A) = \langle x_r(a), r \in \Phi^+, a \in A \rangle$$

$$\mathcal{U}_w(a) = \langle x_r(a), r \in \Phi_w, a \in A \rangle$$

Theorem (Chevalley)

$$\psi : A^{\Phi^+} \xrightarrow{\sim} \mathcal{U}(A), \quad (t_r)_{r \in \Phi^+} \mapsto \prod_{r \in \Phi^+} x_r(t_r)$$

$$\psi_w : A^{\Phi_w} \xrightarrow{\sim} \mathcal{U}_w(A), \quad (t_r)_{r \in \Phi_w} \mapsto \prod_{r \in \Phi_w} x_r(t_r)$$

Chevalley schemes as graded gadgets

$$\mathfrak{g}_{\mathbb{C}} \iff (L, \Phi, n_r) \iff \mathfrak{G} = \mathfrak{G}(L, \Phi, n_r)$$

$$G = (\underline{G}, \mathfrak{G}(\mathbb{C}), e_G)$$

$$\underline{G} = \coprod_{k \geq 0} \underline{G}^{(k)} : \mathcal{F}_{ab}^{(2)} \longrightarrow \mathcal{S}ets$$

$$\mathcal{F}_{ab}^{(2)} \ni (D,\epsilon), \quad \epsilon^2=1$$

$$\underline{G}(D,\epsilon) = \underline{\mathbb{A}}^{\Phi^+}(D) \times \coprod_{w \in W} (p^{-1}(w) \times \underline{\mathbb{A}}^{\Phi_w}(D))$$

$$\mathcal{N}_{D,\epsilon}(L,\Phi) \xrightarrow[p]{} W$$

$$e_G(D,\epsilon) : \underline{G} \rightarrow \mathsf{Hom}(\mathsf{Spec}\, \mathbb{C}[D,\epsilon], \mathfrak{G}(\mathbb{C}))$$

$$e_G(D,\epsilon)(a,n,b) = \psi(e_{\Phi^+}(a))\, e_{\mathcal{N}}(n)\, \psi_w(e_{\Phi_w}(b))$$

$$\underline{\mathbb{A}}^{\Phi^+}(D) \xrightarrow{e_{\Phi^+}} \mathbb{C}^{\Phi^+} \xrightarrow{\psi \sim} \mathcal{U}(\mathbb{C}), \, \mathcal{N}_{D,\epsilon}(L,\Phi) \xrightarrow{e_{\mathcal{N}}} \mathcal{N}(\mathbb{C})$$

$$\underline{\mathbb{A}}^{\Phi_w}(D) \xrightarrow{e_{\Phi_w}} \mathbb{C}^{\Phi_w} \xrightarrow{\psi_w \sim} \mathcal{U}_w(\mathbb{C})$$

Chevalley schemes as graded varieties over \mathbb{F}_{1^2}

Theorem The graded gadget $G = (\underline{G}, \mathfrak{G}(\mathbb{C}), e_G)$ defines a variety over \mathbb{F}_{1^2} .

$$G_{\mathbb{Z}} = \mathfrak{G}, \quad G \hookrightarrow \mathcal{G}(\mathfrak{G}) \text{ immersion of gadgets} \\ \text{(by construction)}$$

The universal property in the definition of (graded) affine varieties over \mathbb{F}_{1^2} can be checked by applying

Proposition (Chevalley) $\mathcal{U} \times p^{-1}(w_o) \times \mathcal{U} \xrightarrow{\theta} \mathfrak{G}$
 $\theta(u, n, v) = unv$

$$\exists! w_o \in W, w_o(\Phi^+) = -\Phi^+, \mathfrak{G} \ni w'_o \xrightarrow{p} w_o \in W$$

θ open embedding onto $\Omega = \text{Spec}(\mathcal{O}_\Omega) \subset \mathfrak{G}$

$$\mathcal{O}_\Omega = \mathcal{O}_{\mathfrak{G}}[d^{-1}], \quad d(w'_o) = 1$$

Zeta function of $G = (\underline{G}, \mathfrak{G}(\mathbb{C}), e_G)$ over \mathbb{F}_1

$$|\mathfrak{G}(\mathbb{F}_q)| = N(q) = q^N \prod_{i=1}^{\ell} (q^{d_i} - 1)$$

$$(\text{Weil}) \quad Z_G(q, T) = \exp\left(\sum_{r \geq 1} N(q^r) \frac{T^r}{r}\right)$$

$$\begin{aligned} &(\text{Soul\'e}) \quad \chi = N(1) \\ &\lim_{q \rightarrow 1} Z_G(q, q^{-s})^{-1} (q-1)^{-\chi} = \prod_{i=N}^{2N+\ell} (s-N-D_J)^{a_i} \\ &N(x) = \sum_i a_i x^i \end{aligned}$$

$$\boxed{\zeta_G(s) = \prod_{i=N}^{\dim \mathfrak{g}} (s-N-D_J)^{(-1)^{\ell+i}}, \quad s \in \mathbb{R}}$$

$$D_J = \sum_{j \in J, \ J \subset \{1, \dots, \ell\}} d_j, \quad \ell + N = \sum_{j=1}^{\ell} d_j, \quad N = \#\Phi^+$$