On a New Generalization of the Expected Discounted Penalty Function

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Toronto, November 2008

Main Objective:

Talk about the potential of the EDPF (and GEDPF) and their features in financial applications.

Secondary Objectives:

Talk about two results:

1. Studying the EDPF under a slightly more general Lévy risk model than the ones studied so far.

2. Generalizing the EDPF in a new direction. We introduce a general EDPF that depends on a new random variable that is not *local* at ruin time.

What's the EDPF and Why can be seen as an interesting object in financial application?

1. Because of the models for which has been studied,

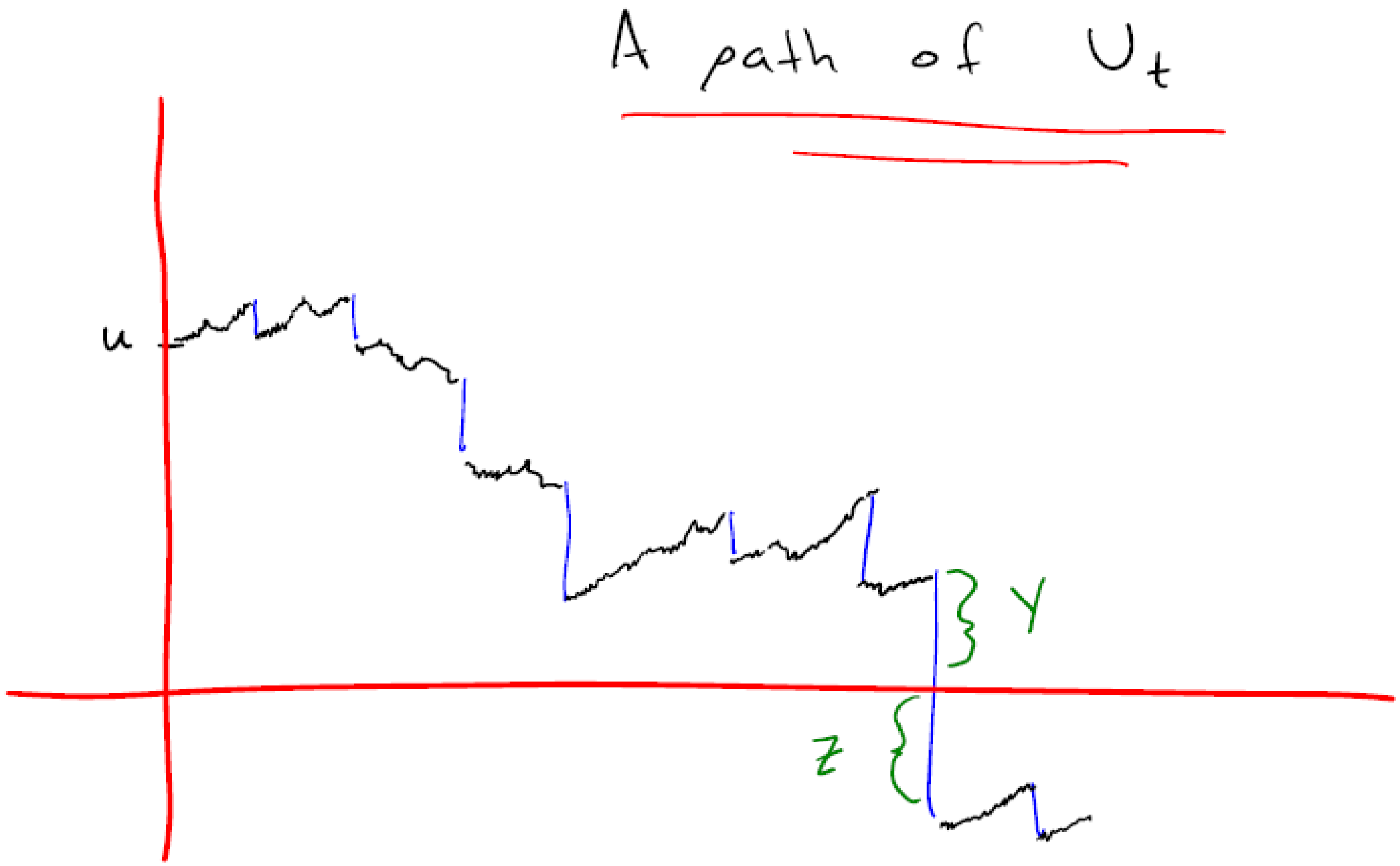
2. Because of the very definition as a discounted cashflow.

Definition 1 *For a surplus process U_t, the EDPF is:*

$$\phi(u) = \mathbb{E}\left[w\left(U(\tau_-), |U(\tau)|\right) e^{-\delta\tau} \mathbb{I}_{\{\tau<\infty\}} | U(0) = u\right] , \qquad (1)$$

where $w(x, y)$ is a nonnegative function and τ is the time of ruin.

EDPF



What are the limitations of the EDPF and What is a GEDPF?

1. Penalty (cashflow) depends on variables known at ruin time,

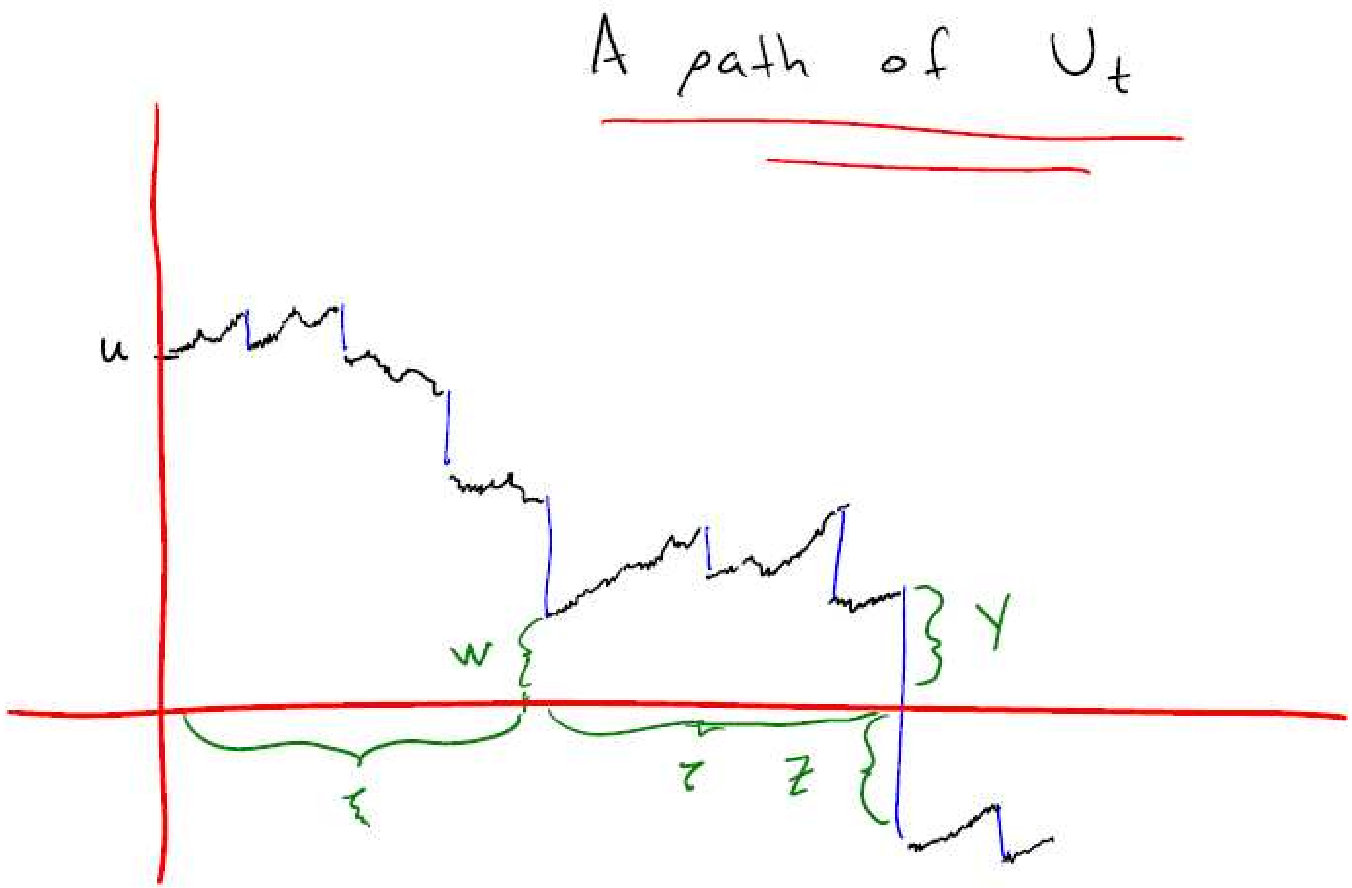
2. A GEDPF incorporates new variables that make it path-dependant.

Definition 2 *For a surplus process U_t, the GEDPF is:*

$$\phi(u) = \mathbb{E}\left[w\left(U(\tau_{-}), |U(\tau)|, Y(\tau-)\right) e^{-\delta\tau}\mathbb{I}_{\{\tau<\infty\}}|U(0) = u\right], \quad (2)$$

where $w(x, y, z)$ is a nonnegative function and τ is the time of ruin.

GEDPF



Historical Context of Our Model

Lévy Processes in Risk Theory:

The classical risk process [see Grandell (1991)]

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i \,, \qquad t \geqslant 0 \,, \tag{3}$$

A diffusion approximation [Iglehart (1969) and Grandell (1977)].

$$U_D(t) = u + ct + \sigma W(t) \,, \qquad t \geqslant 0 \,. \tag{4}$$

A perturbed model [Dufresne and Gerber (1991)].

$$U_P(t) = u + ct - \sum_{i=1}^{N(t)} X_i + \sigma W(t) \,, \qquad t \geqslant 0 \,. \tag{5}$$

All of these are of the form

$$U_L(t) = u + ct + Z(t) \, , \qquad t \geqslant 0 \, ,$$

where Z is a LP.

A first model with PIIS is Dufresne, Gerber and Shiu (1991) [discussed examples are Gamma and IG].

Furrer (1998) α-stable risk process.

Yang and Zhang (2001) for spectrally negative LP.

Morales (2003), Garrido and Morales (2006) for some subordinators.

Huzak *et al.* (2004) for a general perturbed case.

Morales (2007) for a subordinator with a Brownian perturbation.

Lévy Processes

Lévy processes are stochastic processes with IS increments in a one-to-one correspondance with ID distributions.

If X is a LP then $\mathbb{E}\left[e^{izX(t)}\right] = e^{-t\Psi(z)}$, with

$$\Psi(z) = \mathrm{i}mz + \frac{\sigma^2}{2}z^2 + \int_{\mathbb{R}} \left[1 - e^{\mathrm{i}zx} + \mathrm{i}zx\mathbb{I}_{\{(-1,1)\}}(x)\right] \nu(dx) \,, \tag{6}$$

A LP $X(t)$ can be written as

$$X(t) = at + bW(t) + J(t) \,, \qquad t > 0 \,, \tag{7}$$

Expected Discounted Penalty Function

Gerber and Shiu (1998a) introduced the concept of discounted penalty function as a mean to study the distribution of the time to ruin, the amount at and prior to ruin. ϕ is defined as follows:

$$\phi(u) = \mathbb{E}\left[w\left(U(\tau_-), |U(\tau)|\right) e^{-\delta\tau} \mathbb{I}_{\{\tau<\infty\}} | U(0) = u\right] , \qquad (8)$$

where $w(x, y)$ is a nonnegative function and τ is the time of ruin.

If $\delta = 0$ and $w(x, y) = 1$ then $\phi(u) = \psi(u)$.

If $\delta > 0$ and $w(x, y) = 1$ then $\phi(u)$ is the LT of time to ruin τ.

If $\delta = 0$ and $w(x_0, y_0) = 1$ (zero elsewhere) $\phi(u) = f(x_0, y_0|u)$ is the joint density of surplus prior and at ruin.

Convolution Structure for $\phi(u)$

For the classical case we have that ϕ satisfies:

$$\phi(u) = h(u) * \sum_{k=0}^{\infty} g^{*(k)}(u) \ , \quad u \geq 0 \ , \tag{9}$$

for some functions h and g.

This implies that ϕ is the solution of

$$\phi(z) = \int_0^z \phi(x) g(z-x) dx + h(z) \ , \qquad z > 0 \ . \tag{10}$$

ϕ can also be expresed in terms of its LT $\widehat{\phi}$, i.e.

$$\widehat{\phi}(s) = \widehat{\phi}(s)\widehat{g}(s) + \widehat{h}(s) \ , \qquad s \geqslant 0 \ ,$$

or

$$\widehat{\phi}(s) = \sum_{k=0}^{\infty} [\widehat{g}(s)]^k \widehat{h}(s) = \frac{\widehat{h}(s)}{1-\widehat{g}(s)} \qquad s \geqslant 0 \ .$$

Renewal Equation for $\phi(u)$

Extensions of ϕ have been worked out for a model

$$U(t) = u + c\,t - S(t) + \sigma W(t)\ , \qquad t \geq 0\ ,$$

where

1. S a compound Poisson and W is a standard Brownian motion [Tsai and Willmot (2002)].

2. S a subordinator and W is a standard Brownian motion [Morales (2007)].

Here, the EDPF takes the form

$$\begin{aligned}\phi_D(u) &= w_0 \mathbb{E}\left[e^{-\delta\tau}\mathbb{I}_{\{\tau<\infty, U(\tau)=0\}}|U(0)=u\right] \\ &\quad + \mathbb{E}\left[w\left(U(\tau_-), |U(\tau)|\right) e^{-\delta\tau}\mathbb{I}_{\{\tau<\infty, U(\tau)<0\}}|U(0)=u\right] ,\end{aligned} \tag{11}$$

where $w_0 = w(0,0)$ is a positive penalty for ruin caused by hitting zero.

This has to do with *creeping*:

$$\mathbb{P}(X_\tau = 0) > 0 .$$

A spectrally negative Lévy process with positive drift does not creep downwards unless $\sigma > 0$.

Convolution structure for $\phi(u)$

[Morales (2007)]

$$\phi_D(u) = \int_0^u \phi_D(u-y) g_D(y) dy + w_0 e^{-\rho u}[1 - K(u)] + H_w(u) \ , \quad u \geqslant 0 \ ,$$

where

$$g_D(y) = \frac{1}{1+\theta} \int_0^y e^{-\rho(y-s)} k(y-s) \int_s^\infty e^{-\rho(x-s)} dm(x) \ ds \ .$$

The parameter ρ is the unique non-negative solution of

$$cr + \Psi_{S-W}(r) = \delta \ .$$

with $\rho = 0$ when $\delta = 0$.

The function H_w is given by

$$H_w(u) = \frac{1}{1+\theta} \int_0^u e^{-\rho(u-s)} k(u-s) \int_s^\infty e^{-\rho(x-s)} w(x) \ dx \ ds \ ,$$

where

$$w(x) = \int_x^\infty w(x, y - x) dm(y) \ , \qquad x \geqslant 0 \ .$$

The distribution functions K and M (with density functions k and m) are, respectively, $K(x) = 1 - e^{-(2c/\sigma^2)u}$ and $m(x) = \int_x^\infty \frac{\nu(s)}{\int_0^\infty \Pi(z) dz} ds$.

This generalizes Tsai and Willmot (2002). If $\nu(dx) = \lambda dF(x)$, it reduces to the case of a compound Poisson perturbed by a Brownian motion.

General Model

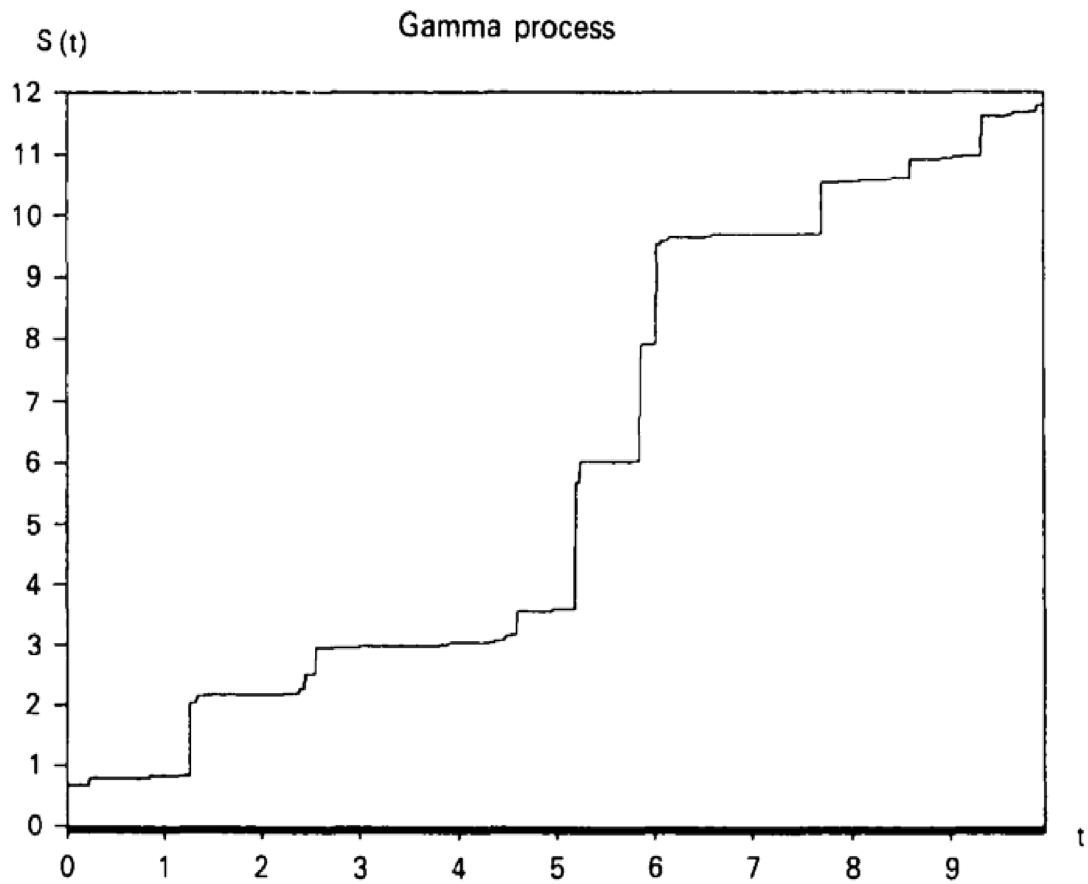
We would like to work out extensions of ϕ for a slightly more general model

$$U(t) = u + c\,t - S(t) + Z(t)\,, \qquad t \geq 0\,. \tag{12}$$

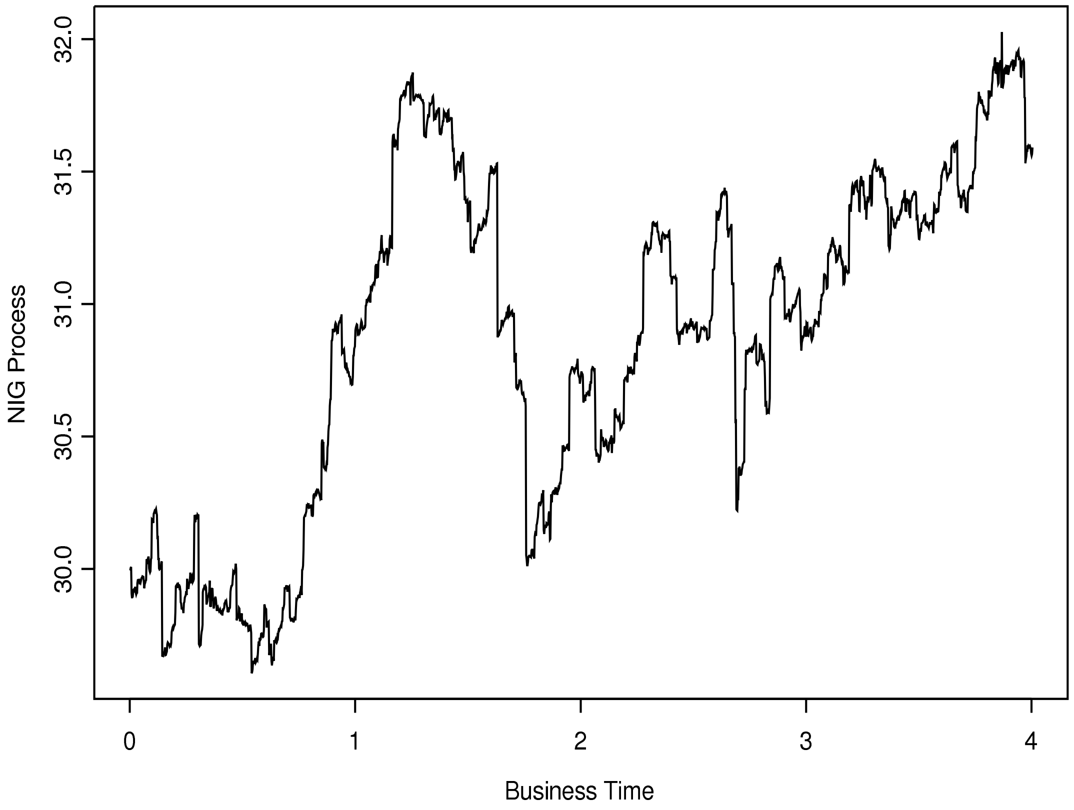
- Aggregate claims S are now modeled by a subordinator,

- Perturbation Z is now a zero-mean spectrally negative Lévy process.

Gamma Process



Spectrally Negative Lévy Process.



First Result

Morales and Olivares (2007) The EDPF satisfies

$$\phi_P(u) = \int_0^u \phi_P(u-y)g_P(y)dy + w_0 e^{-\rho u}[1-K(u)] + H_w(u) \ , \qquad u \geqslant 0 \ ,$$

where

$$g_P(y) = \frac{1}{c}\int_0^y e^{-\rho(y-s)}k(y-s)\left[\int_s^\infty e^{-\rho(x-s)}\nu_S(dx) + G_\rho(s)\right] ds \ ,$$

$$H_w(u) = \frac{1}{c}\int_0^u e^{-\rho(u-s)}k(u-s)\int_s^\infty e^{-\rho(x-s)}\chi(x)\ dx\ ds \ ,$$

$$\chi(x) = \int_x^\infty w(x, y-x)\nu_S(dy) + \int_x^\infty w(x, y-x)\nu_Z(dy) \ , \qquad x \geqslant 0 \ ,$$

the function G_ρ is defined through its Laplace transform

$$\int_0^\infty e^{-\xi x} G_\rho(x)dx = \frac{\Psi_{-J}(\xi) - \Psi_{-J}(\rho)}{\rho - \xi} \ , \qquad \xi \geqslant 0 \ ,$$

ρ is still the unique non-negative solution of the generalized Lundberg equation

$$cr + \Psi_{S-Z}(r) = \delta \quad \text{with } \rho = 0 \text{ when } \delta = 0 \; ,$$

and K (k) is an exponential distribution (density) with mean $\sigma^2/2c$, i.e. $K(x) = 1 - e^{-(2c/\sigma^2)u}$.

We recuperate the form

$$\phi(u) = h(u) * \sum_{k=0}^{\infty} g^{*(k)}(u) \; , \quad u \geq 0 \; . \tag{13}$$

This reduces to well-known cases for particular choices of S and Z.

Proof

This result is obtained as a direct application of recent developments in the understanding of first-passage times for Lévy processes.

The first-passage time problem

Let X be a spectrally positive Lévy process with Laplace exponent $\Psi_X(s) = \frac{1}{t} \ln \mathbb{E}[e^{-sX_t}]$ and LT denoted by $[a, \sigma^2, \Pi_X(dx)]$.

The first-passage time τ_x across a level x is given by

$$\tau_x = \inf\{t > 0 \mid X_t > x\} \ . \tag{14}$$

Definitions

Let Φ_X denote the right inverse of this Laplace exponent, i.e.

$$\Phi_X(\delta) = \sup\{\beta \geqslant 0 \mid \Psi_X(\beta) = \delta\} \ .$$

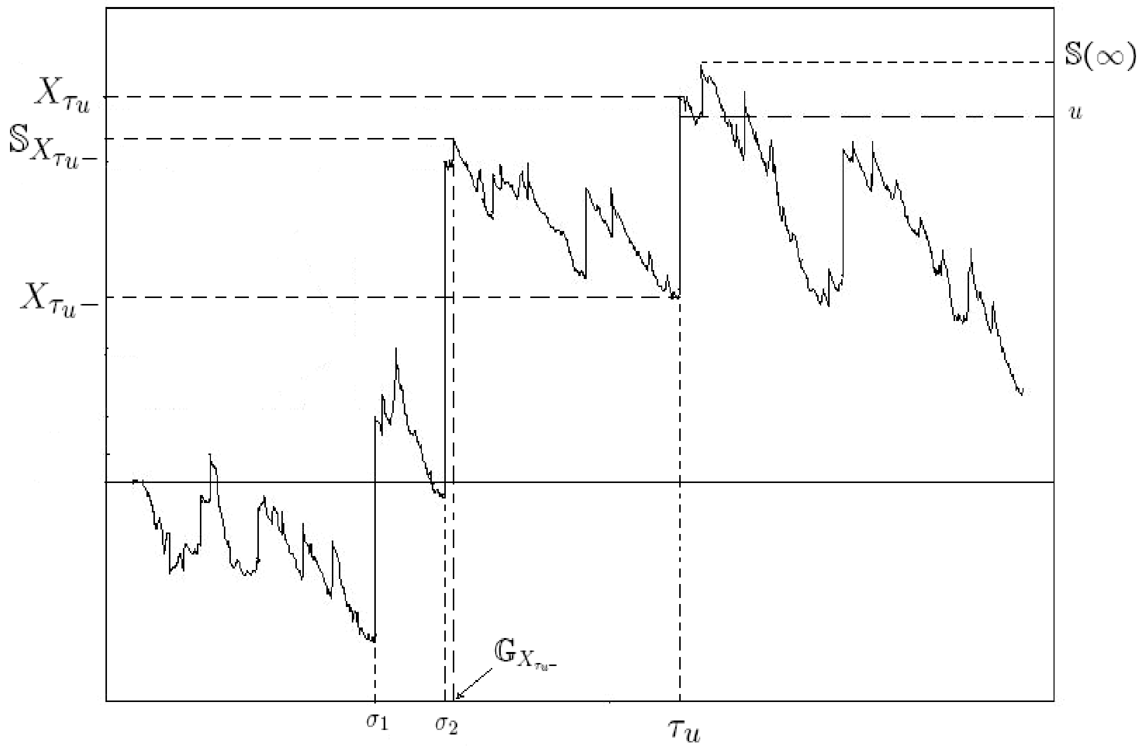
The running supremum process $\bar{X}$ is defined as

$$\bar{X}_t = \sup_{0 \leqslant s \leqslant t} X_s \ . \tag{15}$$

The time-at-the-maximum process $\bar{G}$ is defined as

$$\bar{G}_t = \sup\{s < t \mid \bar{X}_t = X_s\} \ . \tag{16}$$

Trajectory of X



Associated with τ_x we have

- $\bar{G}_{\tau_x -}$ is the time of last maximum prior to first passage (last minimum at ruin),
- $\tau_x - \bar{G}_{\tau_x -}$ is the time elapsed between the last maximum and the time of first passage (Time from last minimum to ruin),
- $X_{\tau_x} - x$ is the overshoot at first passage (deficit at ruin),
- $x - X_{\tau_x -}$ is the undershoot at first passage (surplus prior to ruin),
- $x - \bar{X}_{\tau_x -}$ is the undershoot of last maximum at first passage (last minimum at ruin).

Quintuple law

Doney and Kyprianou (2006)

For each $x > 0$ we have on $u > 0$, $v \geqslant y$, $y \in [0, x]$ and $s, t \geqslant 0$,

$$\mathbb{P}(\bar{G}_{\tau_x -} \in dt,\ \tau_x - \bar{G}_{\tau_x -} \in ds,\ X_{\tau_x} - x \in du\ , x - X_{\tau_x -} \in dv\ , x - \bar{X}_{\tau_x -} \in dy)$$

$$= \mathcal{U}(ds, x - dy)\, \widehat{\mathcal{U}}(dt, dv - y)\, \Pi_X(du + v)\, dv\ , \tag{17}$$

where the bivariate measures $\mathcal{U}$ and $\widehat{\mathcal{U}}$ are defined through their bivariate Laplace transforms

$$\int_0^\infty \int_0^\infty e^{-\alpha s - \beta x}\, \mathcal{U}(ds, dx) = \frac{\Phi_X(\alpha) - \beta}{\alpha - \Psi_X(\beta)} \tag{18}$$

and

$$\int_0^\infty \int_0^\infty e^{-\alpha s - \beta x}\, \hat{\mathcal{U}}(ds, dx) = \frac{1}{\Phi_X(\alpha) + \beta}\ . \tag{19}$$

The general perturbed risk model can be written as

$$R(t) = x - X(t)\,, \qquad t \geqslant 0\,, \tag{20}$$

where $X(t) = S(t) - Z(t) - c\,t$ with S, Z, c and x.

Clearly X is a spectrally positive Lévy process and the ruin time problem is assimilated to the first-passage time problem, i.e. the ruin time τ is the first-passage time of X (τ_x) accross a level x.

Let $f_x(s,t,u,v,y)$ be the joint density of the quintuple law on $u \geqslant 0$, $v \geqslant y$, $y \in [0,x]$ and $s,t \geqslant 0$.

We need to take into account that $\mathbb{P}[X_{\tau_x} = x] > 0$ (upward creeping of the process X when $\sigma^2 > 0$), which implies that $f_x(s,t,u,v,y)$ has an atom at $\bar{0}$ and the EDPF is given by (recall $w(0,0) = w_0$)

$$
\begin{aligned}
\phi(x) &= w_0 \, \mathbb{E}\left[e^{-\delta\tau_x} \, \mathbb{I}_{\{\tau_x = x\}}\right] \\
&\quad + \int_{0+}^{x} \int_{y}^{\infty} \int_{0+}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} e^{-\delta(s+t)} \, w(u,v) \, f_x(s,t,u,v,y) ds \, dt \, du \, dv \, dy \, .
\end{aligned} \tag{21}
$$

The main results is obtained from (21) with a straight-forward integration and a couple of results from the theory of fluctuations for Lévy processes.

Second result

Biffis and Morales (2008)

Let

$$R(t) = x - X(t) \ , \qquad t \geqslant 0 \ . \tag{22}$$

If we let $Y_t = x - \bar{X}_t$, we can define the GEDPF ϕ_G as follows,

$$\phi_G(x) = \mathbb{E}\left[e^{-\delta\tau} w\left(|R(\tau)|,\ R(\tau_-),\ Y_{\tau-}\right) \mathbb{I}_{\{\tau<\infty\}} | R(0) = x\right] \ , \tag{23}$$

with w a non-negative function on $\mathbb{R}^3_+ \cup \{(0,0,0)\}$ such that $w(0,0,0) = w_0 > 0$.

The quintuple law allows us to find a renewal equation for this GEDPF.

Remarks

- The GEDPF still characterizes the risk associated with the surplus process. The key feature is that, unlike the classical EDPF, the GEDPF does not only depend on local characteristics at ruin but on a path-dependent ruin-related random variable as well. The extra information brought by the new variable gives a better description of the underlying process and therefore of the embedded risk.

- This new variable is fundamentally different from the others considered up to now. *Deficit* and *surplus prior* are only observed at ruin and they have little potential as predictive tools. The *last minimum* is not local at ruin.

Remarks

- The new EDPF contains information on the marginal distribution of the last minimum before ruin. This could be used to set up warning barriers for instance. If we denote this distribution by F_m^x, we can see that any given level a such that $F_m^x(a) = \alpha$ can be interpreted as: $\alpha \times 100\%$ *of the times when ruin occurs, we know that the last minimum is smaller than* a.

Biffis and Morales (2008) For a generalized perturbed risk model, the GEDPF ϕ_G is given by the following DRE

$$\phi_G(x) = \int_0^x \phi_G(x-y) g_G(y) dy + H_G(x) + w_0 e^{-\rho x}[1-K(x)] \ , \qquad x \geqslant 0 \ , \tag{24}$$

where

$$g_G(y) = \frac{1}{c}\int_0^y e^{-\rho(y-s)} k(y-s) \left[\int_s^\infty e^{-\rho(x-s)} \nu_S(dx) + G_\rho(s)\right] ds \ , \tag{25}$$

$$H_G(u) = \frac{1}{c}\int_0^u e^{-\rho(u-s)} k(u-s) \int_s^\infty e^{-\rho(x-s)} \chi_G(x,s) \ dx \ ds \ , \tag{26}$$

$$\chi_G(x,s) = \int_x^\infty w(y-x,x,s)\nu_S(dy) + \int_x^\infty w(y-x,x,s)\nu_Z(dy) \ , \qquad x > 0 \ . \tag{27}$$

The function G_ρ is defined through its Laplace transform

$$\int_0^\infty e^{-\xi x} G_\rho(x) dx = \frac{\Psi_{-J}(\xi) - \Psi_{-J}(\rho)}{\rho - \xi} \ , \qquad \xi \geqslant 0 \ ,$$

The constant ρ is still the unique non-negative solution of the generalized Lundberg equation

$$cr + \Psi_{S-Z}(r) = \delta \ ,$$

and K (k) is an exponential distribution (density) with mean $\sigma^2/2c$, i.e. $K(x) = 1 - e^{-(2c/\sigma^2)u}$.

Once again, we recuperate the convolution form

$$\phi(u) = h(u) * \sum_{k=0}^{\infty} g^{*(k)}(u) \ , \quad u \geq 0 \ . \tag{28}$$

This reduces to well-known cases for particular choices of w, S and Z.

Proof

Let us denote the joint density of the quintuple law by $f_x(s,t,u,v,y)$ on $u \geqslant 0$, $v \geqslant y$, $y \in [0,x]$ and $s,t \geqslant 0$. Then a GEDPF is given by (recall $w(0,0,0) = w_0$ and upward creeping when $\sigma^2 > 0$)

$$\begin{aligned}\phi_G(x) &= w_0\, \mathbb{E}\left[e^{-\delta\tau_x}\, \mathbb{I}_{\{\tau_x = x\}}\right] \\ &+ \int_{0+}^{x}\int_{y}^{\infty}\int_{0+}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} e^{-\delta(s+t)}\, w(u,v,y)\, f_x(s,t,u,v,y) ds\, dt\, du\, dv\, dy\ .\end{aligned} \tag{29}$$

Straight forward integration yields the following result for this GEDPF.

Further Generalizations

We could study and even more general EDPF:

$$\phi_G(x) = \mathbb{E}\left[e^{-(\alpha\zeta+\beta\xi)} w\left(|R(\tau)|,\ R(\tau_-),\ Y_{\tau-}\right) \mathbb{I}_{\{\tau<\infty\}} | R(0) = x\right], \tag{30}$$

where ζ is the time of the last minimum and ξ is the time elapsed between the last minimum and ruin (i.e. $\tau = \zeta + \xi$).

or

$$\phi_G(x) = \mathbb{E}\left[e^{-\delta\tau} w\left(|R(\tau)|,\ R(\tau_-),\ Y_{\tau-},\ Y_{\zeta-}\right) \mathbb{I}_{\{\tau<\infty\}} | R(0) = x\right], \tag{31}$$

where ζ is the time of the last minimum and therefore $Y_{\zeta-}$ is the second to last minimum.

Further Work

We could study the GEDPF in a financial context:

1. Work out expressions for examples that are suitable for financial applications,

2. Work out expressions for choices of the penalty function w that make sense as pay-offs.

3. Study further the potential of the sequence of minimums leading to ruin (the barrier) in financial applications.

Conclusions

- We have presented expressions for EDPF in a slightly more general case that includes a subordinator perturbed by a spectrally negative Lévy process.

- We have presented an extended version of the EDPF that includes a third ruin-related random variable. Expressions for this GEDPF have been worked out showing that the well-known convolution structure is still preserved.

- Further work: Study this new GEDPF within those settings where the classical EDPF plays a role.

- Further work: EDPF and GEDPF for more general examples where the perturbation also jumps upwards and not only downwards [Mordecki (2005)].

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A Little Bit of History

Earlier forms for the EDPF:

1. Dufresne and Gerber (1988): *The surpluses immediately before and at ruin, and the amount of the claim causing ruin*

2. Dickson (1992): *On the distribution of the surplus prior to ruin*

3. Gerber and Shiu (1998): *The joint distribution of the time of ruin, the surplus immediately before ruin, and the deficit at ruin*

A Little Bit of History

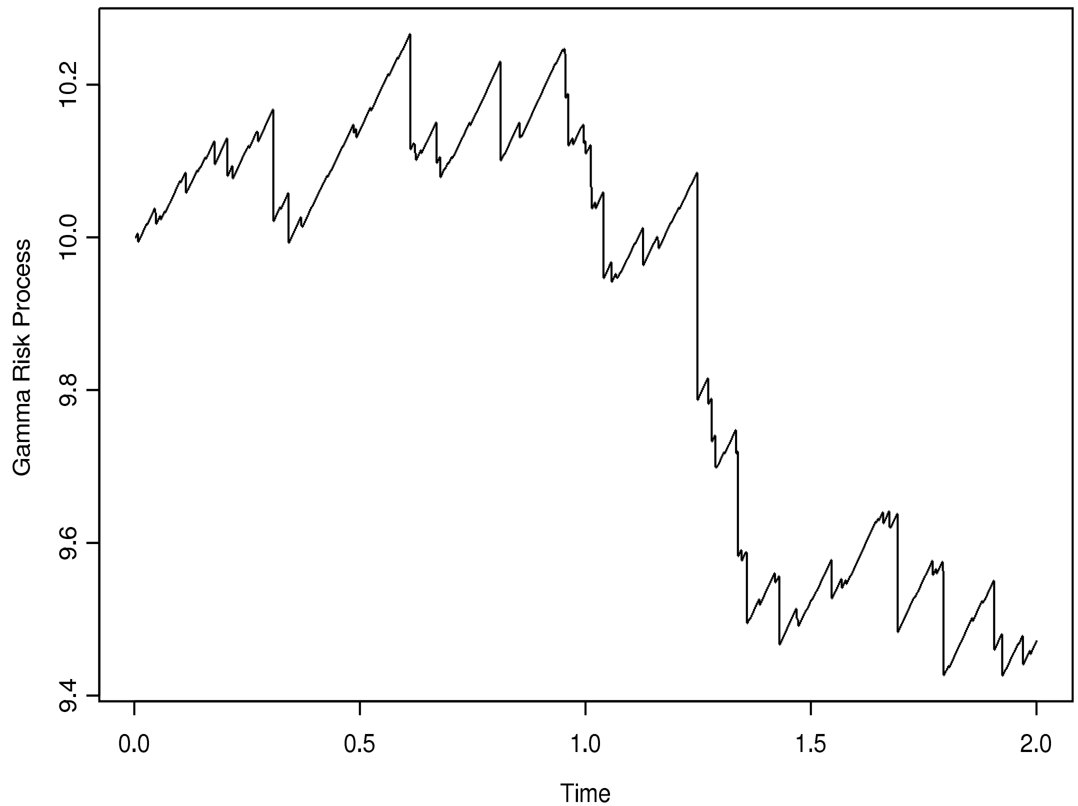
Earlier forms for the quintuple law:

1. Gusak (1969): On the joint distribution of the first exit time and exit value for homogeneous process with independent increments.

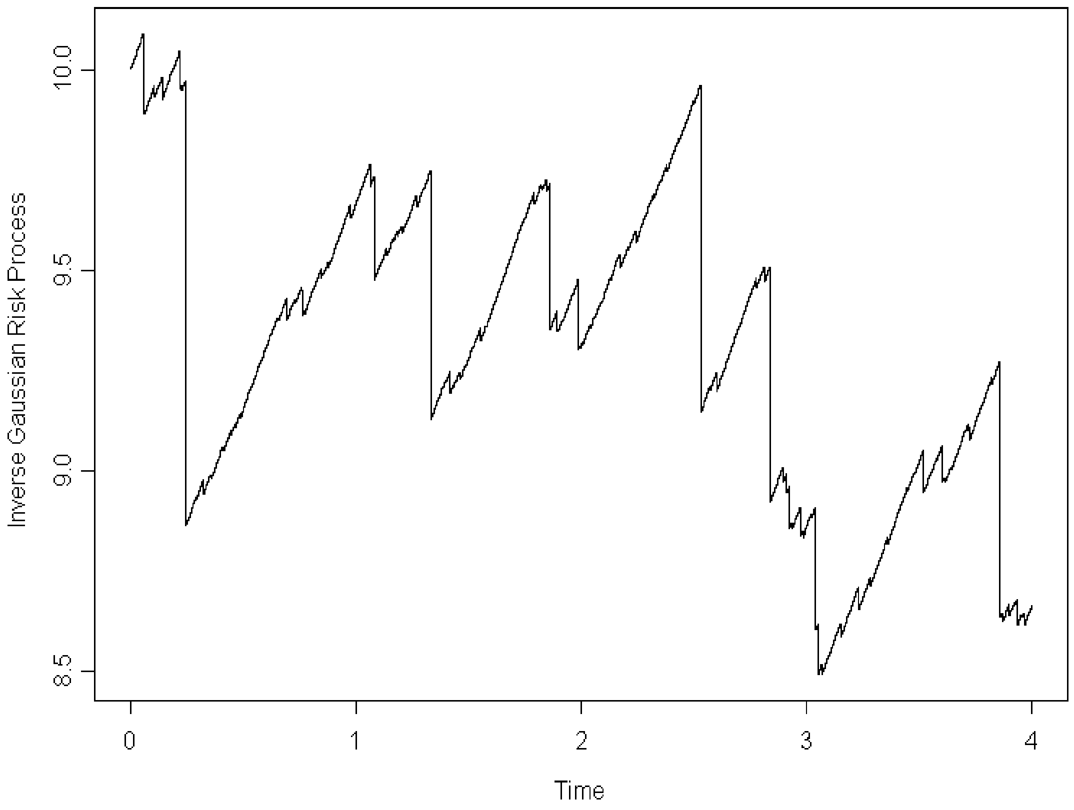
2. Gusak and Korolyuk (1969): On the first passage time across a given level for processes with independent increments.

3. Korolyuk (1975): *On ruin problem for compound Poisson process.*

Gamma Risk Process



Inverse Gaussian Risk Process



Classical Model

$$U(t) = u + ct - \sum_{i=1}^{N(t)} Y_i \ .$$

$$Y_i \sim F \ , \qquad\qquad \sum_{i=1}^{N(t)} Y_i \sim \sum_{n=0}^{\infty} p(n) F^{*(n)}(x) \ .$$

Subordinator Model

$$U(t) = u + ct - S(t) \ .$$

$$S(t) \sim \text{Gamma, IG or GIG} \ , \qquad\qquad Y_i \sim \frac{\int_\epsilon^x \nu_\epsilon(s) ds}{\int_\epsilon^\infty \nu_\epsilon(s) ds} \ .$$

Other results

The EDPF satisfies the following IDE

$$\frac{\sigma^2}{2}\,\phi''_P(u) + c\,\phi'_P(u) + \int_0^u [\phi_P(u-y) - \phi_P(u)]\,\nu_S(dy) + \chi_S(u)$$

$$+\int_0^\infty \left[\phi_P(u-y) - \phi_P(u) + y\,\phi'_P(u)\right]\nu_Z(dy) = [\Pi_S(u) + \delta]\,\phi_P(u)\,, \tag{32}$$

together with

$$\lim_{u\to\infty}\phi_P(u) \;=\; 0 \tag{33}$$

$$\phi_P(u-z) \;=\; \begin{cases} \phi_P(u-z) & \text{if} \quad u-z>0\,, \\ w(u,z-u) & \text{if} \quad u-z\leqslant 0\,. \end{cases} \tag{34}$$

Where $\Pi_S(u) = \int_0^u \nu_S(dy)$ is the integrated tail of ν_S and $\chi_S(u) \equiv \int_u^\infty w(u,y-u)\nu_S(dy)$.

More Results

The LT of the functions ϕ_D, g_D and H_w are

$$\widehat{\phi}_D(s) = \frac{\widehat{H}_w(s) + w_0\widehat{A}(s)}{1-\widehat{g}_D(s)}, \qquad s \geqslant 0 ,$$

$$\widehat{g}_D(s) = \frac{[\Psi_S(s) - \Psi_S(\rho)](s+\rho)}{(\rho - s)\Psi_{c+W}(s+\rho)}, \qquad s \geqslant 0 ,$$

$$\widehat{H}_w(s) = \frac{[\widehat{\chi}_\nu(s) - \widehat{\chi}_\nu(\rho)](s+\rho)}{(\rho - s)\Psi_{c+W}(s+\rho)}, \qquad s \geqslant 0 ,$$

where

$$\widehat{\chi}_\nu(s) = \int_0^\infty e^{-sx}\left[\int_x^\infty w(x, y-x)\nu(dy)\right]\, dx , \qquad s \geqslant 0 ,$$

$$A(x) = e^{-\rho x}[1 - K(x)] , \quad x \geqslant 0 ,$$

Remark

The functions m and K can be seen to be equal to those in generalized ladder–height decomposition.

Huzak *et al.* (2004) Let $X(t) = U(t) - u$, then, its associated ruin probability satisfies :

$$1 - \psi(u) = \frac{\theta}{1+\theta} \sum_{n=0}^{\infty} \left(\frac{1}{1+\theta} \right)^n M^{*n} * K^{*(n+1)}(u) \;, \qquad u \geqslant 0 \;, \tag{35}$$

where M is a ladder–height distribution, with LT given by

$$\widehat{m}(s) = \int_0^{\infty} e^{-s\,x}\, dM(x) = \frac{\Psi_S(s)}{s\mathbb{E}[S(1)]} \;,$$

and K is the distribution with Laplace transform given by

$$\widehat{k}(s) = \int_0^{\infty} e^{-s\,x}\, dK(x) = \frac{cs}{\Psi_{c+Z}(s)} \;.$$

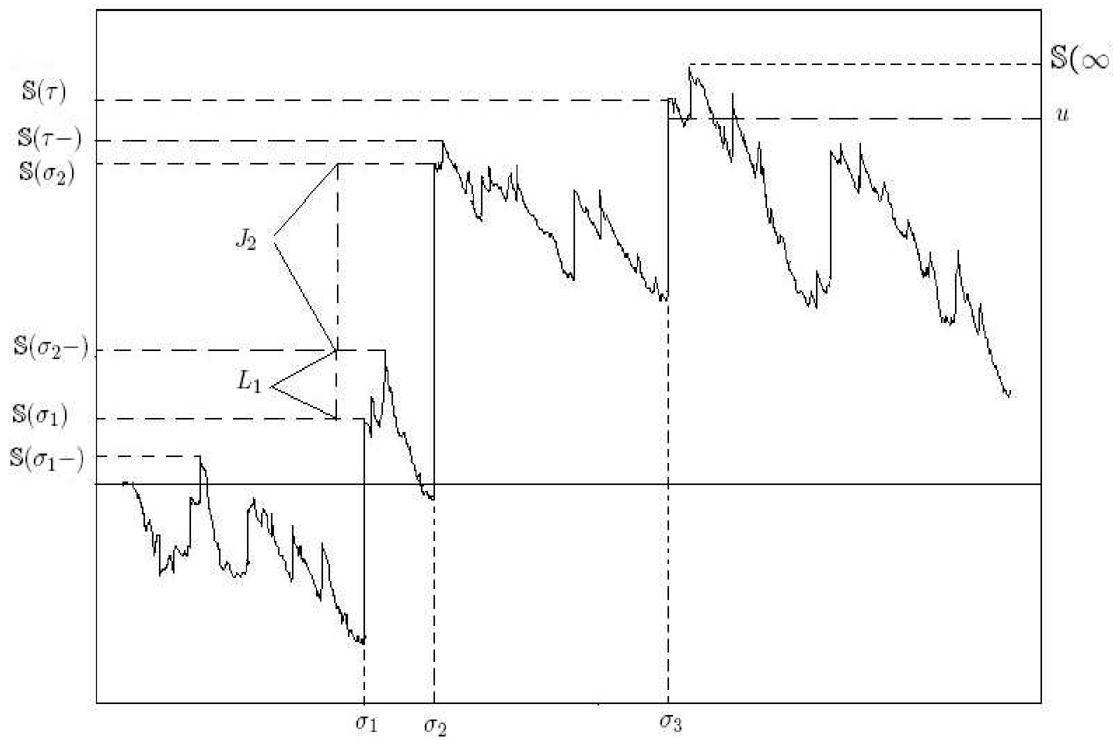
Interpretation

Let us consider

$$u - U(t) = X(t) = S(t) - c\,t - Z(t)\ , \qquad t \geq 0\ , \qquad (36)$$

Let also $\mathbb{S}(t) = \sup_{0 \leqslant s \leqslant t} X(s)$ be the running supremum of X at t.

Ladder-Height Decomposition



Interpretation

There is no first claim. But we can condition on σ_1. Let y be the size of the first overshoot causing a new record in $X(t) = S(t) - Z(t) - ct$. There are three things that could have happened:

I) This jump of the subordinator did not cause ruin i.e. $\mathbb{S}(\sigma_1) < u$. This implies that ruin was not caused previously by a jump of Z,

II) This jump, regardless of whether or not it caused ruin, was preceded by a jump from the perturbation Z that caused ruin i.e. $\mathbb{S}(\sigma_1 -) > u$.

III) This jump caused ruin, i.e. $\mathbb{S}(\sigma_1) > u$ and ruin had not been previously caused by the perturbation, i.e. $\mathbb{S}(\sigma_1 -) < u$.

In general the ruin problem is known for any Lévy process.

Consider X a Lévy process. We can decompose its path into two paths X^+ and X^-.

$$X_t^+ = S_t = \sup_{0 \leqslant s \leqslant t} X_s \ ,$$

$$X_t^- = X_t - S_t \ .$$

If we take the supremum at an exponential random time η (parameter q) then the paths X_η^+ and X_η^- are independent and we have the Wiener-Hopf factorization

$$\mathbb{E}\left[e^{zX_\eta}\right] = \mathbb{E}\left[e^{zX_\eta^+}\right]\mathbb{E}\left[e^{zX_\eta^-}\right] = \Psi^+(q,z)\Psi^-(q,z)\ ,$$

where Ψ^+ and Ψ^- are the so called Wiener-Hopf factors of X.

If we construct a process Lévy risk process $U(t) = u + X_t$ then the ruin probability

$$\psi_t(u) = \mathbb{P}[\inf_{t>s>0} X_s > -u] \ .$$

It turns out that

$$\Psi^-(q,z) = qz \int_0^\infty e^{-qt} \int_0^\infty e^{-zu}[1 - \psi_t(u)] du dt \ ,$$

i.e., the WH factor Ψ^- is the double Laplace transform of the survival probability.

The problem involves inverting a double Laplace transform of a WH factor.