p-adic “hermitian” line bundles

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Motivation and setup

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$\mathcal{L}/X$ a line bundle

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When $K = \mathbb{C}$, a metric on $\mathcal{L}$ plays an important role in Arakelov Geometry
$[K : \mathbb{Q}_p] < \infty$,
$\log$ - choice of a p-adic log,
Q: What is the analogue of a metric in this case

Motivation: The theory of p-adic height pairings

Coleman integration

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Q: How to match local solutions?

Frobenius invariant path

A: Use Frobenius ϕ and impose

Condition: $\phi^*(y_0, y_2, \cdots)$ is a solution of the equation above with ω_i replaced by $\phi^*\omega_i$.

Technically:

- An equation above is a vector bundle with a unipotent connection. These form a Tannakian category.

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- An equation above is a vector bundle with a unipotent connection. These form a Tannakian category.
- Local solutions form fiber functors. Automorphisms of a fiber functor are loops. Isomorphisms between two fiber functors are paths between points.
- Frobenius acts on paths and loops. Coleman integration corresponds to extending local solutions along (unique) Frobenius invariant paths.

Coleman functions

Above gives
(M,∇) - an integrable connection on X. $\Rightarrow$ Canonical paralel translation $v_{x,y} : M_x \longrightarrow M_y$ for $x, y \in X(K)$, commuting with everything you can think of.

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Definition: An abstract Coleman function on X with values in $\mathcal{F}$ (coherent $\mathcal{O}_X$-module) is a fourtuple $(M,\nabla,(m_x \in M_x)_{x\in X(\bar{K})}, s)$ s.t.

- (M,∇) as before
- $v_{x,y}(m_x) = m_y$
- $s \in \mathrm{Hom}(M,\mathcal{F})$.

Coleman functions are connected components of the category of abstract Coleman functions.
They give rise to actual locally analytic functions.

Notations: $\mathcal{O}_{\mathrm{Col}}(X, \mathcal{F})$, $\mathcal{O}_{\mathrm{Col}}(X) := \mathcal{O}_{\mathrm{Col}}(X, \mathcal{O}_X)$,

$\Omega^i_{\mathrm{Col}}(X) := \mathcal{O}_{\mathrm{Col}}(X, \Omega^i_X)$.

Basic properties

Key property: The sequence

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathrm{Col}}(X) \xrightarrow{\mathrm{d}} \Omega^1_{\mathrm{Col}}(X) \xrightarrow{\mathrm{d}} \Omega^2_{\mathrm{Col}}(X)$$

is exact.

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Consequence: $\mathcal{O}_{\mathrm{Col}}(X)$ contains iterated integrals, e.g,

$$\int (\eta \cdot \int \omega)$$

Locally every Coleman function looks like this.

Base change: we can work over $\overline{\mathbb{Q}_p}$.

The p-adic $\bar{\partial}$ operator

Definition: $\mathcal{O}_{\mathrm{Col},1}(X,\mathcal{F}) \subset \mathcal{O}_{\mathrm{Col}}(X,\mathcal{F})$, subset of functions with $E_1 \subset M$, $E_2 = M/E_1$, E_i trivial.

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- This is well defined
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Theorem: There is a short exact sequence,

$$0 \to \mathcal{F}(X) \to \mathcal{O}_{\mathrm{Col},1}(X,\mathcal{F}) \xrightarrow{\bar{\partial}} H^{\otimes}_{\mathcal{F}}(X) .$$

If X is affine, then this sequence is exact on the right.

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Conversly, the (α_i) come from $H^{\otimes}_{\mathcal{F}}(X)$ if and only if $\Psi((\alpha_i))$ is in the image of $\cup$.

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$\mathcal{L}^* = (\mathrm{Tot}(\mathcal{L}) - 0) \xrightarrow{\pi} X$
Definition: A log function on $\mathcal{L}$ is $\log_{\mathcal{L}} \in \mathcal{O}_{\text{Col}}(\mathcal{L}^*)$ such that:

- On a fiber $\mathcal{L}_x$ we have $\log_{\mathcal{L}}(\alpha\ell) = \log(\alpha) + \log_{\mathcal{L}}(\ell)$
- $d\log_{\mathcal{L}} \in \mathcal{O}_{\text{Col},1}(\mathcal{L}^*, \Omega^1)$.

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Theorem: (X complete)

1. If $ch_1(\mathcal{L}) \in \mathrm{Im}(\cup : H^1_{\mathrm{dR}}(X) \otimes \Omega^1(X) \to H^2_{\mathrm{dR}}(X))$, then $\mathrm{Curve}(\log_{\mathcal{L}})$ exists and $\cup \mathrm{Curve}(\log_{\mathcal{L}}) = ch_1(\mathcal{L})$.
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Proof of 1: Choose an affine covering (U_i).
Local curvatures pull back from $\alpha_i \in H^{\otimes}(U_i)$
Computation: $\Psi((\alpha_i)) = c_1(\mathcal{L})$, hence the condition for Glueing the α_i.

Example: Green functions

$X/\overline{\mathbb{Q}_p}$ smooth complete curve.
We define a canonical log function on $O(\Delta)/X\times X$.
We fix splitting $H^1_{\mathrm{dR}}(X) = W \oplus \Omega^1(X)$.
$\pi_1,\ \pi_2 : X\times X \to X$
$\{\omega_1,\ldots,\omega_g\}$ a basis of $\Omega^1(X)$
$\{\bar{\omega}_1,\ldots,\bar{\omega}_g\}\subset W$, $\mathrm{tr}(\bar{\omega}_i\cup\omega_j)=\delta_{ij}$

$$\mu = \frac{1}{g}\sum_{i=1}^{g}\bar{\omega}_i\otimes\omega_i \in H^{\otimes}(X)\ ,$$

$$\Phi \in H^{\otimes}(X\times X)$$

$$\Phi = \pi_1^*\mu + \pi_2^*\mu - \sum_{i=1}^{g}(\pi_1^*\bar{\omega}_i\otimes\pi_2^*\omega_i + \pi_2^*\bar{\omega}_i\otimes\pi_1^*\omega_i)$$

$\cup\Phi = c_1(\mathcal{O}(\Delta)) \Rightarrow \mathcal{O}(\Delta)$ has a log function with curvature Φ (not canonical yet)

Set $G = \log_{\mathcal{O}(\Delta)}(1)$.

G can be made canonical (up to const) by imposing:

- $G(x,y) = G(y,x)$
- Residue condition.

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Then $F(-z) = -F(z) \Rightarrow F^2(-z) = F^2(z) \Rightarrow F^2$ descends to $\mathbb{P}^1$ but is not a Coleman function there.
I have no general theory for this kind of functions yet

Idea: Should consider connections which are extensions of torsion line bundles with connection.

Example

Consider $\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ s.t. $2(x, y) = (\phi(x), ?)$. $\deg \phi = 4$

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Let $\mathcal{L}' = \pi^* \mathcal{L}$

Easy: $\alpha = \omega \otimes [\eta]$ for any $\omega \in \Omega^1(E)$ and $[\eta]$ with $\omega \cup [\eta] = \deg \mathcal{L}'$ is a curvature form for $\mathcal{L}'$ such that $[2]^* \alpha = 4\alpha$

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$[2]^*$ acts by 2 on $\Omega^1(E) \Rightarrow \alpha$ lifts uniqly to a metric on $\mathcal{L}'$ such that $[2]^* \mathcal{L}' \cong (\mathcal{L}')^4$ is an isometry

The metric descends to $\mathbb{P}^1$

Hope: get analogues of the real theory, including equidistribution results

Higher rank bundles?

$\mathcal{V}$ - vector bundle on X.
Q: What should be a p-adic hermitian structure on $\mathcal{V}$?
First idea: $\mathcal{L}^*$ is just the principal bundle associated with $\mathcal{L}$.
$d\log$ is a connection form on $\mathcal{L}$.
$\mathcal{P}$ - Frame bundle associated with $\mathcal{V}$
Over $\mathbb{R}$ - $F(v_1, \cdots, v_n) := (\langle v_i, v_j \rangle)$.
F is positive definite $\Rightarrow \log(F)$ exists.
Q: If there a p-adic analogue?

Problem: $\log(F)$ does not seem to satisfy any reasonable differential equation.

More concretely: Can we recover the equation for the metric from the associated connection over $\mathbb{R}$?

For $\mathcal{L}$

t - A section of $\mathcal{L}$, $f = \langle t, t \rangle$, $\nabla t = \omega t$

$$df = 2\langle \nabla t, t \rangle = 2\omega f$$

So $d\log(f) = 2\omega$ and the equation for f factors via $\log(f)$.

For $\mathcal{V}$

t_i - local basis

$\nabla t_i = \sum \omega_{ij} t_j$

$F_{ij} = \langle t_i, t_j \rangle$

$$dF = F\Omega + \Omega^t F$$

Which no longer factors via $\log F$.

Secondary characteristic classes

$X/\mathbb{C}$
Real Deligne cohomology

$$H^i_{\mathcal{D}}(X,\mathbb{R}(n)) = H^i(X, MF(\Omega_X^{\geq n} \xrightarrow{z \to z \pm \bar{z}} C^\infty(X) \otimes \mathbb{R}(n-1)))$$

Hermitian metric on $\mathcal{V}$ is $\mathcal{V} \cong \overline{\mathcal{V}^*}$ (does not incoporate positivity).

∇ a connection on $\mathcal{V} \Rightarrow \overline{\nabla^*}$ on $\mathcal{V}$
$\mathrm{tr}(\mathrm{Curve}\,\overline{\nabla^*})^n) = \pm \overline{\mathrm{tr}(\mathrm{Curve}\,\nabla)^n)} \Rightarrow$ can use transgression to obtain $c_n(\mathcal{V}) \in H^{2n}_{\mathcal{D}}(X,\mathbb{R}(n))$.

Conclusion: a Hermitian metric is a tool to compute characteristic classes in Deligne cohomology.

Hope: p-adic Hermitian metric is a tool to compute characteristic classes in syntomic cohomology.

Syntomic cohomology

$X/\mathcal{O}_K$
$\phi : X \to X$ a lift of Frobenius of degree q.
$DR^\bullet$ - a complex computing de Rham cohomology for X_K/K
$H^i_{\text{syn}}(X,n) := H^i(MF((\phi^* - q^n) : F^n DR^\bullet \to DR^\bullet))$
Characteristic classes $c_n : K_0(X) \to H^{2n}_{\text{syn}}(X,n)$.

c_1 for line bundles

$\{U_i\}$ - covering of X
$\mathcal{L}/X$ associated with a cocycle (g_{ij})
$DR^\bullet$ - Cech complex for $\{U_i\}$
$\phi : X \to X$ fixes $\{U_i\}$
Then $c_1(\mathcal{L})$ is represented by

$$\left[(d \log(g_{ij}), (\log(\phi^* g_{ij}/g_{ij}^q)) \right]$$

Key observation: $\phi^* g_{ij}/g_{ij}^q \equiv 1 \pmod p$

Syntomic transgression

We try to mimic previous considerations in Deligne cohomology

ϕ^* on cohomology comes from ϕ^* on $\mathcal{V}$

q^n on $H^i(\bullet, n)$ comes from ϕ_q - the q's Adams operation.

Can't expect $\phi^* = \psi_q$

But this is true in characteristic p

Idea: a metric on $\mathcal{V}$ is a deformation of $\phi^*\mathcal{V}$ to $\psi_q\mathcal{V}$

Having that we can deform the connection and obtain the required transgression

A toy example: line bundles again

In previous setup Chose $s_i \in \mathcal{L}(U_i)$ s.t. $s_i/s_j = g_{ij}$
$\psi_q \mathcal{L} = \mathcal{L}^q$
$\phi^* \mathcal{L} / \mathcal{L}^q$ given by cocycle $(h_{ij} := \phi^* g_{ij}/g_{ij}^q)$
Deform $\phi^* \mathcal{L} / \mathcal{L}^q$ to trivial bundle via family $\mathcal{L}_t$ with cocycles (h_{ij}^t) where t goes from 1 to 0.

Note: by the congruence on h_{ij} this is well defined.

Deformation of connection

$\nabla = (\nabla_i)$ - ∇_i a connection on $\mathcal{L}/U_i$

"Curvature" - $[(\nabla_i^2), (\nabla_i - \nabla_j)]$.

E.g. Define ∇_i by $\nabla_i(s_i) = 0$

Curvature is $[(0), (d\log(s_i/s_j))]$ - represents de Rham c_1.

Deformation - Take the same definition for each ∇_t on $\mathcal{L}_t$.

Transgression

$$d/dt\,\mathrm{Curve}\,\nabla_t = d/dt(d\log h_{ij}^t) = d(\log(h_{ij}))$$

So transgression gives the right class.