

Optimal Risk Sharing with Distorted Probabilities

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OUTLINE

- Optimal Risk Sharing
- Distortion Risk Measures
- Constrained Risk Sharing
- Examples
- Open Problems

RISK SHARING

- $n \geq 2$ agents with risky endowments X_i for $i = 1, 2, \dots, n$ wish to reallocate their risk exposures. Let $X \triangleq \sum_{i=1}^n X_i$.
- V_i is the subjective valuation (preference) functional of the i -th agent.
- Collection of allocations of X is

$$\mathcal{A}(X) \triangleq \{\mathbf{Y} := (Y_1, Y_2, \dots, Y_n) : X = \sum_{i=1}^n Y_i, V_i(Y_i) \text{ finite}\}.$$

- Risk Sharing Problem: Find an *optimal* $\mathbf{Y}^* \in \mathcal{A}(X)$: (a) **Pareto optimal**; (b) satisfies a **rationality constraint**.

PARETO OPTIMALITY

- An allocation \mathbf{Y} is **Pareto optimal** if it is impossible to make some agent better off without making another agent worse off.

\Leftrightarrow if $V_i(Y'_i) \leq V_i(Y_i)$, $\forall i$ then $Y'_i = Y_i$.

- It follows that if \mathbf{Y} is Pareto optimal, then

$$\mathbf{Y} = \arg \min_{\mathcal{A}(X)} \sum_{i=1}^n \lambda_i V_i(Y_i) \quad \text{for some weights } \lambda_i \geq 0.$$

- In general multiple optima exist.
- Rationality constraint means that want $V(Y_i) \leq V(X_i)$.

EXISTING LITERATURE

- Classical results where V_i 's are of expected-utility form: Borch (1962), Arrow (1963).
- Dual theory of Yaari: Young and Browne (2000); Choquet preferences: Tallon et al. (2000).
- Recently: from point of view of monetary measures of risk. Entropic risk preferences: Barrieu and El Karoui (2005); convex law-invariant risk measures: Jouini et al. (2006).
- Extensions: when only a given set of transfer instruments is available: Filipovic and Kupper (2008b); under constraints: Bernard and Tian (2008).
- Market equilibrium: Acciaio (2007), Filipovic and Kupper (2008a),...

DISTORTION RISK MEASURES

- Traditional measures of risk distort the effective payoff.
- In the dual theory instead distort tail probabilities $\{X > t\}$.
- Thus risk-adjustment is not in terms of "decreasing marginal utility" but about "poor outcomes are more likely".
- Given a distortion function g , define the distorted probability H_g by

$$\begin{aligned}
 H_g(Y) &= \int Y d(g \circ \mathbb{P}) = \int_0^1 S_Y^{-1}(p) dg(p) \\
 &= \int_{-\infty}^0 (g[S_Y(t)] - 1) dt + \int_0^{\infty} g[S_Y(t)] dt, \quad \forall Y \in \mathcal{P}.
 \end{aligned}
 \tag{1}$$

- Works for any a.s.-finite random variable $\mathcal{P} = \{Y : \mathbb{P}[-\infty < Y < \infty] = 1\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- Have been used for 20+ years in actuarial mathematics. Origins in non-additive probability measures (Denneberg (1994)).

PROPERTIES OF DISTORTION RISK MEASURES

- If $g(p) = p$ then $H_g(Y) = \mathbb{E}Y$.
- **Value-at-Risk** at level $1 - \alpha^{-1}$: $g(p) = 1_{\{p > \alpha^{-1}\}}$.
- **Average Value-at-Risk** at level $1 - \alpha^{-1}$ (AVaR): $g(p) = \min(\alpha p, 1)$.
- Any H is a weighted average of the AVaR (Kusuoka 2001):

$$H(Y) = \int_0^1 \text{AVaR}_\alpha(Y) \mu(d\alpha),$$

for some probability measure μ on $[0, 1]$.

- Consequently also called **spectral** risk measure or Weighted VaR.

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PROPERTIES (CONT.)

- Subset of coherent risk measures that are *law-invariant* and *comonotone*.
- Two random variables Y and $Z \in \mathcal{P}$ are said to be **comonotone** if

$$(Y(\omega_1) - Y(\omega_2))(Z(\omega_1) - Z(\omega_2)) \geq 0, \quad \mathbb{P}(d\omega_1) \times \mathbb{P}(d\omega_2) - a.s.$$

- Equivalently, $\exists V \in \mathcal{P}$ and non-decreasing f_Y and f_Z s.t. $Y = f_Y(V)$ and $Z = f_Z(V)$ a.s. In other words, Y and Z *move together*.

AXIOMATIC CONSTRUCTION

Definition

A function $H : \mathcal{P} \rightarrow \mathbb{R}$ is called a *law-invariant, comonotone, monetary risk measure* (or **distortion risk measure**) if H satisfies the following:

- (a) $H(Y)$ depends only on the law of $Y \in \mathcal{P}$.
- (b) H is monotone in the natural order of \mathcal{P} .
- (c) H is cash equivariant: $H(Y + a) = H(Y) + a$ for any $a \in \mathbb{R}$.
- (d) H is subadditive and comonotone-additive: For $Y, Z \in \mathcal{P}$, $H(Y + Z) \leq H(Y) + H(Z)$, with equality for any Y, Z comonotone.
- (e) H is continuous. ($\lim_{d \rightarrow -\infty} H[\max(Y, d)] = H(Y)$; for $Y \geq 0$, $\lim_{d \rightarrow 0^+} H[\max(Y - d, 0)] = H(Y)$; $\lim_{d \rightarrow \infty} H[\min(Y, d)] = H(Y)$)

One-to-one equivalence with H_g for **concave** g 's.

OUR SETUP

- Study the risk-sharing problem where the effective random loss is $Z_i = Y_i + (a_i + b_i Y_i + c_i \mathbb{E} Y_i) = (1 + b_i) Y_i + a_i + c_i \mathbb{E} Y_i$.
- $a_i \geq 0$ is a fixed cost for transferring the risk X_i to the coalition of agents.
- $b_i \geq 0$ represents proportional costs associated with the actual size of the random loss Y_i , for example, investigative costs.
- $c_i \in \mathbb{R}$ represents costs that reflect the *expected* size of the payout Y_i .
- Overall, a_i, b_i, c_i represent **market frictions** and transaction costs.
- The case $c_i = -(1 + \theta)$ can be interpreted as the risk-loaded premium received by the agent (as in Arrow (1963)).
- Agent i , for $i = 1, 2, \dots, n$, seeks to minimize $H_{g_i}(Z_i)$ for some concave distortion function g_i .
- This is equivalent to minimizing $V_i(Y_i) := (1 + b_i)H_{g_i}(Y_i) + c_i \mathbb{E} Y_i$.

SOME EASY LEMMAS

Lemma (Pareto optimality is cash-equivariant)

If $\mathbf{X}^ = (X_1^*, X_2^*, \dots, X_n^*) \in \mathcal{A}(X)$ is Pareto optimal, then so is $(X_1^*, X_2^*, \dots, X_j^* + \beta, \dots, X_k^* - \beta, \dots, X_n^*) \in \mathcal{A}(X)$ for any $\beta \in \mathbb{R}$ and any $j, k = 1, 2, \dots, n$.*

Lemma

Suppose there exist $i, j = 1, 2, \dots, n$ such that $1 + b_i + c_i \neq 0$ and $(1 + b_i + c_i)(1 + b_j + c_j) \leq 0$, then no Pareto optimal allocation in $\mathcal{A}(X)$ exists.

STRUCTURE OF PARETO OPTIMAL ALLOCATIONS

- From Lemma 2, cash equivariance implies
 $\lambda_i(1 + b_i + c_i)\beta + \lambda_j(1 + b_j + c_j)(-\beta) = 0$ for all i, j .

Theorem

Suppose $(1 + b_i + c_i)(1 + b_j + c_j) > 0$ for all $i, j = 1, 2, \dots, n$. Pareto optimal allocations \mathbf{X}^ are obtained by minimizing*

$$\sum_{i=1}^n V_i(Y_i) / |1 + b_i + c_i| \quad \text{over } \mathbf{Y} \in \mathcal{A}(X). \quad (2)$$

COMONOTONICITY

- An allocation $\mathbf{Y} \in \mathcal{A}(X)$ is comonotone if Y_i and X are comon. $\forall i$.
- Ludkovski and Rüschendorf (2008) show that any integrable non-comonotone allocation $\mathbf{X} \in \mathcal{A}(X)$, $X_i \in L^1(\mathbb{P})$ is **dominated** by some comonotone \mathbf{X}^* , $V_i(X_i^*) \leq V_i(X_i)$, $i = 1, 2, \dots, n$.
- Follows from the fact that V_i (like all distortion risk measures) preserve the stochastic convex (ssd) order \leq_{cx} .

\Rightarrow Can restrict attention to

$$\mathcal{C}(X) \triangleq \{(f_1(X), \dots, f_n(X)) \in \mathcal{A}(X) : f_i \text{ cont., non-decreasing, } \sum_{i=1}^n f_i(x) = x \text{ for } x \in \mathbb{R}\}.$$

- So an optimal risk allocation necessarily satisfies the mutuality principle, whereby the share of each agent depends **only** on the total risk X .

INTEGRATION BY PARTS TRICK

- Suppose $Y = f(X)$ for a continuous, non-decreasing real-valued function f on \mathbb{R}_+ with $f(0) = 0$.
- Integrating by parts

$$(1 + b)H_g(Y) + c\mathbb{E}Y = (1 + b) \int_0^1 S_{f(X)}^{-1}(p) dg(p) + c \int_0^1 S_{f(X)}^{-1}(p) d(p)$$

- Thus, minimizing expression (1) is equivalent to minimizing

$$\sum_{i=1}^n \int_0^\infty \frac{[(1 + b_i)g_i + c_i](S_X(t))}{|1 + b_i + c_i|} df_i(t).$$

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- Thus, minimizing expression $\mathbb{E}Y$ is equivalent to minimizing

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 &= (1+b) \int_0^\infty g[S_X(t)] df(t) + c \int_0^\infty S_X(t) df(t) \\
 &= \int_0^\infty [(1+b)g + c](S_X(t)) df(t).
 \end{aligned}$$

- Thus, minimizing expression ▶ (2) is equivalent to minimizing

$$\sum_{i=1}^n \int_0^\infty \frac{[(1+b_i)g_i + c_i](S_X(t))}{|1+b_i+c_i|} df_i(t).$$

EXPLICIT CHARACTERIZATION

- Therefore $\mathbf{X}^* = (f_1^*(X), f_2^*(X), \dots, f_n^*(X)) \in \mathcal{C}(X)$ is a Pareto optimal allocation if and only if

$$\sum_{i \in \mathcal{I}} (f_i^*)'(t) = 1 \text{ for } \mathcal{I} = \operatorname{argmin}_{k=1,2,\dots,n} \frac{(1 + b_k)g_k(S_X(t)) + c_k S_X(t)}{|1 + b_k + c_k|}, \quad (3)$$

and $(f_i^*)'(t) = 0$ otherwise.

- Optimal contract consists of a collection of tranches.
- Never have proportional sharing.
- Similar to the result in Jouini et al. (2006). There convex duality was used to establish same theorem for $X \in L^\infty$.
- We have a direct method and also an explicit formula for f^* .

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TWO AGENT EXAMPLE

- Insurer (agent 1) pays an **indemnity** $f(X)$ to the buyer (agent 2) in exchange for **premium** $(1 + \theta)\mathbb{E}[f(X)]$.
- Take $b_1 > 0$, $c_1 = -(1 + \theta)$, $b_2 = 0$, $c_2 = -(1 + \theta)$.
- Concave distortion functions g_1 and g_2 .
- Main theorem implies that optimal contract satisfies

$$(f^*)'(t) = \begin{cases} 1, & \text{if } g_1(S_X(t)) - S_X(t) < \frac{\theta - b_1}{\theta(1+b_1)} [g_2(S_X(t)) - S_X(t)]; \\ \beta, & \text{if } g_1(S_X(t)) - S_X(t) = \frac{\theta - b_1}{\theta(1+b_1)} [g_2(S_X(t)) - S_X(t)]; \\ 0, & \text{otherwise.} \end{cases}$$

for arbitrary $0 < \beta < 1$.

DEDUCTIBLE INSURANCE

Proposition

If $(g_1(p) - p)/(g_2(p) - p)$ increases for $p \in (0, 1)$, then deductible insurance is optimal, that is, $f^*(x) = (x - d)_+$ is optimal with the deductible d given by

$$d = \inf \left\{ t : \frac{g_1(S_X(t)) - S_X(t)}{g_2(S_X(t)) - S_X(t)} \leq \frac{\theta - b_1}{\theta(1 + b_1)} \right\}. \quad (4)$$

If no such d exists, then $f^* \equiv 0$.

This proposition covers the following important cases:

Average VaR	$g_i(p) = \min(\alpha_i p, 1)$	$1 < \alpha_1 < \alpha_2$
Prop. Hazards	$g_i(p) = p^{c_i}$	$0 < c_2 < c_1 < 1;$
Dual Power Distortion	$g_i(p) = 1 - (1 - p)^{d_i}$	$1 < d_1 < d_2.$

RISK SHARING WITH CONSTRAINTS

- Often risk sharing is constrained by third-party regulators.
- Thus, amount of risk transfer is **limited**.
- Suppose that each agent faces a constraint of the form $H_{h_i}(Y_i) \leq B_i$.
- If h_i are concave then optimal allocations must still be **comonotone**.
- Otherwise not true, see an example with VaR constraints in Bernard and Tian (2008).

Theorem

The optimal risk allocation for the constrained problem is obtained by finding minimizers of

$$\sum_{i=1}^n \int_0^\infty \frac{[(1 + b_i)g_i + \lambda_i h_i + c_i](S_X(t))}{|1 + b_i + c_i + \lambda_i|} df_i(t) =: \int_0^\infty \sum_i Q_i(S_X(t)) df_i(t), \quad (5)$$

*in which $\lambda_i \geq 0$ is a **Lagrange multiplier** for the i -th constraint.*

ANOTHER TWO AGENT EXAMPLE

- Let

$$\begin{cases} g_1(p) = \min(\alpha_1 p, 1), \\ g_2(p) = \min(\alpha_2 p, 1), \\ h_1(p) = \min(\beta p, 1). \end{cases}$$

- Agent 1 is the **insurer** with the AVaR distortion function g_1 that faces a regulator constraint based on the H_{h_1} risk measure.
- Agent 2 is the **buyer** with the AVaR distortion function g_2 .
- The relevant terms in the sum are given by

$$\begin{cases} Q_1(p) = [(1 + b_1) \min(\alpha_1 p, 1) - (1 + \theta)p + \lambda \min(\beta p, 1)] / |b_1 + \lambda - \theta| \\ Q_2(p) = [\min(\alpha_2 p, 1) - (1 + \theta)p] / \theta. \end{cases}$$

- For a given Lagrange multiplier $\lambda \geq 0$, the optimal contract satisfies $(f^\lambda)'(S_X(t)) = 1$ if $Q_1(p) < Q_2(p)$; else $(f^\lambda)'(S_X(t)) = 0$.
- Solve for λ from $H_{h_1}(f^*(X)) = B$ (if constraint binds).

ANOTHER TWO AGENT EXAMPLE

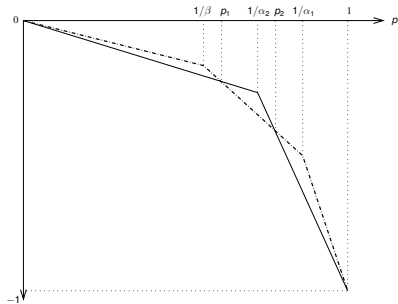
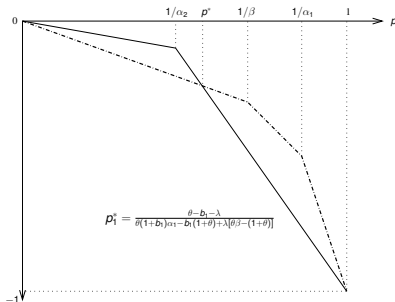
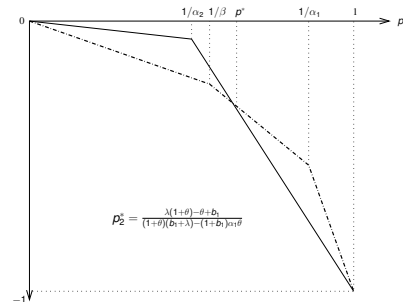
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- Solve for λ from $H_{h_1}(f^*(X)) = B$ (if constraint binds).



Risk functions for 2-agent optimal risk sharing with third-party constraint. The solid line represents $Q_1(p) = [(1 + b_1) \min(\alpha_1 p, 1) - (1 + \theta)p + \lambda \min(\beta p, 1)] / |b_1 + \lambda - \theta|$, and the dashed line is $Q_2(p) = [\min(\alpha_2 p, 1) - (1 + \theta)p] / \theta$ from (5). The crossing points correspond to the tranche levels of optimal contracts. The top two panels are for $\alpha_2 > \beta > \alpha_1 > 1$ (Case (a) on the left, Case (b) on the right), and the bottom panel is for the case $\beta > \alpha_2 > \alpha_1 > 1$.

TWO AGENT EXAMPLE (CONT.)

- Q_i 's are piecewise linear.
- When $\theta > \lambda + b_1$ and $\alpha_2 > \beta > \alpha_1 > 1$ then Q_1 and Q_2 cross at most **once**.

⇒ have deductible insurance, $f^*(x) = (x - d)_+$ and $d = S_X^{-1}(p^*)$ (see top panels).

- If $\beta > \alpha_2 > \alpha_1 > 1$, then Q_1 and Q_2 may have **two** crossing points, i.e. **capped deductible** insurance is optimal, $f^*(x) = (x - d_1)_+ \wedge d_2$ (see bottom panel).
- In the latter case B can be interpreted as the quantile amount of risk the insurer can cover.

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SINGLE AGENT OPTIMIZATION WITH CONSTRAINT

- To further illustrate the effects of constraints, consider a **single-agent** optimization problem.
- A buyer of insurance wishes to minimize her exposure given that the insurer is able to only take on limited risk.
- \implies Minimize $(1 + b)H_g(X - f(X)) + (1 + \theta)\mathbb{E}f(X)$, subject to the regulatory constraint $H_h(f(X)) \leq B$.
- Can completely classify all the possible cases for AVaR distortions.
- Explicit formulas for d and λ when X has $Exp(\mu)$ distribution.
- Counter-intuitively, there are situations where f^* is **not unique** and there is a continuum of Pareto optimal contracts.

$\theta > (1+b)\alpha - 1$			
$B > 0$	Case 5	$d = +\infty$	$\lambda = 0$
$\theta = (1+b)\alpha - 1$			
$B > 0$	Case 4a	non-unique optimum	$\lambda = 0$
$(1+b)\beta - 1 \leq \theta < (1+b)\alpha - 1$			
$\mu B \leq \beta/\alpha$	Case 4b	non-unique optimum	$\lambda = ((1+b)\alpha - (1+\theta))/\beta$
$\beta/\alpha < \mu B < \frac{(1+b)\beta}{1+\theta}$	Case 3b	$d = (1/\mu) \ln \left(\frac{\beta}{\mu B} \right)$	$\lambda = \frac{1+b}{\mu B} - \frac{1+\theta}{\beta} > 0$
$\mu B \geq \frac{(1+b)\beta}{1+\theta}$	Case 3a	$d = (1/\mu) \ln \left(\frac{1+\theta}{1+b} \right)$	$\lambda = 0$
$b < \theta < (1+b)\beta - 1$			
$\mu B \leq \beta/\alpha$	Case 4b	non-unique optimum	$\lambda = ((1+b)\alpha - (1+\theta))/\beta$
$\beta/\alpha < \mu B < 1$	Case 3b	$d = (1/\mu) \ln \left(\frac{\beta}{\mu B} \right)$	$\lambda = \frac{1+b}{\mu B} - \frac{1+\theta}{\beta} > 0$
$1 \leq \mu B < 1 + \ln \left(\frac{\beta(1+b)}{1+\theta} \right)$	Case 2b2	$d = -B + \frac{1+\ln \beta}{\mu}$	$\lambda = (1+b) - \frac{1+\theta}{\beta} e^{\mu B - 1}$
$\mu B \geq 1 + \ln \left(\frac{\beta(1+b)}{1+\theta} \right)$	Case 2a	$d = 1/\mu \ln \left(\frac{1+\theta}{1+b} \right)$	$\lambda = 0$
$\theta \leq b$			
$\mu B \leq \beta/\alpha$	Case 4	non-unique optimum	$\lambda = ((1+b)\alpha - (1+\theta))/\beta$
$\beta/\alpha < \mu B < 1$	Case 3b	$d = (1/\mu) \ln \left(\frac{\beta}{\mu B} \right)$	$\lambda = \frac{1+b}{\mu B} - \frac{1+\theta}{\beta} > 0$
$1 \leq \mu B < 1 + \ln \beta$	Case 2b1	$d = -B + \frac{1+\ln \beta}{\mu}$	$\lambda = (1+b) - \frac{1+\theta}{\beta} e^{\mu B - 1}$
$\mu B \geq 1 + \ln \beta$	Case 1	$d = 0$	$\lambda = 0$

Table: Classification of Pareto optimal allocations for single-agent constrained optimization. Here $X \sim \text{Exp}(\mu)$, $g(p) = \min(\alpha p, 1)$, $h(p) = \min(\beta p, 1)$.

CONCLUSION

- Extend results of Jouini et al. (2006) to include
 - More general risk allocations $X \in L^1$;
 - Market frictions/transaction costs.
 - Third-party constraints.
- Have a direct method that allows explicit computations for several classes of risk preferences.
- Easy proof of deductible-insurance optimality.

LOOKING AHEAD

Many further questions remain.

- It should be possible to extend these ideas to **rank dependent expected utility** (distortion + utility), aka Savage preferences.
- E.g. exponential-distortion risk measure, see Tsanakas and Desli (2003):

$$H(X) = \frac{1}{\gamma} \ln \left\{ \int_{-\infty}^0 (g[S_{e^{\gamma Y}}(t)] - 1) dt + \int_0^{\infty} g[S_{e^{\gamma Y}}(t)] dt \right\}.$$

Convex and \leq_{cx} -consistent but no longer coherent or comon.-additive.

- Analysis goes through but can no longer do the local optimization after the **integration-by-parts** trick.
- Conjecture: will get a ladder of tranches for any risk measures that are \leq_{cx} -consistent.
- How to generalize to **multi-period** problems?

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