# § 1 Lefschetz theorems

compare the geometry / topology of a (smooth) projective variety $X \hookrightarrow \mathbb{P}^N$ and of a hyperplane section $Y := X \cap H$,
($H \hookrightarrow \mathbb{P}^N$ hyper-plane / surface)

# Lefschetz theorems on arithmetic schemes

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October 20, 2008

Workshop on arithmetic geometry

1 Lefschetz theorems

2 Lcf and G3 on arithmetic surfaces

joint work with A. El [illegible]

3 Leff on arithmetic three-folds

work in progress

. Lefschetz's version

1924 L'analysis situs et la géométrie algébrique

$V_n \hookrightarrow \mathbb{P}^N$ smooth projective complex variety

$H \hookrightarrow \mathbb{P}^N$ hyperplane

$i \leq n-1 \Rightarrow H_i(V_n, V_n \cap H) = 0$

i.e.

$H_i(V_n \cap H) \xrightarrow{\simeq} H_i(V)$ if $i < n-1$

$\twoheadrightarrow$ if $i = n-1$

. Precursors

? "Bertini" ? $V_n$ projective irreducible

$n \geq 2 \Rightarrow V_n \cap H$ connected

Picard 1889 $H_1(V_2 \cap H) \twoheadrightarrow H_1(V_2)$

Castelnuovo - Enriques 1906 $H_1(V_n \cap H) \simeq H_1(V_n)$ if $n \geq 3$

. Further developments over $\mathbb{C}$

. $\pi_i$ instead of $H_i$ ⟵ Morse theory
. singular spaces

Thom, Bott, Andreotti - Frankel

Fulton, Hansen, Barth, Hamm, ...

# Algebraic approaches to L.T.

- Zariski

  ~ 1950 $\pi_0$ and formal functions

  "principle of degeneration"

- Grothendieck

  "Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux."

  Séminaire de géométrie algébrique du Bois Marie, 1962

  "SGA 2"

  cohomological methods : formal geometry and duality

# The conditions Lef and Leff

$X$ algebraic variety / scheme over $k$

$Y \hookrightarrow X$

$\hat{X} :=$ formal completion of $X$ along $Y$
$= \varinjlim X_n$

if $\mathcal{F}$ is a coherent sheaf over a neighbhd of $Y$, then $\hat{\mathcal{F}}$ denotes its "restriction" $\varprojlim \mathcal{F}_{|Y_n}$ to $\hat{X}$.

**Lef $(X, Y)$** Lefschetz condition

> For any vector bundle $E$ over an open nghbhd $U$ of $Y$ in $X$,
>
> $$\Gamma(U, E) \xrightarrow{\sim} \Gamma(\hat{X}, \hat{E})$$

**Leff $(X, Y)$** condition de Lefschetz effective

> Any vector bundle $E$ for over $\hat{X}$ "extends" to a vector bundle $E$ over some open nbhd $U$ of $Y$ in $X$

[ i.e. there exists $U$ ..., $E$ locally free coherent sheaf over $U$, and an isom. $\hat{E} \simeq E$ for of $\mathcal{O}_{\hat{X}}$ - modules over $Y$ ]

**Theorem** ( SGA 2, exposé XII, in a simple case)

$X$ smooth projective variety over $k$

$Y$ ample hypersurface in $X$

[i.e hyperplane section of $X$ for some projective embedding]

$\dim X \geq 2 \Rightarrow$ Lef $(X, Y)$

$\dim X \geq 3 \Rightarrow$ Leff $(X, Y)$

**Comments** :

1) actual version in SGA 2:

relative version, under weaker regularity hypotheses ; namely

$$\begin{array}{ccc} Y & \hookrightarrow & X \\ \downarrow & & \downarrow \pi \text{ proper} \\ S & = & S \end{array}$$

with $Y$ (relative) Cartier divisor, ample relative to $\pi$, and similar assumptions on the dimension / depth of the fibers of $\pi$

2) local versions

idea : $Y \hookrightarrow X \hookrightarrow \mathbb{P}^N$

$0 \in CY \hookrightarrow CX \hookrightarrow \mathbb{A}^{N+1}$

vector bundle over $X$

"homogeneous" vector bundle over $CX \setminus \{0\}$

$\rightsquigarrow$ natural to investigate $L_{ef}$ and $L_{eff}$ for

$CY \setminus \{0\} \hookrightarrow \acute{C}X \setminus \{0\}$

and more generally

$(f = 0) \setminus \{m_A\} \hookrightarrow \operatorname{Spec} A \setminus \{m_A\}$

$A$ noetherian local ring

cf. SGA 2, exposé IX

extended notably by

M.me Raynaud Ann. Sci. ENS 7 1974

G. Faltings Ann. of Math. 110 1979

# 3) geometric applications

ample hypersurface ↝ $Y \hookrightarrow \hat{X} \hookrightarrow X$ ↜ smooth, projective over $k$ dimension $d$

| | | | | | |
|---|---|---|---|---|---|
| $\pi_0$ | | $\simeq$ | $\simeq$ | $\Leftarrow$ Lef$(X,Y)$ | $d \geq 2$ |
| $\pi_1^{et}$ | | $\simeq$ | $\twoheadrightarrow$ | $\Leftarrow$ Lef$(X,Y)$ | $d \geq 2$ |
| $\pi_1^{et}$ | | $\simeq$ | $\simeq$ | $\Leftarrow$ Leff$(X,Y)$ | $d \geq 3$ |
| Pic | $\forall i>0, H^i(Y, N^{\vee \otimes i}) \neq 0 \Rightarrow$ | $\longleftarrow$ | $\longleftarrow$ | $\Leftarrow$ Lef$(X,Y)$ | $d \geq 3$ |
| Pic | $\forall i>0, H^i(Y, N^{\vee \otimes i}) = 0$, $H^2(Y, N^{\vee \otimes i}) = 0$ | $\simeq$ | $\simeq$ | $\Leftarrow$ Leff$(X,Y)$ | $d \geq 4$ |

$k = \mathbb{C}$, $Y$ sm. ⇑ "Kodaira vanishing"

$N := N_Y X \simeq \mathcal{O}(Y)_{|Y}$ ample line bundle over $Y$

Pic := { line bundles / iso. }

4) Related work in Hironaka - Matsumura, J. Math. Soc. Japan 20, 1968

irreducible variety over $k$

$$Y \hookrightarrow \hat{X} \hookrightarrow X$$

$$k(\hat{X}) \longleftarrow k(X)$$

$\|$ $\|$

formal rational functions on $\hat{X}$ — rational functions on $X$

$Y$ is G3 in $X$ $\iff$ $k(\hat{X}) = k(X)$

Examples of G3 embeddings:

- $Y \hookrightarrow \mathbb{P}^N$ $\quad \dim Y > 0$
- $Y_1 \hookrightarrow X_2$

  ample curve — projective surface
- over $\mathbb{C}$, relation with pseudoconcavity

Hartshorne, Gieseker

Ehoiz, Faltings

# Lefschetz theorems on arithmetic schemes ?

$$\begin{array}{ccc} Y & \hookrightarrow & \mathcal{X} \\ \| & & \downarrow f \\ \mathcal{X} \cap \mathbb{P}^{N-1}_{\mathbb{Z}} & & \mathbb{P}^{N}_{\mathbb{Z}} \end{array}$$

projective flat schemes over $\mathbb{Z}$ ; $\mathcal{X}$ regular

hyperplane section

- SGA 2

  Lef $(\mathcal{X}, Y)$ holds when $\dim \mathcal{X}_{\mathbb{Q}} \geq 2$

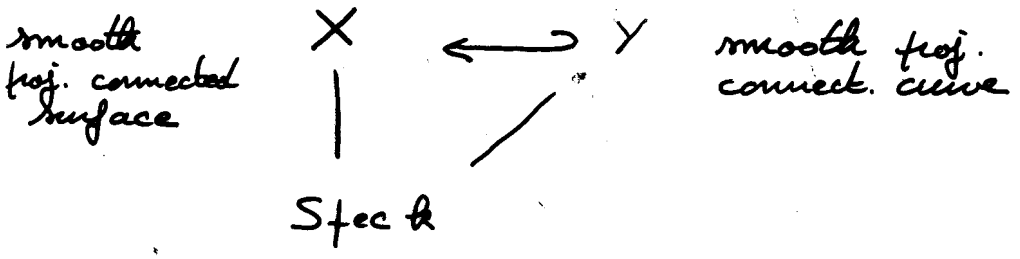
  Leff $(\mathcal{X}, Y)$ " " $\dim \mathcal{X}_{\mathbb{Q}} \geq 3$

- analogy " number fields - function fields"
  "Arakelov's philosophy"

arithmetic analogue — Lef$(\mathcal{X}, Y)$ when $\dim \mathcal{X} \geq$
involving complex analysis/geometry of
of $Y(\mathbb{C}) \hookrightarrow \mathcal{X}(\mathbb{C})$ — Leff$(\mathcal{X}, Y)$ when $\dim \mathcal{X} \geq$

# 2 Lef and G3 on arithmetic surfaces

# Geometric case




HGeo $\quad Y.Y = \deg_Y N_Y X > 0$

"big and nef"

$\Rightarrow$ $\exists$ finite family $(C_i)_{i\in I}$ of projective curves in $X$ s.t. $C_i \cap Y = \emptyset$

$U := X \setminus \bigcup_{i\in I} C_i$

$\hat{X} := \varinjlim_n Y_n$ ; $E$ v.b. on $X$, $\hat{E} := \varprojlim E_n$

Thm (Grothendieck, Hartshorne, Hironaka–Matsumura ...)

When HGeo holds,

$k(X) \xrightarrow{\sim} \kappa(\hat{X})$ ;

for any vector bundle $E$ over $U$,

$\Gamma(U, E) \xrightarrow{\sim} \Gamma(\hat{X}, \hat{E})$

# Arithmetic counterpart

data : $\mathcal{Y} \hookrightarrow \mathcal{X}$ projective regular integral arithmetic surface

$\pi$ : $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ number field.

$\nu$ : $\mathcal{Y} \to \operatorname{Spec} \mathcal{O}_K$

for any $\sigma : K \hookrightarrow \mathbb{C}$,

$$\mathcal{Y}_\sigma \in \Omega_\sigma \hookrightarrow \mathcal{X}_\sigma(\mathbb{C})$$

such that $\Omega_{\bar{\sigma}} = \Omega_\sigma^{c.c.}$

$g_{\mathcal{Y}_\sigma, \Omega_\sigma}$ Green function of $\mathcal{Y}_\sigma$ in $\Omega_\sigma$ :

- $g_{\mathcal{Y}_\sigma, \Omega_\sigma}$ harmonic on $\Omega_\sigma \setminus \{\mathcal{Y}_\sigma\}$ ; vanishes "on" $\partial \Omega_\sigma$
- $g_{\mathcal{Y}_\sigma, \Omega_\sigma} = \log |z - z(\mathcal{Y}_\sigma)|^{-1} + \sigma + O(|z - z(\mathcal{Y}_\sigma)|)$

$\| \cdot \|^{can}_{\mathcal{Y}_\sigma, \Omega_\sigma}$ norm on $T_{\mathcal{Y}_\sigma} \mathcal{X}_\sigma$ :

$$\left\| \frac{\partial}{\partial z} \right\|^{can}_{\mathcal{Y}_\sigma, \Omega_\sigma} := e^{-\sigma}$$

HAr $\quad \widehat{deg}_{\mathcal{Y}} \left( N_{\mathcal{Y}} \mathcal{X}, \left( \| \cdot \|^{can}_{\mathcal{Y}_\sigma, \Omega_\sigma} \right) \right) > 0$



12 T

HAr

$\Downarrow$ Fekete, Siegel, ..., Rumely ...

$\exists$ finite family $(C_i)_{i\in I}$ of 1-dimensional closed integral subschemes s.t. $C_i \cap Y = \emptyset$

$C_{i,\sigma} \cap \Omega_\sigma = \emptyset$

$$\mathcal{U} := \mathcal{X} \setminus \bigcup_{i\in I} C_i$$

**Theorem** (A. Chambert-Loir + JBB)

When HAr holds,

$$\chi(\mathcal{X}) \simeq \left\{ f \in \Gamma(Y, \mathrm{Frac}\, \mathcal{O}_{\hat{\mathcal{X}}_Y}) \;\middle|\; \begin{array}{l}\text{for any } \sigma: K \hookrightarrow \mathbb{C},\\ f_\sigma \text{ meromorphic on } \Omega_\sigma\end{array} \right\}$$

for any vector bundle $\mathcal{E}$ over $\mathcal{U}$,

$$\Gamma(\mathcal{U}, \mathcal{E}) \xrightarrow{\sim} \left\{ s \in \Gamma(\hat{\mathcal{X}}, \hat{\mathcal{E}}) \;\middle|\; \begin{array}{l}\text{for any } \sigma: K \hookrightarrow \mathbb{C},\\ s_\sigma \text{ holomorphic ...}\\ \text{section of } \mathcal{E}_\sigma \text{ over } \Omega_\sigma\end{array} \right\}$$

# Comments

1) Basic example

$$\begin{array}{ccc} 0 & \hookrightarrow & \mathbb{P}^1_{\mathbb{Z}} \\ & \searrow^{\sim} & \downarrow \\ & & \operatorname{Spec}\mathbb{Z} \end{array}$$

$\Omega := \mathring{D}(0,R)$

$g_{0,\Omega} = \log^+ \frac{R}{|z|}$

$\left\| \frac{\partial}{\partial z} \right\|^{\mathrm{can}}_{0,\Omega} = \frac{1}{R}$

$\widehat{\deg}\left(N_0 \mathbb{P}^1_{\mathbb{Z}}, \|\cdot\|^{\mathrm{can}}_{0,\Omega}\right) = \log R$

$HA_\Omega \iff R > 1 \qquad \mathcal{U} = \mathbb{A}^1_{\mathbb{Z}}$

Borel's theorem:

$f \in \mathbb{Z}[[X]]$ meromorphic on $\mathring{D}(0,R)$, $R>1$

$\Rightarrow f \in \mathbb{Q}(X)$.

2) Established by using

(i) Diophantine approximation techniques, and

(ii) Hodge Index Theorem (Faltings–Hriljac) on arithmetic surfaces.

Admits more general versions, involving $p$-adic rigid analytic geometry. cf. "Manin Festschrift"

Applications to $p$-adic and classical modular forms

3) Application to $\pi_1$, extending earlier work of Ihara

$$\begin{array}{ccc} Y & \hookrightarrow & \mathcal{X} \\ & & \downarrow \\ & & \operatorname{Spec} \mathbb{Z} \end{array} \qquad Y_{\mathbb{C}} \in \Omega = \Omega^{c.c.} \hookrightarrow \mathcal{X}(\mathbb{C})$$

$$\text{HA}_{\mathcal{X}} \Rightarrow \hat{\pi}_1(\Omega, x) \twoheadrightarrow \pi_1(\mathcal{X}, x)$$

Example (variant): $\pi_1\left(y^2 + y = x^{2g+1}\right) = \{0\}$

$$\downarrow$$

$$\mathbb{A}^2_{\mathbb{Z}}$$

cf. JBB Ann. Sc. ENS 32 1999

3 Leff on arithmetic 3-folds

<u>Data</u> :

$\mathcal{X}$ integral regular projective scheme over Spec $\mathbb{Z}$, s.t. $\dim \mathcal{X} = 3$

$\uparrow$

$\mathcal{Y}$ smooth divisor in $\mathcal{X}$

$\mathcal{Y}(\mathbb{C}) \hookrightarrow \Omega \hookrightarrow \mathcal{X}(\mathbb{C})$ analytic open subset invariant under c.c.

$\rightsquigarrow$ may define a ringed topos $(\tilde{\mathcal{X}}, \tilde{\mathcal{O}})$, designed to deal with the "combination" of formal geometry on $\hat{\mathcal{X}} := \varinjlim \mathcal{Y}_n$ and analytic geometry on $\mathcal{X}(\mathbb{C})$ :

. Grothendieck topology

$$\tilde{\mathcal{U}} := (V, \mathcal{U}^{an}) \quad \text{s.t. } \mathcal{U}^{an} \cap \mathcal{Y}(\mathbb{C}) = V(\mathbb{C})$$

Zariski $\downarrow$ $\qquad$ $\downarrow$ analytic

$\mathcal{Y} \qquad \mathcal{X}(\mathbb{C})$

...

. sheaf of rings

$$\tilde{\mathcal{O}}(\tilde{\mathcal{U}}) = \{(\hat{s}, s^{an}) \in \hat{\mathcal{O}}(V) \times \mathcal{O}^{an}(\mathcal{U}^{an}) \mid \hat{s}_{\mathbb{C}} = \widehat{s^{an}} \text{ on } \mathcal{X}(\mathbb{C})^{\wedge}_{V(\mathbb{C})}\}$$

<u>Example</u> : vector bundle on $\tilde{\mathcal{X}}$ := locally free sheaf of $\mathcal{O}_{\tilde{\mathcal{X}}}$-module of finite rank

$\tilde{\mathcal{E}} \longleftrightarrow (\hat{\mathcal{E}}, \mathcal{E}^{an}, \varphi)$ $\quad \hat{\mathcal{E}} \sim \mathcal{E}^{an\wedge}$

"Conjecture"

Assume that there exists

$\overline{\mathcal{L}} := (\mathcal{L}, \|\cdot\|)$ hermitian line bundle on $\mathcal{X}$

$s \in \Gamma(\mathcal{X}, \mathcal{L}) \setminus \{0\}$

such that

$\overline{\mathcal{L}}$ is nef $\qquad \hat{h}_{\overline{\mathcal{L}}} \geq 0$ : >and !

$\overline{\mathcal{L}}$ is relatively positive $\mathcal{L}$ ample, $c_1(\overline{\mathcal{L}}) >$

$Y = \operatorname{div} s$

$\|s\| \leq 1$ on $\mathcal{X}(\mathbb{C})$

$\Omega = \{ x \in \mathcal{X}(\mathbb{C}) \mid \|s(x)\| < 1 \}$.

Then any vector bundle $\tilde{\mathcal{E}}$ over $\tilde{\mathcal{X}}$

is algebraizable.

namely : it is the "restriction" of a vector bundle $\mathcal{E}$ on $\mathcal{X} \setminus F$, for some finite set $F$ of closed points in $\mathcal{X}$;
$F = \phi$ when $\operatorname{rk} \mathcal{E} = 1$

" . " : OK modulo a "conjecture"
in Kosarew-Peternell Compositio 74 (1990)

# Comments

1) Basic example

$$\mathbb{P}^1_{\mathbb{Z}} \hookrightarrow \mathbb{P}^2_{\mathbb{Z}} \qquad \Omega_R := \{(x_0 : x_1 : x_2) \in \mathbb{P}^2(\mathbb{C}) \mid |x_2|^2 < R^2(|x_0|^2 + |x_1|^2)\}$$

$$\| \ (X_2 = 0)$$

with $R > 1$

N. B. : $\mathrm{Pic}(\hat{\mathbb{P}}^2_{\mathbb{Z}}, \hat{\mathcal{O}})$ "huge" groups

$\mathrm{Pic}(\Omega_R, \mathcal{O}^{an})$

2) Toy model for "arithmetic algebraization theorems" concerning, say, line bundles.

Actually, here also, more sophisticated variants involving $p$-adic rigid geometry at finite places.

Motivation : Conjecture of Ogus LNM 900

$X$ smooth projective variety over a number field $K$

$$c_1 : CH^1(X)_{\mathbb{Q}} \longrightarrow H^2_{dR}(X/K)$$

For almost every $p$, $H^2_{dR}(X/K) \otimes K_p$ is equipped with the crystalline Frobenius $\Phi_p$

$$c_1(CH^1(X)_{\mathbb{Q}}) \subset \{\alpha \in H^2_{dR}(X/K) \mid \text{"}\forall\text{"}\, p,\ \Phi_p \alpha = p\alpha\}$$

## 3) Applications to $\pi_1$

$$\text{“}\ \pi_1(Y) *_{\pi_1(Y_c)} \hat{\pi}_1(\Omega)\ \text{”} \simeq \pi_1(X)$$

“ . ” : take care of base points and complex conjugation

$$h^i_{\mathcal{D}}(\overline{E})$$

$\overline{E}$ := hermitian vector bundle over Spec $\mathbb{Z}$
a. k. a. euclidean lattice

$$\dim_{\mathbb{F}_1} \Gamma(\overline{\operatorname{Spec}\mathbb{Z}}, \overline{E}) = ??$$

$$h^0(\overline{E}) := \log \# \{ v \in E \mid \|v\| \leq 1 \}$$

see e.g. Gillet, Soulé Israel J.M. 74 1991

$$h^0_{\mathcal{D}}(\overline{E}) := \log \sum_{v \in E} e^{-\pi \|v\|^2}$$

Rössler 1993
van der Geer - Schoof 2000

Motivations

i) "Riemann - Roch = Poisson"
functional equation : Hecke, F. K. Schmidt, Tate

$$h^1_{\mathcal{D}}(\overline{E}) := h^0_{\mathcal{D}}(\overline{E}^\vee)$$

$$h^0_{\mathcal{D}}(\overline{E}) - h^1_{\mathcal{D}}(\overline{E}) = \widehat{\deg}\, \overline{E}$$

More generally, for any number field $K$ and any hermitian vector $\overline{E} = (E; (\|\cdot\|_\sigma)_{\sigma : K \hookrightarrow \mathbb{C}})$ on Spec $\mathcal{O}_K$, we let

$$\pi_* \overline{E} := (E, \| \ \|) \qquad \|e\|^2 := \sum_{\sigma : K \hookrightarrow \mathbb{C}} \|e\|^2_\sigma$$

$$h^i_{\mathcal{D}}(\overline{E}) := h^i_{\mathcal{D}}(\pi_* \overline{E}), \quad i = 0, 1$$

$\overline{\omega} := (\omega, (\| \cdot \|_\omega))$ "canonical" hermitian line bundle on $\operatorname{Spec} \mathcal{O}_K$

$\omega := \operatorname{Hom}_{\mathbb{Z}}(\mathcal{O}_K, \mathbb{Z})$

$\| \operatorname{tr}_{K/\mathbb{Q}} \|_\omega := 1$

Then : $h^1_{\mathcal{D}}(\overline{E}) = h^0_{\mathcal{D}}(\overline{E}^\vee \otimes \overline{\omega})$

$$h^0_{\mathcal{D}}(\overline{E}) - h^1_{\mathcal{D}}(\overline{E}) = \hat{\deg}\, \overline{E} - \operatorname{rk} E \, \tfrac{1}{2} \log |D_K|$$

ii) classical results in the theory of euclidean lattices :

- coding cf. Conway - Sloane
- modular forms associated to integral lattices
- geometry of numbers

Banaszczyk Math. Ann. 296 1993
Gasbarri Topology 38 1999

iii) $$\int_{\mathbb{R}^N} e^{-\pi (x_1^2 + \dots + x_N^2)} \, dx_1 \dots dx_N = 1$$

## Facts :

1) $\hat{h}^0_B(\bar{E}) \in [0, +\infty[$

2) $-\pi \leq \hat{h}^0_B(\bar{E}) - \hat{h}^0(\bar{E}) \leq \frac{1}{2} r \log r + \log(1-\frac{1}{2^r})^{-1}$

$r := \mathrm{rk}\, E$

1) + 2) + "Riemann-Roch" $\Rightarrow$ Minkowski

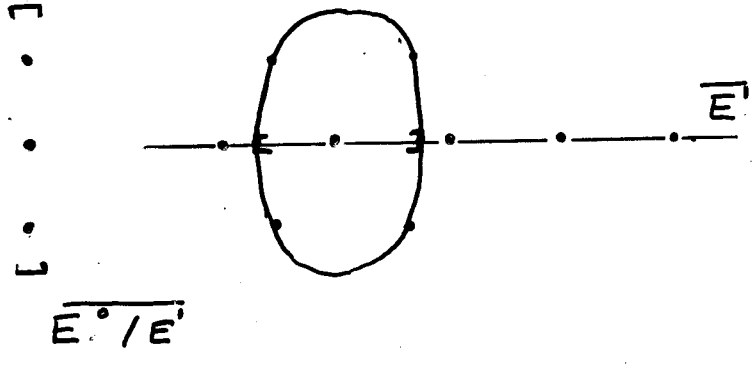
3) filtration $E = E^0 \supset E^1 \supset \dots \supset E^N = \{0\}$

$$\hat{h}^0_B(\bar{E}) \leq \sum_{i=1}^{N} \hat{h}^0_B(\overline{E^{i-1}/E^i})$$

$\Leftarrow$ $e^{-\pi \|x\|^2}$ of positive type

NB : . Gillet, Mazur, Soulé Bull. LMS 23 1991
"$\hat{h}^0$"; product ball, Blichfeldt
. does not hold for $\hat{h}^0$

$E = E^0 \supset E^1 \supset E^2 = \{0\}$




$\hat{h}^0(\bar{E}) = \log 5$ ; $\hat{h}^0(\overline{E^0/E^1}) = \log 3$

# Algebraization proofs

Kodaira - Spencer, GAGA (Serre)

"existence theorem in formal geometry" (Grothendieck)

basic argument :

$X$ smooth projective complex algebraic variety

$L$ analytic line bundle over $X$

Poincaré - Lefschetz : "$L$ is algebraic"

Cohomological approach of Kodaira - Spencer

$$Y \hookrightarrow X \hookrightarrow \mathbb{P}^N \qquad Y = \operatorname{div} s$$

$$\uparrow \text{hyperplane section} \qquad \mathcal{O}(1) \qquad s \in \Gamma(X, \mathcal{O}(1)) \setminus \{0\}$$

may assume $L_{|Y}$ algebraic

want to show $L \otimes \mathcal{O}(D)$ admits a non-zero analytic section for $D \gg 0$.

exact sequence of analytic sheaves:

$$0 \to L \otimes \mathcal{O}(D-1) \xrightarrow{s\cdot} L \otimes \mathcal{O}(D) \to L \otimes \mathcal{O}(D)_{|Y} \to 0$$

"large" if $D \gg$

$$0 \to H^0(X^{an}, L \otimes \mathcal{O}(D-1)) \xrightarrow{s^0_{tr}} H^0(X^{an}, L \otimes \mathcal{O}(D)) \to H^0(Y, L \otimes \mathcal{O}(D))$$

$$\hookrightarrow H^1(X^{an}, L \otimes \mathcal{O}(D-1)) \xrightarrow{s^1_{tr}} H^1(X^{an}, L \otimes \mathcal{O}(D)) \to H^1(Y, L \otimes \mathcal{O}(D))$$

$= 0$ if $D \gg 0$

key analytic point:

$$\dim_{\mathbb{C}} H^1(X^{an}, L \otimes \mathcal{O}(D)) < \infty$$

non increasing for $D \gg 0$ ...
stationary " "
$s^1_{tr}$ iso " "

algebraization of $\widetilde{\xi}$ vector bundle over $\widetilde{\mathcal{X}}$ :

$$\overline{H^1} := \left( H^1(\widetilde{\mathcal{X}}, \widetilde{\xi} \otimes \mathcal{O}(D)), \|\cdot\|_{L^2} \right)$$

↑ Hodge theory over $\Omega(\mathbb{C})$

- $H^1(\hat{\mathcal{X}}, \hat{\xi} \otimes \mathcal{O}(D))$ and $H^1(\Omega(\mathbb{C}), \xi^{an} \otimes \mathcal{O}(D))$ have in general infinite rank ($\Omega(\mathbb{C})$ is pseudo-concave), but are "well behaved" (e.g. $H^1(\Omega(\mathbb{C}), \xi^{an} \otimes \mathcal{O}(D))$ is separated)

$$\sum_{v \in H^1} e^{-\pi \|v\|^2_{L^2}} < \infty$$

in other words

$$h^0(\overline{H^1}) := \log \sum_{v \in H^1} e^{-\pi \|v\|^2_{L^2}}$$

is well defined in $[0, +\infty[$;

this plays the role of the finite dimensionality of $H^1(X, L \otimes \mathcal{O}(D))$ in Kodaira !