

The height of Toric Subvarieties

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Toronto, October 23, 2008

- ① Previous work
- ② Toric Varieties
- ③ Toric subvarieties
- ④ Hermitian line bundles on toric varieties
- ⑤ The height of toric subvarieties
- ⑥ Examples and generalizations

O Previous work

- Bost, Gillet, Soulié: $h_{FS}(\mathbb{P}^n)$
- Maillot: Toric varieties with canonical metrics
- Sombra, Philippsen. Heights of translated toric varieties
- Dan, Casasnovas-Maillot, Berenstein-Yger
height of toric hypersurfaces
- Monroig, height of Hirzebruch surfaces.
- Berman - Boucksom, capacity of smooth toric varieties

①

Toric varieties

Let $N \cong \mathbb{Z}^d$ be a lattice, $N_{\mathbb{R}} = N \otimes \mathbb{R}$, $N_{\mathbb{C}} = N \otimes \mathbb{C}$

Let $M = N^\vee$ be the dual lattice $M_{\mathbb{R}}, M_{\mathbb{C}}$

Let $\Sigma = \{\tau\}$ a fan on $N_{\mathbb{R}}$.

This is a rational finite convex polyh. cone decomposition

* Every $\tau \in \Sigma$ is $\tau = \mathbb{R}_{\geq 0}v_1 + \dots + \mathbb{R}_{\geq 0}v_k$, $v_1, \dots, v_k \in N$.

* If $\tau \in \Sigma$ all faces of τ belong to Σ .

* If $\tau, \gamma \in \Sigma$ then $\tau \cap \gamma$ is a common face.

* The elements of Σ do not contain any line.

To a fan Σ we can associate a variety as follows

For $\Gamma \in \Sigma$ let $\Gamma^\vee \subseteq M_{\mathbb{R}}$ be the *dual cone*:

$$\Gamma^\vee = \{ u \in M_{\mathbb{R}} \mid u(x) \geq 0 \quad \forall x \in \Gamma \}$$

Then $M_\Gamma = \Gamma^\vee \cap M$ is a semigroup.

Let $\mathbb{Z}[M_\Gamma]$ be the *semigroup ring*:

$$\mathbb{Z}[M_\Gamma] = \left\{ \sum_{u \in \Gamma^\vee \cap M} m_u X^u \right\} \text{ finite formal sums}$$

The schemes $X_\Gamma = \text{Spec } \mathbb{Z}[M_\Gamma]$

can be glued together to define a scheme

$$X_\Sigma = \bigcup_{\Gamma \in \Sigma} X_\Gamma$$

In particular we have a dense open subset.

$$X_0 = \text{Spec}(\mathbb{Z}[M]) \cong \text{Spec}(\mathbb{Z}[x_1, x_1^{-1}, \dots, x_d, x_d^{-1}])$$

such that

$\Pi = X_0(\mathbb{C}) = \text{Hom}(M, \mathbb{C}^*)$ is a d-dim torus
that acts on $X_\Sigma(\mathbb{C})$.

In particular we have a parametrization

$$N_{\mathbb{C}} = \text{Hom}(M, \mathbb{C}) \xrightarrow{\text{escp}} \text{Hom}(M, \mathbb{C}^*) = X_0(\mathbb{C})$$

Then $\Pi_{\mathbb{R}} = \text{escp}(iN_{\mathbb{R}})$ is a compact real torus

Properties of Σ and X_Σ .

* Σ is complete : $N_{IR} = \bigcup_{\Gamma \in \Sigma} \Gamma$, iff X_Σ is proper

* Each $\Gamma \in \Sigma$ can be written as

$$\Gamma = R_{\geq 0} v_1 + \dots + R_{\geq 0} v_n$$

with v_1, \dots, v_n part of a basis of N , iff X_Σ is smooth

We will assume that X_Σ is proper and smooth.

Equivariant Cartier divisors on X_Σ

are given by the choice of a monomial rational function

$\chi^{u(\sigma)}$ on each X_σ such that $\frac{\chi^{u(\sigma)}}{\chi^{u(\sigma')}}$ is regular
on $X_\sigma \cap X_{\sigma'}$.

Then $u(\sigma) \in M$ defines a linear function on Γ
and the above condition is equivalent to

$$u(\sigma)|_{\Gamma \cap \sigma'} = u(\sigma')|_{\sigma \cap \sigma'}$$

Thus they define a piecewise linear function

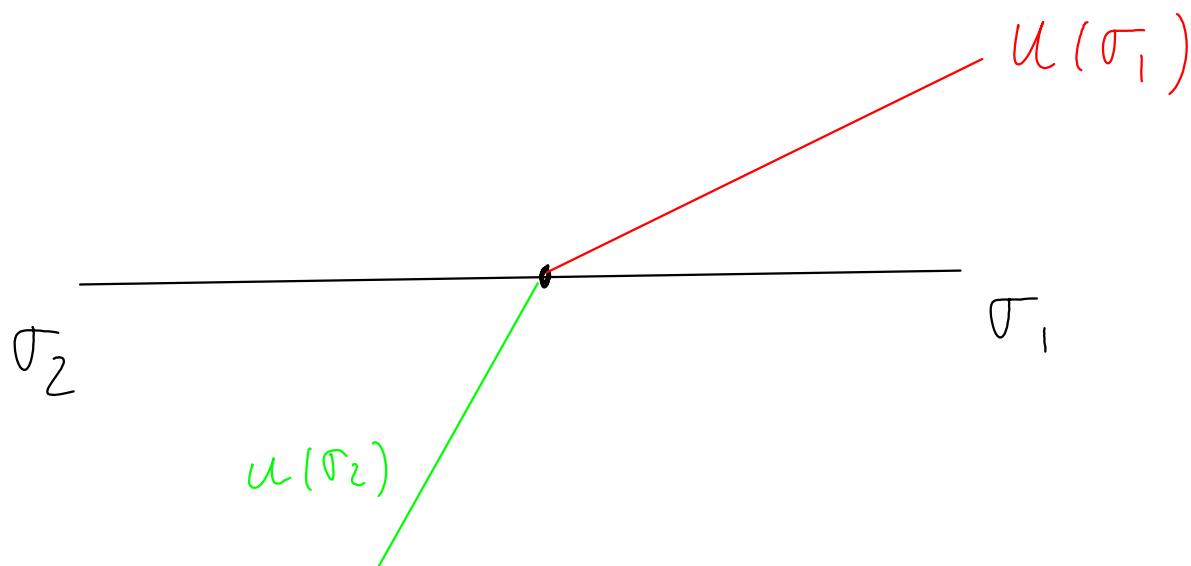
ψ_0 on $N_{\mathbb{R}}$

The Cartier divisor D is ample if and only if
the function Ψ_D is strictly concave.

For d -dimensional cones

$$u(\tau) \Big|_{\tau'} > u(\tau') \Big|_{\tau'}$$

If $d=1$ this means:



If D is ample we call ψ_D a polarization and
we define a polytope $\Delta_D \subseteq M_{IR}$

$$\Delta_D = \{ u \in M_{IR} \mid u(x) \geq \psi_D(x) \quad \forall x \in N_{IR} \}$$

Many properties of X_Σ can be read from Δ_D !

- * The elements of $S_D \cap M$ are a basis of $\Gamma(X_\Sigma, \mathcal{O}(D))$
- * For any Haar measure with $\text{corank}(M) = 1$ it holds

$$\deg(D) = D^d = m! \text{Vol}(\Delta_D)$$

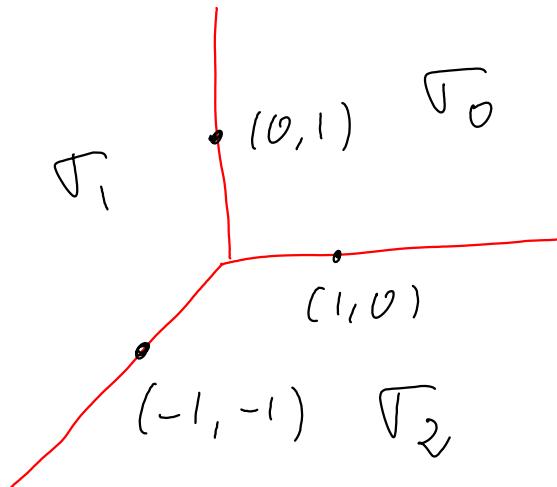
- * If D_1, \dots, D_d as before

$$D_1 \cdot \dots \cdot D_d = m! \text{mscVol}(\Delta_{D_1}, \dots, \Delta_{D_d})$$

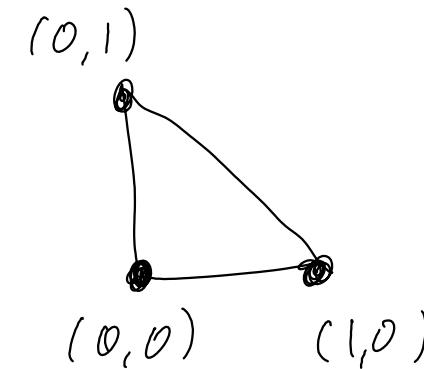
Example

$$N = \mathbb{Z}^2 \quad N_{\mathbb{R}} = \mathbb{R}^2$$

fan Σ



Polytope Δ



Toric variety:

polarization

$$\psi_D = \begin{cases} 0 & (x,y) \in R_0 \\ x & (x,y) \in R_1 \\ y & (x,y) \in R_2 \end{cases}$$

$$\mathbb{P}^2, D = \{x_0 = 0\}$$

$$\begin{cases} \deg(D) = 1 = 2 \text{ Vol } \Delta \\ \dim \mathcal{R}(\mathbb{P}^2, \mathcal{O}(1)) = 3 \end{cases}$$

2 Toric Subvarieties

Example 1. Each $\tau \in \Sigma$ defines a closed subvariety $V(\tau)$ as follows

$$N(\tau) = \frac{N}{N \cap R_{\tau}}$$

$$M(\tau) = M \cap \tau^{\perp} = N(\tau)^{\vee}$$

Then $M(\tau)$ is the maximal subgroup of M_{τ} and there is a map of semigroups $M_{\tau} \rightarrow M(\tau)$ that induces an injection

$$\mathbb{Z}[M_{\tau}] \rightarrow \mathbb{Z}[M(\tau)]$$

Thus $O(\tau) = \text{Spec}(\mathbb{Z}[M(\tau)])$ is a closed subvariety of X_{τ} . Hence locally closed subvariety of X_{Σ} .

$$V(\tau) = \overline{O(\tau)}$$

- * There is a fan decomposition of $N(\Gamma)$ given by Star(Γ), denoted $\Sigma(\Gamma)$.
- * If ψ_D is a polarization of (N, Σ) then " $\psi_D - u(\Gamma)$ " = ψ_{D_0} defines a polarization of $(N(\Gamma), \Sigma(\Gamma))$.
- * The polytope associated to ψ_{D_0} is a face of S_D denoted F_Γ .

Equivariant Weil divisors

Each 1-dimensional cone \mathbb{Z} is called an edge

Every edge \mathbb{Z} has a minimal length element of N denoted $v_{\mathbb{Z}}$.

Every edge defines a prime Weil divisor $D_{\mathbb{Z}} = V(\mathbb{Z})$.

The correspondence between Cartier divisors and Weil divisors is given by

$$\psi_D \rightarrow D = \sum_{\mathbb{Z}} -\psi_D(v_{\mathbb{Z}}) \cdot D_{\mathbb{Z}}$$

The assumption that Σ is smooth implies that $\bigcup_{\mathbb{Z}} D_{\mathbb{Z}}$ is a normal crossings divisor

Example 2

If $Q \subseteq N$ is a saturated mb lattice, then

$$P = \frac{M}{Q^\perp} = Q^\vee.$$

The fan decomposition Σ of N_R induces a fan decomposition of Q_R by intersection, denoted Σ_Q .

Let $\sigma' \in \Sigma_Q$ and $\sigma \in \Sigma$ such that $\sigma' \subseteq \sigma$.

There are semigroup maps

$$M_\sigma = M \cap \sigma^\vee \rightarrow M_{\sigma'} := M_\sigma + Q^\perp \xrightarrow{Q^\perp} P_{\sigma'} := P \cap \sigma'^\vee$$

That induce maps

$$\tilde{Y}_{\sigma'} = \text{Spec}(\mathbb{Z}[P_{\sigma'}]) \rightarrow Y_{\sigma'} = \text{Spec}(\mathbb{Z}[M_{\sigma'}]) \rightarrow X_{\sigma'} = \text{Spec}(\mathbb{Z}[n_{\sigma'}])$$

* These maps glue together to give maps

$$\tilde{Y}_{\bar{\Sigma}_Q} \rightarrow Y_{\bar{\Sigma}_Q} \rightarrow X_{\bar{\Sigma}}$$

Where $Y_{\bar{\Sigma}_Q}$ is a closed subvariety and $\tilde{Y}_{\bar{\Sigma}}$ is the normalization.

The ideal sheaf of $Y_{\bar{\Sigma}_Q}$ is generated by elements

$$x^u - x^v \quad u, v \in M_T \quad u - v \in Q^+.$$

* If ψ_D is a polarization of (N, Σ) , then $\psi_D|_{Q_{IR}}$ is a polarization of $(Q, \bar{\Sigma}_Q)$.

* The polytope associated to $(Q, \psi_D|_{Q_{IR}})$ is $\pi(\Delta_D)$ where $\pi: M_{IR} \rightarrow P_{IR}$ is the projection.

Example 3

We can mix together examples 1 and 2.

* We consider saturated sublattices $Q \subseteq N(\sigma)$.

These will define subvarieties of $V(\sigma)$.

* Alternatively we can consider cones of the fan $\bar{\Sigma}_Q$.

* The polytope of such subvariety will be a projection of a face of S_D , denoted $S_{D,Q}$.

Intersection theory

The intersection of a subvariety γ_{Σ_Q} as in escape 2 and a equivariant Cartier divisor is given as follows
let ψ_D be the polarization function associated to D .

Then $\psi_D|_{Q_{IR}}$ is a polarization of (Q, Σ_Q) .

The associated Weil divisor of $\tilde{\gamma}_{\Sigma_Q}$ is the
intersection subvariety:

$$D \cdot \gamma_{\Sigma_Q} = \sum_{\substack{z' \text{ edges} \\ \text{of } \Sigma_Q}} -\psi_D(v_{z'}) D_{z'}$$

The intersection
multiplicities
can be read
from the polytope

Comments

- * The integral model of $D \cdot Y_{\Sigma_Q}$ is very easy (no vertical components)
- * Each component of $D \cdot Y_{\Sigma_Q}$ is a toric subvariety as in example 3.
- * The polytope of each component of $D \cdot Y_{\Sigma_Q}$ is a face of the polytope of Y_{Σ_Q} .

3. Hermitian line bundles

Consider a equivariant line bundle $\mathcal{O}(D)$, $D = \sum m_z D_z$. This vector bundle has a canonical integral trivializing section e on X_0 .

Assume that $(\mathcal{O}(D))_{\mathbb{C}}$ has smooth metric $\| \cdot \|$ that is invariant with respect to the action of the compact torus π_R .

We consider the function

$$N_R \hookrightarrow N_{\mathbb{C}} \xrightarrow{\exp} \pi \cong X_0(\mathbb{C}) \xrightarrow{-\log \|e\|} \mathbb{R}$$

f

Comments

- * The metric $\|\cdot\|$ is positive if and only if the smooth function f is strictly convex: $\underline{\text{Hess}(f) > 0}$
- * The continuity of the metric implies that $f + \psi_D$ is bounded
- * If $\|\cdot\|$ is positive, the stability set of f , defined as

$$\{u \in M_{IR} \mid u(x) - f(x) \text{ has a maximum}\}$$

is the interior of the polytope Δ_D .

In the sequel we will assume that we have chosen an ample divisor D and a positive metric $\|\cdot\|$.

The Legendre transform

The Legendre transform of f is the map $f^V: \overset{\circ}{\Delta}_D \rightarrow \mathbb{R}$ given by

$$f^V(u) = \max_{x \in N} (u(x) - f(x))$$

The moment map $\mu: N \rightarrow \overset{\circ}{\Delta}_D$ given by

$$\mu(x) = df(x) \in M$$

is a diffeomorphism. Hence it has a differentiable inverse μ^{-1}

Then

$$f^V(u) = u(\mu^{-1}(u)) - f(\mu^{-1}(u)).$$

Legendre transforms and subvarieties

Proposition. The function f^\vee can be extended to a continuous function on S_D . Moreover, if $\sigma \in \Sigma$, let $f_\sigma : N(\sigma) \rightarrow \mathbb{R}$ the function obtained restricting $(\mathcal{O}(D), ||\cdot||)$ to $V(\sigma)$. Then

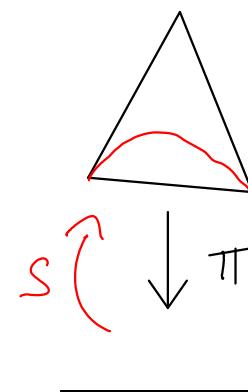
$$f_\sigma^\vee = f^\vee|_{F_\sigma}$$

Let now $Q \subseteq N$ be a sublattice. There is a diagram

$$\begin{array}{ccccc}
 & f & & f^\vee & \\
 & \swarrow & \nearrow M_f & \searrow & \\
 R & & N_{IR} & \xrightarrow{\quad} & S_D \\
 & \uparrow & & & \downarrow \pi \\
 & \swarrow & Q_{IR} & \longrightarrow & S_{D,y} \\
 g & & \nearrow M_g & & \searrow g^\vee
 \end{array}$$

Proposition. There is a unique section $s: S_{D,y} \rightarrow S_D$ that makes the above diagram commutative. This section is given by

$$s(u) = \arg \min_{u' \in \pi^{-1}(u)} f^\vee(u')$$



Legendre transform and small sections

The points of $S_D \cap M$ are an integral basis of $\mathcal{P}(X_\Sigma, \mathcal{O}(D))$.

Then, if $u \in S_D \cap M$ it corresponds to a section e^u

$$\begin{aligned} f^v(u) &= \sup_{x \in N_{IR}} (u(x) - f(x)) \\ &= \sup_{p \in X_\Sigma} (\log X^u + \log \|e^u\|) \\ &= \sup_{p \in X_\Sigma} (\log \|e^u\|) = \log \|e^u\|_{\sup} \end{aligned}$$

Thus if $f^v(u) < 0$ then $\|e^u\|_{\sup} < 1$.

Therefore we obtain a rough estimate on the existence of small sections.

4 The height of a toric subvariety

Theorem (Philippon, Sombra, -) Let X_Σ be a complete smooth toric variety, $\bar{L} = (\mathcal{O}(D), \|\cdot\|)$ be an ample line bundle with a positive hermitian metric. Let Y be a toric subvariety with associated lattices $Q, P = Q^\vee$ and polytope S_D . Let $dVol$ be a Haar measure of $P_{\mathbb{R}}$ such that the fundamental lattice has volume 1. Let $g^\vee: S_D \rightarrow \mathbb{R}$ as before. Then

$$h_{\bar{L}}(Y) = (n+1)! \int_{S_D} g^\vee dVol$$

$$n = \dim(Y)$$

Proof

Induction on the dimension of Y .

Let e be the canonical section of \bar{L} .

* by Bost - Gillet - Saito.

$$h_{\bar{L}}(Y) - h_{\bar{L}}(Y, \text{div } e) = \int_Y \log \|e\| c_1(\bar{L})^m$$

* A \mathbb{Z} -basis of Q determines coordinates x_1, \dots, x_n on Q and dual coordinates u_1, \dots, u_m on $P = Q^\vee$.

* Using the invariance of $\| \cdot \|$ with respect to
 the action of $\mathbb{T}_\mathbb{R}$ and the parametrization of
 γ_0 by $Q_\mathbb{C}$ we obtain

$$\int_Y \log \| \ell \| C_1(\bar{\iota})^m = m! \int_{Q_{\mathbb{R}}} g \cdot \det \text{Hess}(g) dx_1 \wedge \dots \wedge dx_m$$

* By the definition of the moment map μ

$$\mu^*(d\text{Vol}) = \det(\text{Hess } g) dx_1 \wedge \dots \wedge dx_m$$

Thus $\int_Y \log \| \ell \| C_1(\bar{\iota})^m = m! \int_S g(\mu^{-1}(u)) d\text{Vol}$

* Since $g^V(u) = u(\mu^{-1}(u)) - g(\mu^{-1}(u))$

$$\int_Y \log \|e\| C_1(E)^m = m! \int_S -g^V dVol + m! \int_S u(\mu^{-1}(u)) dVol$$

* Consider the form on Δ

$$\lambda = \sum_{j=1}^m (-1)^{j-1} u_j du_1 \wedge \dots \wedge \widehat{du_j} \wedge \dots \wedge du_m$$

Then

$$\begin{aligned} d(g^V \lambda) &= dg^V \cdot \lambda + m g^V dVol \\ &= u(\mu^{-1}(u)) dVol + m g^V dVol \end{aligned}$$

by Stokes theorem

$$\int_{\Delta} u(\mu^{-1}(u)) dVol = -m \int_{\Delta} g^{\vee} dVol + \int_{\partial\Delta} g^{\vee} \lambda$$

Hence

$$h_{\bar{L}}(Y) - h_{\bar{L}}(Y \cdot \text{div } e) = (m+1)! \int_{\Delta} -g^{\vee} dVol + m! \int_{\partial\Delta} g^{\vee} \lambda$$

* Using induction by hypothesis and computing
the intersection $Y \cdot \text{div } e$ by means of the polytope
we can identify.

$$h_{\bar{L}}(Y \cdot \text{div } e) = -m! \int_{\partial\Delta} g^{\vee} \lambda$$

□

5. Examples and generalizations

From a polytope Δ we can recover the polarized toric variety.

From the function g on Δ we can recover the metric.

Example 1 Canonical metrics of Guillemin

$\Delta \subseteq M_{\mathbb{R}}$ polytope . $\exists x_k \in N, \gamma_k \in \mathbb{Z} \quad k=1, \dots, d$

$$\Delta = \{ u \mid u(x_k) - \gamma_k \geq 0 \quad \forall k \}$$

$$g^v = \frac{1}{2} \sum_{k=1}^d (u(x_k) - \gamma_k) \log (u(x_k) - \gamma_k)$$

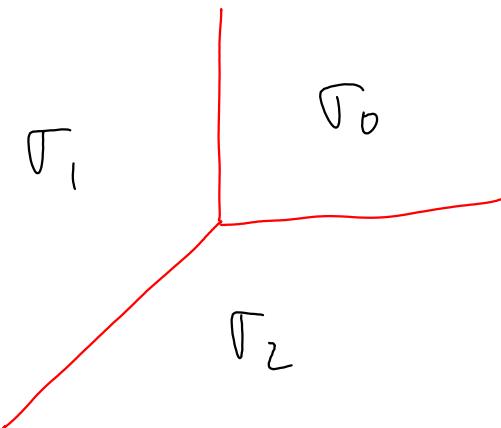
determines a smooth metric on the polarization divisor.

Example 2 $N = \mathbb{Z}^m$, $N_{\mathbb{R}} = \mathbb{R}^m$, we consider the fan generated by the vectors

$$v_0 = (-1, \dots, -1)$$

$$v_i = (0, \dots, \underbrace{1}_{i}, \dots, 1) \quad i = 1, \dots, m$$

In dimension 2:



The corresponding toric variety is \mathbb{P}^n .

We consider the line bundle $\mathcal{O}(1)$ with section d_0 , so

$D = \{d_0 = 0\} = D_{V_0}$, and the Fubini-Study metric.

$$|d_{-0}|^2 = \frac{d_0 \bar{d}_0}{d_0 \bar{d}_0 + \dots + d_m \bar{d}_m}$$

$$f(x_1, \dots, x_m) = \frac{1}{2} \log (1 + e^{2x_1} + \dots + e^{2x_m})$$

$$\mu(x_1, \dots, x_m) = \frac{1}{1 + e^{2x_1} + \dots + e^{2x_m}} (e^{2x_1}, \dots, e^{2x_m})$$

$$f^V(u_1, \dots, u_m) = \frac{1}{2} \left(\sum_{i=1}^m u_i \log u_i + (-u_1 - \dots - u_m) \log (1 - u_1 - \dots - u_m) \right)$$

$$h(\mathbb{P}^m) = \frac{m+1}{2} \sum_{j=2}^{m+1} \frac{1}{j} \quad (\text{BGS})$$

Example 3 Monomial curves. (Philippon-Sombra)

We consider the curve C : $t \mapsto (1; t^{a_1}; \dots; t^{a_m})$
with $(a_1, \dots, a_m) = 1$.

This is a toric subvariety of \mathbb{P}^m given by the
monoid lattice $Q = \langle (a_1, \dots, a_m) \rangle$.

Using the diagram

$$\begin{array}{ccccc} & f & & f^\vee & \\ R & \swarrow & N & \rightarrow & \Delta_{\mathbb{P}^m} \\ & \nearrow g & \uparrow s & \downarrow \pi & \\ Q & \rightarrow & \Delta_C & \xrightarrow{\quad j \quad} & \mathbb{R} \\ & g^\vee & & & \end{array}$$

We can compute g^\vee .

Write $F(\lambda) = 1 + \lambda^{a_1} + \dots + \lambda^{a_m}$

$$G(\lambda) = \frac{\lambda F'}{F}$$

Then

$$\begin{aligned} h_{FS}(c) &= \int_0^\infty \log(F(\lambda)) - G(\lambda) \log(\lambda) \, d\lambda \\ &= \frac{1}{2} \int_0^\infty G(\lambda) (a_m - G(\lambda)) \frac{d\lambda}{\lambda} \end{aligned}$$

If $F(\lambda) = \prod_{\zeta \in Z(F)} (\lambda - \zeta)^{\ell(\zeta)}$.

$$h_{FS}(c) = \sum_{\zeta \in Z(F)} \ell(\zeta)^2 + \frac{1}{2} \sum_{(\zeta_1, \zeta_2) \in Z(F)^2, \zeta_1 \neq \zeta_2} \ell(\zeta_1) \ell(\zeta_2) \frac{\zeta_1 + \zeta_2}{\zeta_1 - \zeta_2} \log(-\zeta_1).$$

We can specialize to the rational normal curve.

$$t \mapsto (1:t:t^2:\dots:t^m)$$

In this case $F(\lambda) = 1 + \lambda + \dots + \lambda^m$ the cyclotomic polynomial. In this case

$$h(C_m) = \frac{m}{2} + \pi \sum_{j=1}^{[m/2]} \left(1 - \frac{2j}{m+1} \right) \cot \left(\frac{\pi j}{m+1} \right)$$

Generalization 1

By continuity we can extend the theorem to the case of admissible metrics in the sense of Maillet.

Example. The canonical metric on a toric variety satisfies $\delta = -\psi_D$ and $\delta^\vee = 0$

we obtain the well known result

$$h_{\text{can}}(x) = 0$$

Generalization 2: Translated toric varieties

X a smooth toric variety, N the lattice.

$p \in X(K)$ K a # field

$Q \subseteq N$ a mb lattice π_Q the corresponding subtorus.

$X_{p, Q} \subseteq X$ "The orbit of p under the action
of the mb torus π_Q ".

The integral models of $X_{p, Q}$ and of $X_{p, Q} \cdot \text{dir}(s)$
are more interesting.

Combining previous work of Philippon-Sombra
 on the height of translated toric varieties
 with the canonical metric we can obtain
 formulas for the height with respect to other metrics.
 There is an adelic formula

$$h(X_{\Sigma, \alpha}) = (m+1)! \sum_v \frac{[\kappa_v : (\mathbb{Q}_v)]}{[\kappa : \mathbb{Q}]} \int_S \Omega_{A, D, d} \, d\text{Vol}$$

We can interpret the integral models as
admissible metrics in the sense of Zhang
 We would like to interpret the integrals corresponding
 to Ω also as degenerate transforms.

Generalization 3

Multi heights

If $\bar{L}_1, \dots, \bar{L}_n$ are hermitian line bundles with positive metric. Then the multiheight of γ is given by

$$h(\gamma; \bar{L}_1, \dots, \bar{L}_n) = M I_n(-f_1^\vee, \dots, -f_m^\vee),$$

where $M I_n$ is the mixed integral introduced by Philippon and Sombra.