On Risk Measures via Gaussian Distributions under Model Uncertainty

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Why normal distributions $\mathcal{N}(\mu, \sigma^2)$ are so widely used?

People use everywhere in finance normal distributions. If someone faces a random variable X of which the distribution is difficult to get, he will first try normal distribution $\mathcal{N}(\mu, \sigma^2)$.

Explanation for these phenomenons is the well-known central limit theorem:

Theorem (Central Limit Theorem (CLT))

$\{X_i\}_{i=1}^{\infty}$ is assumed to be i.i.d. with $\mu = E[X_1]$ and $\sigma^2 = E[(X_1-\mu)^2]$. Then for each bounded and continuous function $\varphi \in C(\mathbb{R})$, we have

$$\lim_{i\to\infty} \mathbb{E}[\varphi(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i-\mu))] = E[\varphi(X)], \quad X \sim \mathcal{N}(0,\sigma^2).$$

The beauty and power of this result come from: the above sum tends to $\mathcal{N}(0,\sigma^2)$ regardless the original distribution of X_i, provided that $X_i \sim X_1$, for all $i = 2, 3, \cdots$ and that $X_1, X_2, \cdots$ are mutually independent.

History of CLM

- Abraham de Moivre 1733,
- Pierre-Simon Laplace, 1812: Théorie Analytique des Probabilité
- Aleksandr Lyapunov, 1901
- Cauchy's, Bessel's and Poisson's contributions, von Mises, Polya, Lindeberg, Lévy, Cramer

‘dirty work’ through “dirty data”?

But in, real world in finance, or in any other human science situation, it is not so often to see and to check that the above $\{X_i\}_{i=1}^{\infty}$ is really i.i.d. Many academic people think that people in finance just widely and deeply abuse this beautiful mathematical result to do ‘dirty work’ through “dirty data”

Explanation and via a new CLT under Distribution Uncertainty

We will largely weaken the above i.i.d. assumption: not only we do not know the distribution of X_i, in fact we don't assume $X_i \sim X_j$, they may have difference unknown distributions. We only assume that the distribution of X_i, $i = 1, 2, \cdots$ are within some subset of distribution functions

$$\mathcal{L}(X_i) \in \{F_\theta(x) : \theta \in \Theta\}.$$

This assumption evidently more realistic.

In the situation of distributional uncertainty and/or probability uncertainty (or model uncertainty) the problem of decision become more complicated. A well-accepted method is the following robust calculation:

$$\sup_{\theta\in\Theta} E_\theta[\varphi(\xi)], \ \inf_{\theta\in\Theta} E_\theta[\varphi(\eta)]$$

and then to compare their values, where E_θ represent the expectation of a possible probability in our uncertainty model.

Our basic mathematical tool:

$$\hat{\mathbb{E}}[X] = \sup_{\theta\in\Theta} E_\theta[X], \quad \textbf{it is easy to check } \hat{\mathbb{E}}[X] : \mathcal{H} \to \mathbb{R}$$

has the properties a)-d):

a) $X \geq Y$ then $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$
b) $\hat{\mathbb{E}}[c] = c$
c) $\hat{\mathbb{E}}[X+Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$
d) $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X], \lambda \geq 0$.

This implies that, conversely, for each sublinear expectation also corresponds a uncertainty subset of probabilities.

The language of uncertainty subset of probability is equivalently to the language of the corresponding sublinear expectation.

For the distribution uncertainty of two random variables X and Y, if for each real function we always have

$$\hat{\mathbb{E}}[\varphi(X)] = \hat{\mathbb{E}}[\varphi(Y)],$$

Then the distribution uncertainties of X and Y is the same. If

$$\hat{\mathbb{E}}[\varphi(X)] \geq \hat{\mathbb{E}}[\varphi(Y)]$$

Then the distribution uncertainty of X is stronger than that of Y.

$$\hat{\mathbb{E}}[\varphi(X)] = \sup_{\theta \in \Theta} \int_{-\infty}^{\infty} \varphi(x) dF_\theta(x)$$

$F_\theta(x)$, $\theta \in \Theta$: the distribution uncertainty subset of X.

Our new central limit theorem under distribution uncertainty by using the language of sublinear expectation.

We are given a sequence $\{X_i\}_{i=1}^{\infty}$ under a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ (meaning that we are in a random world where we have a uncertainty subset of probabilities). We assume that $\{X_i\}_{i=1}^{\infty}$ are identically distributed under $\hat{\mathbb{E}}$:

$$\hat{\mathbb{E}}[\varphi(X_i)] = \hat{\mathbb{E}}[\varphi(X_1)], \quad \forall \varphi, \quad i = 2, 3, \cdots$$

meaning that X_i are within a subset of distributions (uncertainty distribution subset).

Space of random variables

- $(\Omega, \mathcal{F})$: a measurable space;
- $\mathcal{H}$ a linear space of random variables ($\mathcal{F}$-meas. functions on Ω) s.t.

$$X_1, \cdots, X_n \in \mathcal{H} \Rightarrow \varphi(X) \in \mathcal{H}, \quad \forall \varphi \in C_b(\mathbb{R}^n)$$

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Definition

A nonlinear expectation: a functional $\hat{\mathbb{E}}:\mathcal{H}\mapsto\mathbb{R}$

- (a) Monotonicity: if $X\geq Y$ then $\hat{\mathbb{E}}[X]\geq\hat{\mathbb{E}}[Y]$.
- (b) Constant preserving: $\hat{\mathbb{E}}[c]=c$.

A sublinear expectation:

- (c) Sub-additivity (or self–dominated property):

$$\hat{\mathbb{E}}[X]-\hat{\mathbb{E}}[Y]\leq\hat{\mathbb{E}}[X-Y].$$

- (d) Positive homogeneity: $\hat{\mathbb{E}}[\lambda X]=\lambda\hat{\mathbb{E}}[X],\quad\forall\lambda\geq 0.$

Sublinear Expectation

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$\mathcal{H}$ is the set of bounded and measurable random variables on $(\Omega, \mathcal{F})$.

$$\rho(X) := \hat{\mathbb{E}}[-X]$$

Definition–Coherent risk measure

$\rho(X) : \mathcal{H} \mapsto \mathbb{R}$ is a coherent risk measure if it satisfies:

- (a) Monotonicity: if $X \geq Y$ then $\rho[X] \leq \rho[Y]$.
- (b) Constant translatability: $\rho[X + c] = \rho[X] - c$.
- (c) Convexity: (or self–dominated property):

$$\rho[\alpha X + (1-\alpha)Y] \leq \alpha\rho[X] + (1-\alpha)\rho[Y].$$

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Robust representation of sublinear expectations

- Huber Robust Statistics (1981).
- Artzner, Delbean, Eber & Heath (1999)
- Föllmer & Schied (2004)

Theorem

$\hat{\mathbb{E}}[\cdot]$ is a sublinear expectation on $\mathcal{H}$ if and only if there exists a subset $\mathcal{P} \in \mathcal{M}_{1,f}$ (the collection of all finitely additive probability measures) such that

$$\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} E_P[X], \quad \forall X \in \mathcal{H}.$$

(For interest rate uncertainty, see Barrieu & El Karoui (2005)).

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Meaning of the robust representation: Statistic model uncertainty

$$\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} E_P[X], \quad \forall X \in \mathcal{H}.$$

The size of the subset $\mathcal{P}$ represents the degree of model uncertainty: The stronger the $\hat{\mathbb{E}}$ the more the uncertainty

$$\hat{\mathbb{E}}_1[X] \geq \hat{\mathbb{E}}_2[X], \quad \forall X \in \mathcal{H} \iff \mathcal{P}_1 \supset \mathcal{P}_2$$

The distribution of a random vector in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$

Definition (Distribution of X)

Given $X = (X_1, \cdots, X_n) \in \mathcal{H}^n$. We define:

$$\hat{\mathbb{F}}_X[\varphi] := \hat{\mathbb{E}}[\varphi(X)] : \varphi \in C_b(\mathbb{R}^n) \longmapsto \mathbb{R}.$$

We call $\hat{\mathbb{F}}_X[\cdot]$ the distribution of X under $\hat{\mathbb{E}}$.

Fact

$\mathbb{F}_X[\cdot]$ *forms a sublinear expectation on* $C_b(\mathbb{R}^n)$, *thus*

$$\mathbb{F}_X[\varphi] = \sup_{\theta \in \Theta} \int_{\mathbb{R}^n} \varphi(x) F_\theta(dy).$$

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Definition

The distribution of X is said to be stronger than Y if

$$\hat{\mathbb{E}}[\varphi(X)] \geq \hat{\mathbb{E}}[\varphi(Y)], \quad \forall \varphi \in C_b(\mathbb{R}^n).$$

Definition

X,Y are said to be identically distributed, ($X \sim Y$, or X is a copy of Y), if they have same distributions:

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Remark.

X is stronger than Y in distribution means that the uncertainty of X is bigger that Y. $X \sim Y$ means they have the same degree of uncertainty.

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Independence under $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$

Definition

A m-dim. random vector Y is said to be independent a n-dim. vector $X = (X_1, \cdots, X_n)$ if for each $\varphi \in C_b(\mathbb{R}^n \times \mathbb{R}^m)$ such that $\varphi(X, Y) \in \mathcal{H}_1$, we have:

$$\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=Y}].$$

Fact

Meaning: the realization of X does not change (improve) the distribution uncertainty of Y.

Lemma

Given the distributions X,Y, we can make copies $\bar{X}$ and $\bar{Y}$ of X and Y such that $\bar{Y}$ is independent to $\bar{X}$. The distribution of and $(\bar{X}, \bar{Y})$ is uniquely determined.

Fact

The computational complexity of $\hat{\mathbb{E}}[\varphi(X_1, \cdots, X_k)]$ will enormously reduced if X_{i+1} is independent to X_i for each i:

From order m^k to $k \times m$.

Independence under $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ can be subjective

Example

X, Z: two r.v. in $(\Omega, \mathcal{F}, P)$,
$Y = h(\eta(X), Z)$, Z is independent to X under P.
About the function $\eta(x)$: we only know $\eta(x) \in \Theta$.
The robust expectation of $\varphi(X, Y)$ is:

$$\hat{\mathbb{E}}[\varphi(X,Y)] := E_P\left[\left\{\sup_{\theta\in\Theta} E_P[\varphi(x, h(\theta, Z))]\right\}_{x=X}\right].$$

Y is not independent to X w.r.t. P
but is independent under $\hat{\mathbb{E}}$.

Example (An extreme example)

In reality, $Y = X$
but we are in the very beginning
and we know noting about the relation of X and Y,
the only information we know is $X, Y \in \Theta$.
The robust expectation of $\varphi(X, Y)$ is:

$$\hat{\mathbb{E}}[\varphi(X, Y)] = \sup_{x,y \in \Theta} \varphi(x, y).$$

Y is independent to X, X is also independent to Y.

The notion of independence

Fact

Y is independent to X DOES NOT IMPLIES
X is independent to Y

Example

- $\overline{\sigma}^2 := \hat{\mathbb{E}}[Y^2] > \underline{\sigma}^2 := -\hat{\mathbb{E}}[-Y^2] > 0, \quad \hat{\mathbb{E}}[X] = \hat{\mathbb{E}}[-X] = 0.$
  Then
- If Y is independent to X:
$$\hat{\mathbb{E}}[XY^2] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[xY^2]_{x=X}] = \hat{\mathbb{E}}[X^+\overline{\sigma}^2 - X^-\underline{\sigma}^2]$$
$$= \hat{\mathbb{E}}[X^+](\overline{\sigma}^2 - \underline{\sigma}^2) > 0.$$
- But if X is independent to Y:
$$\hat{\mathbb{E}}[XY^2] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[X]Y^2] = 0.$$

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We also assume that X_{i+1} is independent to $\{X_1, \cdots, X_i\}$ under $\hat{\mathbb{E}}$. The notion of independence is defined as follows:

Definition

Given to random vectors X,Y. Y is said to be independent of X if

$$\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}]$$

The meaning of this independence: the realization of value of X does not change the distribution uncertainty of Y.

The conditional distribution of $\varphi(X,Y)$ knowing X is:

$$\hat{\mathbb{E}}[\varphi(X,Y)|X] = \hat{\mathbb{E}}[\varphi(x,Y)]_{x=X}.$$

From our new central limit theorem

Proposition. (CLT) Let $\{X_i\}_{i=1}^{\infty}$ be a i.i.d. in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ in the following sense:

(i) identically distributed:

$$\hat{\mathbb{E}}[\varphi(X_i)] = \hat{\mathbb{E}}[\varphi(X_1)], \quad \forall i = 1, 2, \cdots$$

(ii) independent: for each i, X_{i+1} is independent to $(X_1, X_2, \cdots, X_i)$ under $\hat{\mathbb{E}}$.

We also assume that $\hat{\mathbb{E}}[X_1] = -\hat{\mathbb{E}}[-X_1]$, then we denote

$$\overline{\sigma}^2 = \hat{\mathbb{E}}[X_1^2], \quad \underline{\sigma}^2 = -\hat{\mathbb{E}}[-X_1^2]$$

Then for each convex function φ we have

$$\hat{\mathbb{E}}[\varphi(\frac{S_n}{\sqrt{n}})] \to \frac{1}{\sqrt{2\pi\overline{\sigma}^2}} \int_{-\infty}^{\infty} \varphi(x) \exp(\frac{-x^2}{2\overline{\sigma}^2}) dx$$

and for each concave function ψ we have

$$\hat{\mathbb{E}}[\psi(\frac{S_n}{\sqrt{n}})] \to \frac{1}{\sqrt{2\pi\underline{\sigma}^2}} \int_{-\infty}^{\infty} \psi(x) \exp(\frac{-x^2}{2\underline{\sigma}^2}) dx.$$

An fundamentally important sublinear distribution

Definition

A random variable X in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called normally distributed if

$$aX + b\bar{X} \sim \sqrt{a^2 + b^2}X, \quad \forall a, b \geq 0.$$

where $\bar{X}$ is an independent copy of X.

- We have $\hat{\mathbb{E}}[X] = \hat{\mathbb{E}}[-X] = 0$.
- We also denote $X \sim \mathcal{N}(0, [\underline{\sigma}^2, \overline{\sigma}^2])$, where

$$\overline{\sigma}^2 := \hat{\mathbb{E}}[X^2], \quad \underline{\sigma}^2 := -\hat{\mathbb{E}}[-X^2].$$

Normal distribution under Sublinear expectation

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G-normal distribution: a under sublinear expectation $\mathbb{E}[\cdot]$

- **(1)** For each **convex** φ, we have

$$\hat{\mathbb{E}}[\varphi(X)] = \frac{1}{\sqrt{2\pi\overline{\sigma}^2}} \int_{-\infty}^{\infty} \varphi(y) \exp(-\frac{y^2}{2\overline{\sigma}^2}) dy$$

- **(2)** For each **concave** φ, we have,

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Fact

If $\underline{\sigma}^2 = \overline{\sigma}^2$, *then* $\mathcal{N}(0; [\underline{\sigma}^2, \overline{\sigma}^2]) = \mathcal{N}(0, \overline{\sigma}^2)$.

Fact

The larger to $[\underline{\sigma}^2, \overline{\sigma}^2]$ *the stronger the uncertainty.*

Fact

But the uncertainty subset of $\mathcal{N}(0; [\underline{\sigma}^2, \overline{\sigma}^2])$ *is not just consisting of*

$$\mathcal{N}(0; \sigma), \quad \sigma \in [\underline{\sigma}^2, \overline{\sigma}^2]!!$$

Theorem

$X \sim \mathcal{N}(0, [\underline{\sigma}^2, \overline{\sigma}^2])$ in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ iff for each $\varphi \in C_b(\mathbb{R})$ the function

$$u(t,x) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X)], \quad x \in \mathbb{R}, \quad t \geq 0$$

is the solution of the PDE

$$u_t = G(u_{xx}), \quad t > 0, \quad x \in \mathbb{R}$$
$$u|_{t=0} = \varphi,$$

where $G(a) = \frac{1}{2}(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-)(= \hat{\mathbb{E}}[\frac{a}{2}X^2])$. G-normal distribution.

Law of Large Numbers (LLN), Central Limit Theorem (CLT)

Striking consequence of LLN & CLT

Accumulated independent and identically distributed random variables tends to a normal distributed random variable, whatever the original distribution.

LLN under Choquet capacities:

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Definition

A sequence of random variables $\{\eta_i\}_{i=1}^{\infty}$ in $\mathcal{H}$ is said to converge in law under $\hat{\mathbb{E}}$ if the limit

$$\lim_{i\to\infty} \hat{\mathbb{E}}[\varphi(\eta_i)], \quad \text{for each } \varphi \in C_b(\mathbb{R}).$$

Theorem

Let $\{X_i\}_{i=1}^{\infty}$ in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ be identically distributed: $X_i \sim X_1$, and each X_{i+1} is independent to $(X_1, \cdots, X_i)$. We assume furthermore that

$$\hat{\mathbb{E}}[|X_1|^{2+\alpha}] < \infty \text{ and } \hat{\mathbb{E}}[X_1] = \hat{\mathbb{E}}[-X_1] = 0.$$

$S_n := X_1 + \cdots + X_n$. Then $S_n/\sqrt{n}$ converges in law to $\mathcal{N}(0; [\underline{\sigma}^2, \overline{\sigma}^2])$:

$$\lim_{n\to\infty} \hat{\mathbb{E}}[\varphi(\frac{S_n}{\sqrt{n}})] = \hat{\mathbb{E}}[\varphi(X)], \ \forall \varphi \in C_b(\mathbb{R}),$$

where

$$\text{where } X \sim \mathcal{N}(0, [\underline{\sigma}^2, \overline{\sigma}^2]), \quad \overline{\sigma}^2 = \hat{\mathbb{E}}[X_1^2], \ \underline{\sigma}^2 = -\hat{\mathbb{E}}[-X_1^2].$$

Sketch of Proof: A new method

For a function $\varphi \in C_{Lip}(\mathbb{R})$ and a small but fixed $h > 0$, let V be the solution of the PDE on $(t,x) \in [0,1] \times \mathbb{R}$,

$$\partial_t V + G(\partial^2_{xx} V) = 0,$$
$$V|_{t=1} = \varphi.$$

We have, since $X \sim \mathcal{N}(0, [\underline{\sigma}^2, \overline{\sigma}^2])$,

$$V(t,x) = \hat{\mathbb{E}}[\varphi(x + \sqrt{1-t}X)].$$

Particularly,

$$V(0,0) = \hat{\mathbb{E}}[\varphi(X)], \quad V(1,x) = \varphi(x).$$

$\cdots$ Sketch of Proof $\delta = 1/n$. Thus $\hat{\mathbb{E}}[\varphi(\sqrt{\delta}S_n)] - \hat{\mathbb{E}}[\varphi(X)]$ equals

$$\hat{\mathbb{E}}[V(1,\sqrt{\delta}S_n) - V(0,0)] = \hat{\mathbb{E}}\sum_{i=0}^{n-1}\{V((i+1)\delta,\sqrt{\delta}S_{i+1}) - V(i\delta,\sqrt{\delta}S_i)\}$$

$$\text{by Taylor's expansion} \quad = \hat{\mathbb{E}}\sum_{i=0}^{n-1}(I_\delta^i + J_\delta^i), \quad \hat{\mathbb{E}}[|J_\delta^i|] \le C\delta^{1+\alpha}$$

$$\begin{aligned} I_\delta^i = \quad & \partial_t V(i\delta,\sqrt{\delta}S_i)\delta + \tfrac{1}{2}\partial_{xx}^2 V(i\delta,\sqrt{\delta}S_i)X_{i+1}^2\delta \\ & +\partial_x V(i\delta,\sqrt{\delta}S_i)X_{i+1}\sqrt{\delta}. \end{aligned}$$

We have

$$\begin{aligned} \hat{\mathbb{E}}[I_\delta^i] &= \hat{\mathbb{E}}[\partial_t V(i\delta,\sqrt{\delta}S_i) + \frac{1}{2}\partial_{xx}^2 V(i\delta,\sqrt{\delta}S_i)X_{i+1}^2]\delta \\ &= \hat{\mathbb{E}}[\{\partial_t V + G(\partial_{xx}^2 V)\}(i\delta,\sqrt{\delta}S_i)]\delta = 0 \end{aligned}$$

Cases with mean-uncertainty

What happens if $\hat{\mathbb{E}}[X_1] > -\hat{\mathbb{E}}[-X_1]$?

Definition

A random variable Y in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is $\mathcal{U}([\underline{\mu}, \overline{\mu}])$-distributed $(Y \sim \mathcal{U}([\underline{\mu}, \overline{\mu}]))$ if

$$aY + b\bar{Y} \sim (a+b)Y, \quad \forall a, b \geq 0.$$

where $\bar{Y}$ is an independent copy of Y, where $\overline{\mu} := \hat{\mathbb{E}}[Y] > \underline{\mu} := -\hat{\mathbb{E}}[-Y]$

- We can prove that

$$\hat{\mathbb{E}}[\varphi(Y)] = \sup_{y \in [\underline{\mu}, \overline{\mu}]} \varphi(y).$$

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Definition

A pair of random variables (X, Y) in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is $\mathcal{N}([\underline{\mu}, \overline{\mu}], [\underline{\sigma}^2, \overline{\sigma}^2])$-distributed ($(X, Y) \sim \mathcal{N}([\underline{\mu}, \overline{\mu}], [\underline{\sigma}^2, \overline{\sigma}^2])$) if

$$(aX + b\bar{X}, a^2 Y + b^2 \bar{Y}) \sim (\sqrt{a^2 + b^2} X, (a^2 + b^2) Y), \quad \forall a, b \geq 0.$$

where $(\bar{X}, \bar{Y})$ is an independent copy of (X, Y),

$$\overline{\mu} := \hat{\mathbb{E}}[Y], \ \underline{\mu} := -\hat{\mathbb{E}}[-Y]$$
$$\overline{\sigma}^2 := \hat{\mathbb{E}}[X^2], \ \underline{\sigma}^2 := -\hat{\mathbb{E}}[-X], \ \ (\hat{\mathbb{E}}[X] = \hat{\mathbb{E}}[-X] = 0).$$

Theorem

$(X,Y) \sim \mathcal{N}([\underline{\mu},\overline{\mu}],[\underline{\sigma}^2,\overline{\sigma}^2])$ in $(\Omega,\mathcal{H},\hat{\mathbb{E}})$ iff for each $\varphi \in C_b(\mathbb{R})$ the function

$$u(t,x,y) := \hat{\mathbb{E}}[\varphi(x+\sqrt{t}X, y+tY)], \quad x \in \mathbb{R}, \quad t \geq 0$$

is the solution of the PDE

$$\begin{aligned} u_t &= G(u_y, u_{xx}), \quad t > 0, \quad x \in \mathbb{R} \\ u|_{t=0} &= \varphi, \end{aligned}$$

where

$$G(p,a) := \hat{\mathbb{E}}[\frac{a}{2}X^2 + pY].$$

Central Limit Theorem under Sublinear Expectation

Theorem

Let $\{X_i + Y_i\}_{i=1}^{\infty}$ *be an independent and identically distributed sequence. We assume furthermore that*

$$\hat{\mathbb{E}}[|X_1|^{2+\alpha}] + \hat{\mathbb{E}}[|Y_1|^{1+\alpha}] < \infty \text{ and } \hat{\mathbb{E}}[X_1] = \hat{\mathbb{E}}[-X_1] = 0.$$

$S_n^X := X_1 + \cdots + X_n$, $S_n^Y := Y_1 + \cdots + Y_n$. *Then* $S_n/\sqrt{n}$ *converges in law to* $\mathcal{N}(0; [\underline{\sigma}^2, \overline{\sigma}^2])$:

$$\lim_{n\to\infty} \hat{\mathbb{E}}[\varphi(\frac{S_n^X}{\sqrt{n}} + \frac{S_n^Y}{n})] = \hat{\mathbb{E}}[\varphi(X+Y)], \ \forall \varphi \in C_b(\mathbb{R}),$$

where (X, Y) *is* $\mathcal{N}([\underline{\mu}, \overline{\mu}], [\underline{\sigma}^2, \overline{\sigma}^2])$*-distributed.*

Definition

Under $(\Omega, \mathcal{F}, \hat{\mathbb{E}})$, a process $B_t(\omega) = \omega_t$, $t \geq 0$, is called a G–**Brownian motion** if:

- (i) $B_{t+s} - B_s$ is $\mathcal{N}(0, [\underline{\sigma}^2 t, \overline{\sigma}^2 t])$ distributed $\forall\ s, t \geq 0$
- (ii) For each $t_1 \leq \cdots \leq t_n$, $B_{t_n} - B_{t_{n-1}}$ is independent to $(B_{t_1}, \cdots, B_{t_{n-1}})$.
- For simplification, we set $\overline{\sigma}^2 = 1$.

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Theorem

If, under some $(\Omega, \mathcal{F}, \hat{\mathbb{E}})$, a stochastic process $B_t(\omega), t \geq 0$ satisfies

- For each $t_1 \leq \cdots \leq t_n$, $B_{t_n} - B_{t_{n-1}}$ is independent to $(B_{t_1}, \cdots, B_{t_{n-1}})$.
- B_t is identically distributed as $B_{s+t} - B_s$, for all $s, t \geq 0$
- $\hat{\mathbb{E}}[|B_t|^3] = o(t)$.
- Then B is a G-Brownian motion.

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Fact

Like $\mathcal{N}(0,[\underline{\sigma}^2,\overline{\sigma}^2])$-distribution, the G-Brownian motion $B_t(\omega)=\omega_t$, $t\geq 0$, can strongly correlated under the unknown 'objective probability', it can even be have very long memory. But it is i.i.d under the robust expectation $\hat{\mathbb{E}}$. By which we can have many advantages in analysis, calculus and computation, compare with, e.g. fractal B.M.

Itô's integral of G–Brownian motion

For each process $(\eta_t)_{t\geq 0} \in \mathbb{L}_{\mathcal{F}}^{2,0}(0,T)$ of the form

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega)\mathbf{I}_{[t_j,t_{j+1})}(t), \quad \xi_j \in \mathbb{L}^2(\mathcal{F}_{t_j}) \ (\mathcal{F}_{t_j}\text{-meas. } \& \ \hat{\mathbb{E}}[|\xi_j|^2] < \infty)$$

we define

$$I(\eta) = \int_0^T \eta(s)dB_s := \sum_{j=0}^{N-1} \xi_j(B_{t_{j+1}} - B_{t_j}).$$

Lemma

We have

$$\hat{\mathbb{E}}[\int_0^T \eta(s)dB_s] = 0$$

and

$$\hat{\mathbb{E}}[(\int_0^T \eta(s)dB_s)^2] \leq \int_0^T \hat{\mathbb{E}}[(\eta(t))^2]dt.$$

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Definition

Under the Banach norm $\|\eta\|^2 := \int_0^T \hat{\mathbb{E}}[(\eta(t))^2]dt$,

$$I(\eta) : \mathbb{L}^{2,0}(0,T) \mapsto \mathbb{L}^2(\mathcal{F}_T) \quad \text{is a contract mapping}$$

We then extend $I(\eta)$ to $\mathbb{L}^2(0,T)$ and define, the stochastic integral

$$\int_0^T \eta(s)dB_s := I(\eta), \quad \eta \in \mathbb{L}^2(0,T).$$

Lemma

We have

(i) $\int_s^t \eta_u dB_u = \int_s^r \eta_u dB_u + \int_r^t \eta_u dB_u$.

(ii) $\int_s^t (\alpha\eta_u + \theta_u) dB_u = \alpha \int_s^t \eta_u dB_u + \int_s^t \theta_u dB_u, \ \ \alpha \in \mathbb{L}^1(\mathcal{F}_s)$

(iii) $\hat{\mathbb{E}}[X + \int_r^T \eta_u dB_u | \mathcal{H}_s] = \hat{\mathbb{E}}[X]$, $\forall X \in \mathbb{L}^1(\mathcal{F}_s)$.

Quadratic variation process of G–BM

We denote:

$$\langle B \rangle_t = B_t^2 - 2\int_0^t B_s dB_s = \lim_{\max(t_{k+1}-t_k)\to 0} \sum_{k=0}^{N-1} (B_{t^N_{k+1}} - B_{t_k})^2$$

$\langle B \rangle$ is an increasing process called **quadratic variation process** of B.

$$\hat{\mathbb{E}}[\langle B \rangle_t] = \overline{\sigma}^2 t \quad \text{but} \quad \hat{\mathbb{E}}[-\langle B \rangle_t] = -\underline{\sigma}^2 t$$

Lemma

$B_t^s := B_{t+s} - B_s,\ t \geq 0$ is still a G-Brownian motion. We also have

$$\langle B \rangle_{t+s} - \langle B \rangle_s \equiv \langle B^s \rangle_t \sim \mathcal{U}([\underline{\sigma}^2 t, \overline{\sigma}^2 t]).$$

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We have the following isometry

$$\hat{\mathbb{E}}[(\int_0^T \eta(s)dB_s)^2] = \hat{\mathbb{E}}[\int_0^T \eta^2(s)d\langle B\rangle_s],$$

$$\eta \in M_G^2(0,T)$$

Itô's formula for G–Brownian motion

$$X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \eta_s d\langle B\rangle_s + \int_0^t \beta_s dB_s$$

Theorem.

Let α, β and η be process in $L_G^2(0,T)$. Then for each $t \geq 0$ and in $L_G^2(\mathcal{H}_t)$ we have

$$\begin{aligned}\Phi(X_t) = \Phi(X_0) &+ \int_0^t \Phi_x(X_u)\beta_u dB_u + \int_0^t \Phi_x(X_u)\alpha_u du \\ &+ \int_0^t [\Phi_x(X_u)\eta_u + \frac{1}{2}\Phi_{xx}(X_u)\beta_u^2] d\langle B\rangle_u\end{aligned}$$

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Stochastic differential equations

Problem

We consider the following SDE:

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t h(X_s)d\langle B\rangle_s + \int_0^t \sigma(X_s)dB_s, \ \ t > 0.$$

where $X_0 \in \mathbb{R}^n$ *is given*

$$b, h, \sigma : \mathbb{R}^n \mapsto \mathbb{R}^n \ \ \text{are given Lip. functions.}$$

The solution: a process $X \in M_G^2(0, T; \mathbb{R}^n)$ *satisfying the above SDE.*

Theorem

There exists a unique solution $X \in M_G^2(0, T; \mathbb{R}^n)$ *of the stochastic differential equation.*

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Prospectives

- Risk measures and pricing under dynamic volatility uncertainties ([A-L-P1995], [Lyons1995]) —for path dependent options;
- Stochastic (trajectory) analysis of sublinear and/or nonlinear Markov process.
- New Feynman-Kac formula for fully nonlinear PDE: path-interpretation.

$$u(t,x) = \hat{\mathbb{E}}_x[\varphi(B_t)\exp(\int_0^t c(B_s)ds)]$$

$$\partial_t u = G(D^2 u) + c(x)u, \quad u|_{t=0} = \varphi(x).$$

- Fully nonlinear Monte-Carlo simulation.
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Thank you,

In the classic period of Newton's mechanics, including A. Einstein, people believe that everything can be deterministically calculated. The last century's research affirmatively claimed the probabilistic behavior of our universe: God does plays dice!

Nowadays people believe that everything has its own probability distribution. But a deep research of human behaviors shows that for everything involved human or life such, as finance, this may not be true: a person or a community may prepare many different p.d. for your selection. She change them, also purposely or randomly, time by time.