DEVELOPING A STOCHASTIC MORTALITY MODEL FOR INTERNAL ASSESSMENTS *

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Intro

- Internal assessments
  - ▸ portfolio valuation
  - ▸ capital allocation (solvency investigation)
  - ▸ ...

$\looparrowright$ appraisal of risks

$\looparrowright$ adoption of a stochastic model

- Focus here is on
  - ▸ a life annuity portfolio. Annuities are immediate, in arrears and with fixed benefits
  - ▸ mortality risks only
    - ↳ risk of random fluctuations
    - ↳ longevity risk

    Other risks are disregarded

- Background assumptions
  - the insurer holds the market life table, which represents the best estimate assumption about future mortality
  - the insurer does not have access to data sets and methodologies underlying the construction of the life table
  - possibly some alternative tables, e.g. provided by the institution constructing the best estimate table, are available, without any specific recommendation about their use

- Our tasks
  1. *We describe a mortality model allowing for both random fluctuations and systematic deviations, extending some classical results about the modelling of the number of deaths joint to the modelling of parameter uncertainty*
  2. *We then test the setting within an internal solvency model. A comparison with the relevant requirement proposed within Solvency 2 is performed*

Basic assumptions

- Time of issue (of the portfolio): t_0; entry age: x_0
- Annual outflows

outflows:	$B_1^{(\Pi)}$	$B_2^{(\Pi)}$	$\dots$	$B_t^{(\Pi)}$	$\dots$
time (since issue):	1	2	$\dots$	t	$\dots$

- If we assume the same annual amount to each annuitant: $B_t^{(\Pi)} = b\,N_t$
  - ▸ N_t: number of annuitants at time t
    - ▹ if known: n_t
  - ▸ $N_{t-1} - N_t = D_t$: number of deaths in year $(t-1, t)$
- Given $N_0 = n_0$, we then address

# deaths:	D_1	D_2	$\dots$	D_t	$\dots$
time (since issue):	1	2	$\dots$	t	$\dots$

- In detail

time (since issue)

current age	1	2	$\dots$	t	$\dots$
x_0					
$\vdots$					
x				$D_{x,t}$	
$\vdots$					
ω					
total # deaths	D_1	D_2	$\dots$	D_t	$\dots$

Basic assumptions (cont)

- Let refer to one cohort only

current age \ time (since issue)	1	2	$\dots$	t	$\dots$
x_0	$D_{x_0,1}$				
x_0+1		$D_{x_0+1,2}$			
$\vdots$					
x				$D_{x,t}$	
$\vdots$					
ω					
total # deaths	$D_1 = D_{x_0,1}$	$D_2 = D_{x_0+1,2}$	$\dots$	$D_t = D_{x,t}$	$\dots$

- Similarly, $N_t = N_{x,t}$ when just one cohort is referred to

- The random number of deaths is affected by
  - random fluctuations
  - systematic deviations

- Random fluctuations
  - If the size of the portfolio is large enough, then with high probability $\frac{D_{x,t}}{n_{x,t-1}} \approx \underbrace{q^*_{x,t}}$
    best estimate (BE) mortality rate
  - Due to the actual size of the portfolio, $\frac{D_{x,t}}{n_{x,t-1}} \gtrless q^*_{x,t}$
  - Representation
    - For a cohort: $[D_{x,t}|\ q^*_{x,t}; n_{x,t-1}] \sim \text{Bin}(n_{x,t-1}, q^*_{x,t})$
    - Possibly approximated as:

      $$\boxed{[D_{x,t}|q^*_{x,t}; n_{x,t-1}] \sim \text{Poi}(n_{x,t-1}\, q^*_{x,t})}$$

      - via generalization, this can be applied also in the case of more than one cohort or various benefit amounts

- Systematic deviations
  - High probability that $\frac{D_{x,t}}{n_{x,t-1}}$ is not close to $q^*_{x,t}$ also in very large portfolios
    $\Rightarrow$ *deviation in aggregate mortality*
  - Representation: $\boxed{\text{random mortality rate, } Q_{x,t}}$
  - The deviation in aggregate mortality can be temporary or permanent
  - Temporary deviation
    - typically an upward shock, reasonably independent of previous ones
    - the impact could be age-dependent
  - Permanent deviation
    - the underlying trend, for the whole population or for some cohorts, is other than what described by $q^*_{x,t}$
    - reasonably, deviations are (positively) correlated in time

The mortality rate

- We assume the multiplicative model

$$\boxed{Q_{x,t} = q^*_{x,t}\, Z_{x,t}}$$

  - clearly: $Z_{x,t} > 0$
    and in particular: $Z_{x,t} \gtreqless 1$, but such that $0 \leq Q_{x,t} \leq 1$
  - The coefficient $Z_{x,t}$ should account for both temporary and permanent deviations
  - Possible assumptions about the coefficients $Z_{x,t}$'s
    - Independent or correlated in time/age
    - Shape of the probability distribution (pdf)
      - age- and time-dependent
      - fixed in time (but only when independence in time is accepted)

- Referring to one cohort

current age	time (since issue)				
	1	2	$\dots$	t	$\dots$
x_0	$Z_{x_0,1}$				
x_0+1		$Z_{x_0+1,2}$			
$\vdots$					
x				$Z_{x,t}$	
$\vdots$					
ω					

  - We test two assumptions
    - Independence among the $Z_{x,t}$'s, which are further assumed to be identically distributed
    - Correlation assumption:
      $Z_{x_0,1} \Rightarrow Z_{x_0+1,2} \Rightarrow \ \dots \ \Rightarrow Z_{x,t} \Rightarrow \ \dots$

Probability distribution of systematic deviations

- We assume

$$\boxed{Z_{x,t} \sim \text{Gamma}(\alpha_{x,t}, \beta_{x,t})}$$

- It follows

$$Q_{x,t} \sim \text{Gamma}\left(\alpha_{x,t}, \frac{\beta_{x,t}}{q^*_{x,t}}\right)$$

- For the number of deaths, setting $Q_{x,t} = q$ we let

$$[D_{x,t} |\ q; n_{x,t-1}] \sim \text{Poi}(n_{x,t-1}\, q)$$

Then we can show that

$$\boxed{[D_{x,t} |\ n_{x,t-1}] \sim \text{NBin}\left(\alpha_{x,t}, \frac{\theta_{x,t}}{\theta_{x,t}+1}\right)}$$

$$\theta_{x,t} = \frac{\beta_{x,t}}{n_{x,t-1}\, q^*_{x,t}}$$

Probability distribution of systematic deviations (cont)

- We note that

$$\mathbb{E}[D_{x,t}|q^*_{x,t};n_{x,t-1}] = n_{x,t-1}\, q^*_{x,t}$$

whilst

$$\mathbb{E}[D_{x,t}|n_{x,t-1}] = \underbrace{\frac{\alpha_{x,t}}{\beta_{x,t}}}\, n_{x,t-1}\, q^*_{x,t}$$

magnitude of the
systematic deviation

Assuming independence in time of systematic deviations

- Assumption: the $Z_{x,t}$'s are independent in time, and identically distributed
- Rationale
  - the mortality dynamics is mainly affected by temporary deviations
  - the insurer's mortality experience is not reliable for detecting the underlying trend

- For the solvency investigation, we take: $Z_{xt} \sim \text{Gamma}(0.75\beta, \beta)$. It follows
  - $\mathbb{E}[Q_{xt}] = 0.75\, q^*_{xt}$
  - $\mathbb{E}[D_t |\, n_{t-1}] = 0.75\, n_{t-1}\, q^*_{xt}$

  consistently with the relevant assumption in Solvency 2

Assuming correlation in time of systematic deviations. Updating parameters to experience

- Assumption: the $Z_{x,t}$'s are correlated in time, identically distributed
- Further assumption: the mortality experience from the portfolio is reliable as an evidence of the trend of the cohort (or the population)

$\Rightarrow$ *An inferential procedure is adopted for updating the parameters of the pdf of $Z_{x,t}$ to experience*

- **Steps of the inferential procedure** (one cohort is referred to)
  - Valuation at time 0 (issue time; no previous experience available)
    - $\boxed{Z_{x,t} \sim \text{Gamma}(\bar{\alpha}, \bar{\beta})}$ for all times t (and ages $x = x_0 + t$)
    - $\boxed{D_{x_0,1} \sim \text{NBin}\left(\alpha_{x_0,1}, \frac{\theta_{x_0,1}}{\theta_{x_0,1}+1}\right)}$

$$\alpha_{x_0,1} = \bar{\alpha}$$

$$\theta_{x_0,1} = \frac{\bar{\beta}}{n_{x_0,0}\, q^*_{x_0,1}}$$

- Valuation at time 1
  - Let $D_{x_0,1} = d_{x_0,1}$ the observed number of deaths in $(0,1)$
  - Then $n_{x_0+1,1} = n_{x_0,0} - d_{x_0,1}$
  - We can calculate the posterior pdf of $Q_{x_0,1}$, conditional on $D_{x_0,1} = d_{x_0,1}$. It turns out

$$\boxed{[Q_{x_0,1} | D_{x_0,1} = d_{x_0,1}] \sim \text{Gamma}\left(\bar{\alpha} + d_{x_0,1}, \frac{\bar{\beta}}{q^*_{x_0,1}} + n_{x_0,0}\right)}$$

and hence:

$$\boxed{[Z_{x,t} | D_{x_0,1} = d_{x_0,1}] \sim \text{Gamma}(\bar{\alpha} + d_{x_0,1}, \beta + n_{x_0,0}\, q^*_{x_0,1})}$$

  - We then have

$$\boxed{[D_{x_0+1,2} |\, n_{x_0,0}, d_{x_0,1}] \sim \text{NBin}\left(\alpha_{x_0+1,2}, \frac{\theta_{x_0+1,2}}{\theta_{x_0+1,2}+1}\right)}$$

$$\alpha_{x_0+1,2} = \bar{\alpha} + d_{x_0,1} \qquad \theta_2 = \frac{\bar{\beta} + n_{x_0,0}\, q^*_{x_0,1}}{n_{x_0+1,1}\, q^*_{x_0+1,2}}$$

- Valuation at time $t-1$
  - Having observed

$$D_{x_0,1} = d_{x_0,1}, D_{x_0+1,2} = d_{x_0+1,2}, \ldots, D_{x_0+t-2,t-1} = d_{x_0+t-2,t-1}$$

and then

$$n_{x_0+h,h} = n_{x_0+h-1,h-1} - d_{x_0+h-1,h} \qquad \text{at time } h = 1, 2, \ldots, t-1$$

it turns out

$$\boxed{[D_{x_0+t-1,t} | \, n_{x_0,0}, d_{x_0,1}, d_{x_0+1,2}, \ldots, d_{x_0+t-2,t-1}] \sim \text{NBin}\left(\alpha_{x_0+t-1,t}, \frac{\theta_{x_0+t-1,t}}{\theta_{x_0+t-1,t+1}}\right)}$$

$$\alpha_{x_0+t-1,t} = \bar{\alpha} + \sum_{h=1}^{t-1} d_{x_0+h-1,h}$$

$$\theta_{x_0+t-1,t} = \frac{\bar{\beta} + \sum_{h=1}^{t-1} n_{x_0+h-1,h-1} \, q^*_{x_0+h-1,h}}{n_{x_0+t-1,t-1} \, q^*_{x_0+t-1,t}}$$

- For the expected number of deaths, we have

$$\mathbb{E}[D_{x_0+t-1,t} | \; n_{x_0,0}, d_{x_0,1}, d_{x_0+1,2}, \ldots, d_{x_0+t-2,t-1}]$$
$$= \frac{\bar{\alpha} + \sum_{h=1}^{t-1} d_{x_0+h-1,h}}{\bar{\beta} + \sum_{h=1}^{t-1} n_{x_0+h-1,h-1} \, q^*_{x_0+h-1,h}} \; n_{x_0+t-1,t-1} \, q^*_{x_0+t-1,t}$$

Depending on experience, $\frac{\bar{\alpha}+\sum_{h=1}^{t-1} d_{x_0+h-1,h}}{\bar{\beta}+\sum_{h=1}^{t-1} n_{x_0+h-1,h-1} \, q^*_{x_0+h-1,h}} \gtreqless \frac{\bar{\alpha}}{\bar{\beta}}$

- For the solvency investigation, we set $\bar{\alpha} = 0.75\bar{\beta}$

Capital allocation

- Notation: let $Y_t^{(\Pi)}$ be the present value of future payments for the current portfolio (at a given interest rate)
- **A regulatory requirement: the Solvency 2 proposal**
  - We refer to the SCR (Solvency Capital Requirement) and we consider only the requirement for insurance contracts where the sum at risk is negative
    - a capital charge for longevity risk is required
    - the SCR reduces to such a capital charge
  - Capital charge at time z: change in the net value of assets minus liabilities (ΔNAV) against a permanent 25% decrease in mortality rates for each age
    - Under our assumptions, this reduces to

$$\boxed{\text{Life}_{\text{long},z} = V_z^{(\Pi)[-25\%]} - V_z^{(\Pi)[BE]}}$$

$$\mathbb{E}[Y_z^{(\Pi)} | \{0.75\, q^*_{xt}\}] \qquad\qquad \mathbb{E}[Y_z^{(\Pi)} | \{q^*_{xt}\}]$$

- Portfolio reserve

$$V_z^{(\Pi)} = V_z^{(\Pi)[BE]} + RM_z$$

where RM_z is a risk margin, assessed according to a Cost-of-Capital logic. In particular

$$\boxed{RM_z = 0.06 \cdot \sum_{h=z+1}^{m} \text{SCR}_h \, (1 + r_f)^{-h}}$$

where

0.06: spread

m: “maturity” of the portfolio (i.e. maximum residual lifetime of in-force policies)

r_f: risk-free rate

in our implementation, $\text{SCR}_h = \text{Life}_{\text{long},h}$ as expected according to the BE mortality table

- **Rules for internal models**
  - Let A_t be the amount of portfolio assets at time t

$$A_t = A_{t-1}\,(1+i) - B_t^{(\Pi)} \qquad (t = z+1, z+2, \ldots)$$

    with A_z given at the valuation time z and i the investment yield (assumed to be the risk-free rate)
    Then

$$\boxed{M_t = A_t - V_t^{(\Pi)[BE]}}$$

    represents the assets available to meet risks (to be split into risk margin and required capital)

  - Let
    ε accepted default probability
    T time-horizon for solvency ascertainment

- A reasonable solvency rule

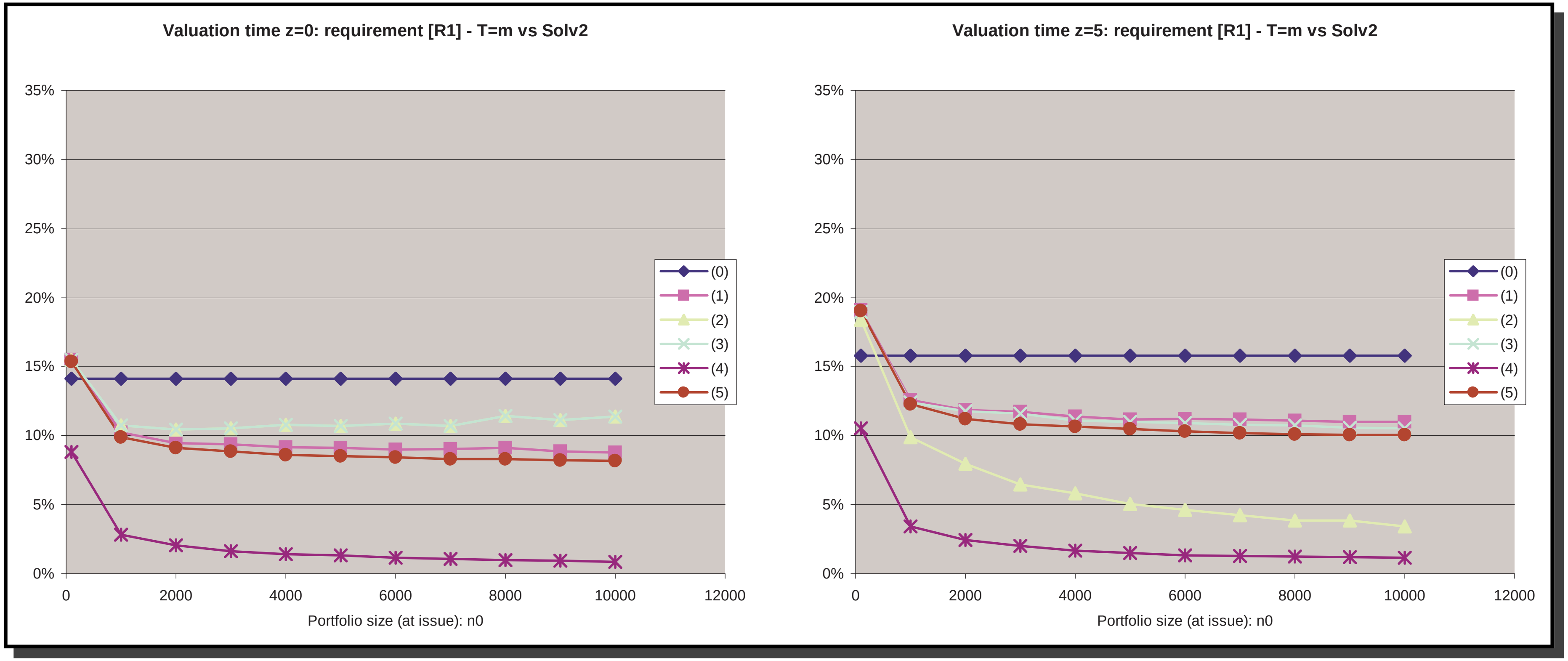
$$[R1] \boxed{\mathbb{P}[(M_{z+1} \geq 0) \wedge (M_{z+2} \geq 0) \wedge \cdots \wedge (M_{z+T} \geq 0)] = 1 - \varepsilon}$$

- We note that in Solvency 2
  - The accepted default probability is 0.005. So we set: $\varepsilon = 0.005$
  - Implicitly, for longevity risk $T = m$
- Requirement $[R1]$ needs a stochastic model

Some numerical investigations

- Input data
  - One cohort; initial age: $x_0 = 65$; males
  - Best estimate life table: IPS55 (projected life table for Italian males, cohort 1955)
  - Maximum age: $\omega = 119$, whence the maturity of the portfolio at time z is: $m = 119 - 65 - z$
  - (Initial) parameters of the pdf of $Z_{x,t}$: $\beta = \bar{\beta} = 100$, so that $\mathbb{CV}(Q_{x,t}) = \frac{\sqrt{\mathbb{V}\text{ar}(Q_{x,t})}}{\mathbb{E}(Q_{x,t})} = 10\%$
  - Risk-free rate and investment yield: 3% p.a.
  - Annual amount: $b = 1$

Some numerical investigations (cont)



(0) Solvency 2: $\frac{M_z^{[Solv2]}}{V_z^{(\Pi)[BE]}} = \frac{\text{Life}_{\text{long},z} + RM_z}{V_z^{(\Pi)[BE]}}$

(1)–(5) Rule $[R1]$, with $T = m$: $\frac{M_z^{[R1]}}{V_z^{(\Pi)[BE]}}$

(1) with fixed parameters for the pdf of $Z_x(t)$

(2) with updated parameters, experience as the best estimate life table

(3) with updated parameters, experience as the Solvency 2 stress scenario (i.e. BE-25%)

(4) allowing for random fluctuations only (mortality rate certain, given by q^*_{xt})

(5) allowing for random fluctuations and systematic deterministic deviations (mortality rate certain, given by $0.75\, q^*_{xt}$)

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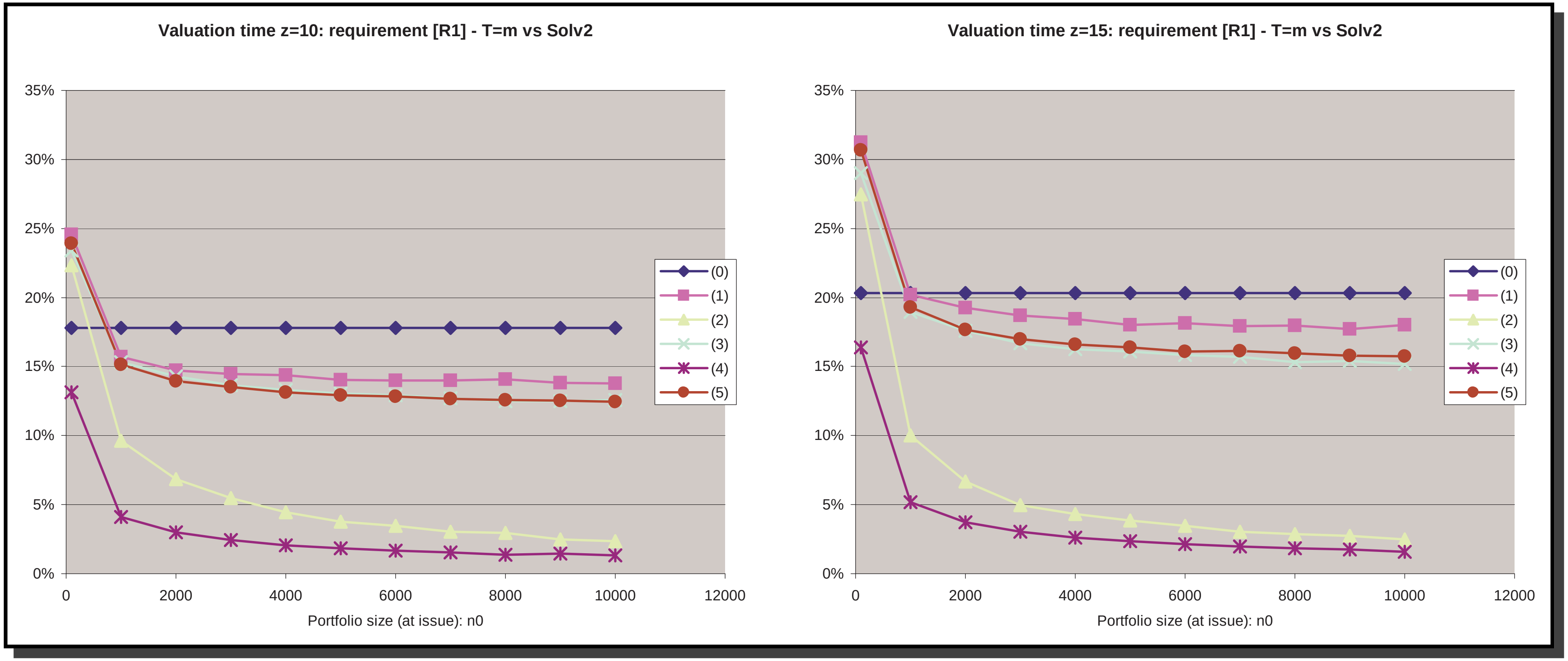

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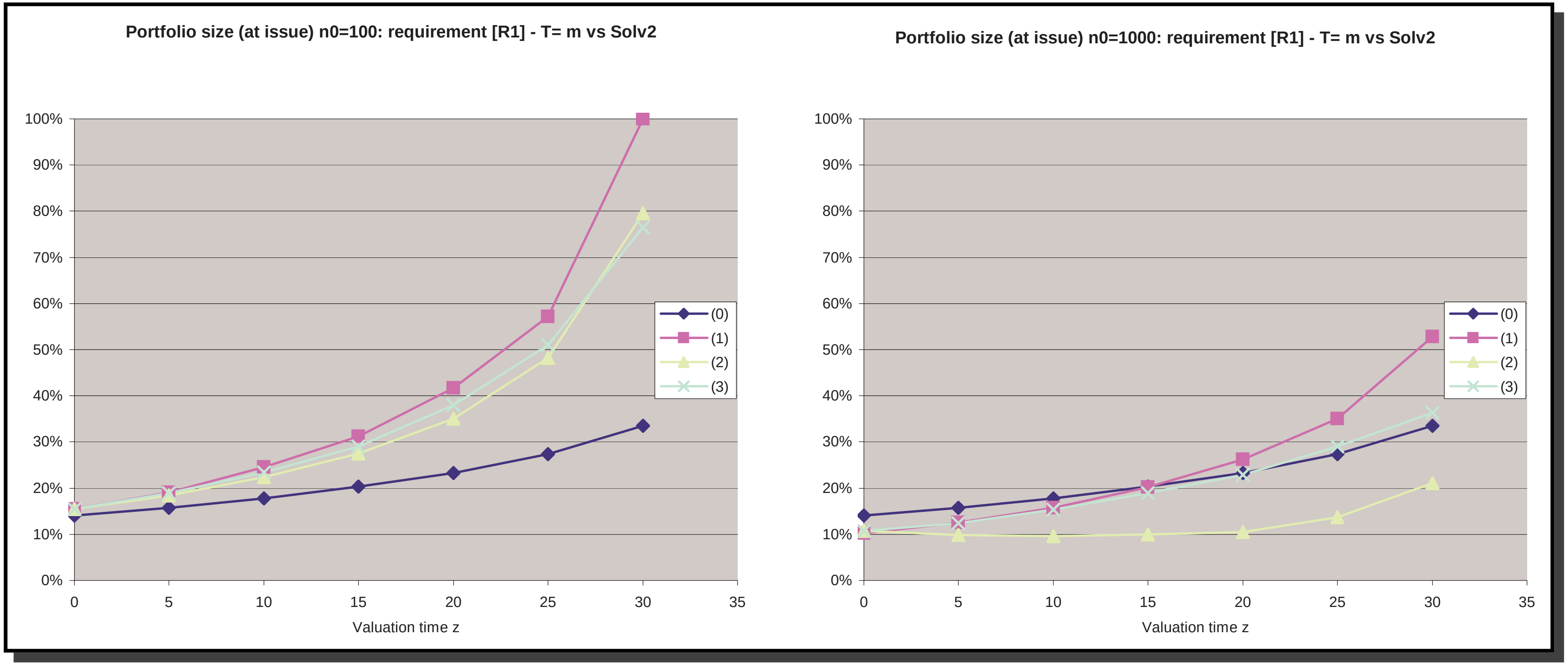

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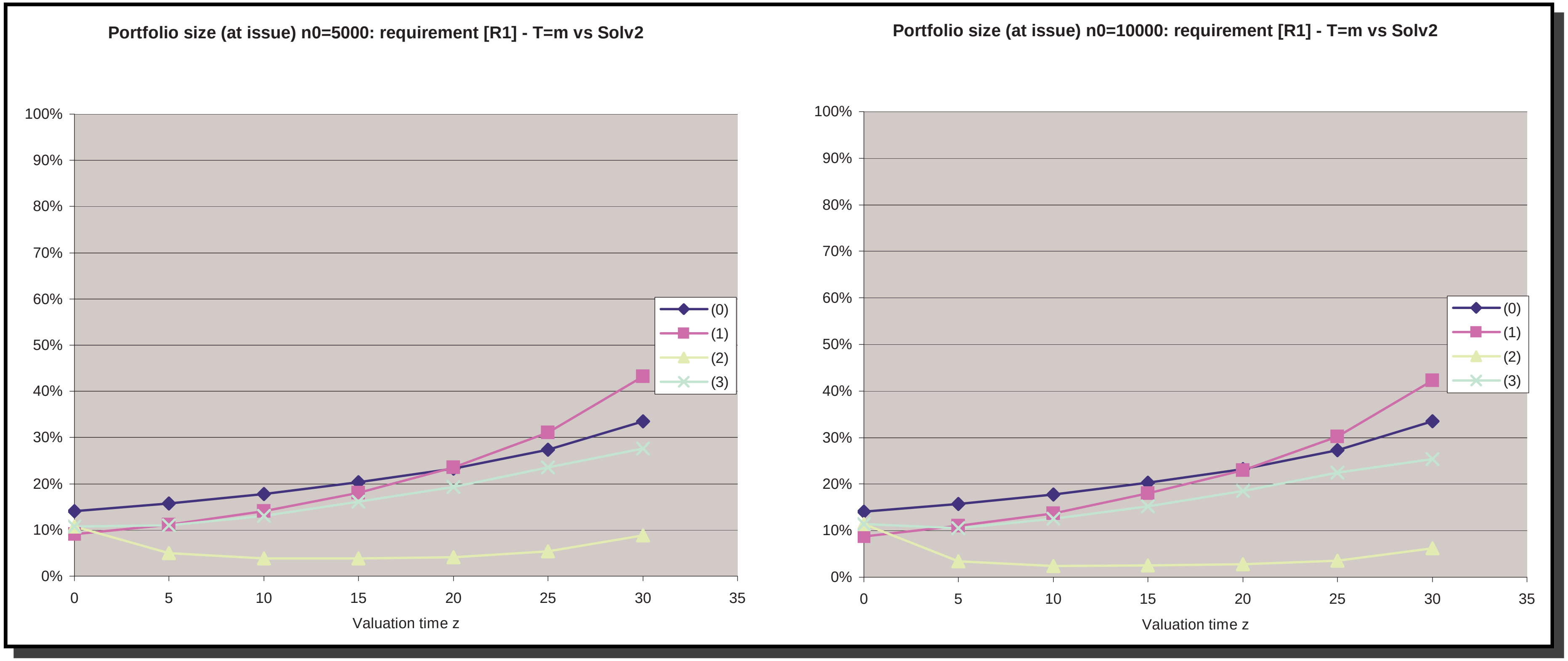

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Concluding remarks

- In Solvency 2, an allowance for the systematic mortality risk is only involved, which is represented in a deterministic way
- The rule is very simple to implement, but the capital charge may result either too large or too low in time or in respect of the portfolio size
- Adoption of internal rules is possible, but validation by the supervisory authority must be obtained
- Even though the insurer does not have the expertise to deal with the methodologies underlying the best estimate table and, in general, with stochastic mortality models, a simple structure may lead to a satisfactory assessment of the impact of mortality risks, including both random fluctuations and longevity risk
- If the insurer prefers to adopt the standard Solvency 2 rule, the proposed inferential procedure may suggest an update of the parameters for the stress scenario (also in this case, a validation by the supervisory authority would be required)

- Further investigations
  - More than one cohort
  - Age-dependence
  - Calibration
  - . . .

- For details, see
  A. Olivieri, E. Pitacco (2008)
  Stochastic mortality: the impact on target capital
  Available at `http://ssrn.com/abstract=1287688`