# HOW TO INVEST IF YOU MUST:
## Active portfolio management with investment goals and shortfall constraints

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- Goal of active portfolio management is to *beat* a benchmark/index

  (passive managers *track* an index)

  - Benchmark/Index is some specific portfolio strategy (eg liability)
  - $\Rightarrow$ Goal of active manager is to *beat* another portfolio strategy

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If you *have* to beat the index, then it is a

- Survival Problem (Investing if you must ala Dubins and Savage)

If you *can* beat the index, then you can find a

- Growth Strategy (Kelly type)

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- Objectives of interest: for a given investment "goal" and "shortfall" level,
  - Maximize probability of achieving goal before shortfall (survival) ...
  - Minimize expected time until goal reached, etc.. (growth) ...,
- For fixed finite horizon problems, option strategies are optimal (Browne *Adv. Appl. Prob.*, *IAFE* and *J. Portf. Management*), but are very "risky"
- Constant proportions is optimal for many objectives over an infinite horizon (Browne *Finance & Stochastics*) , but is considered "too simple" an idea to be used.

Today

- Background: Gambling, portfolio theory
- Continuous-time models, stochastic control
- Financial models where 'Dubins and Savage' as well as 'Kelly' apply
  - External risk (insurance funds), liabilities, benchmarking
  - Finite-time goal problem and Connections with Digital options
  - New objectives for risk (shortfall probability)/ return (time to goal) tradeoff
    - Policy is no longer constant proportion
    - Framework allows consideration of risk-constraints
  - Two - player game theoretic versions
    - Contrast with discrete-time results

Gambling/Investing

$$\{Z_n : n \geq 1\} \sim \text{iid}, \;\; E(Z_i) = \mu, \;\; Var(Z_i) = \sigma^2$$

Let $\pi_n$ = fraction of wealth invested on $n$th trial.

Wealth after $n$ gambles: $X_n = X_{n-1} + (\pi_n X_{n-1}) Z_n = \cdots = X_0 \prod_{i=1}^{n} (1 + \pi_i Z_i)$

Portfolio Theory:

* $k$ risky securities, & 1 risk-free, with return $R$.
  - $\boldsymbol{Z}_n = [Z_{i,n}, \ldots, Z_{k,n}]$, $Z_{i,n}$=return on security $i$ over period $n$.
  - Random walk model: $E(\boldsymbol{Z}_n) = \boldsymbol{\mu}, Var(\boldsymbol{Z}_n) = \boldsymbol{\Sigma}$.
* Portfolio weights: $\pi_{i,n}$=% of wealth invested in security $i$ over period $n$

$$\boldsymbol{\pi}_n = (\pi_{1,n}, \ldots, \pi_{k,n})'$$

* Wealth process:

$$X_n = X_{n-1} [1 + \boldsymbol{\pi}_n' \boldsymbol{Z}_n + (1 - \boldsymbol{\pi}_n' \mathbf{1}) R] = X_{n-1} [1 + R + \boldsymbol{\pi}_n' (\boldsymbol{Z}_n - R\mathbf{1})]$$

| | Econ/Finance | Gambler/Probabilist |
|---|---|---|
| Objectives | max $E\left[U(X_N)\right]$ | "Goal Problems" (survival, growth) |
| $\mu < 0$ | $\pi_n = 0$, all $n$ | Survival: Dubins and Savage<br>$\pi_n^* = \min\left\{1, \frac{\text{goal}}{X_{n-1}} - 1\right\}$ |
| $\mu > 0$ | if $U(x) = \frac{\delta}{\delta-1}x^{1-1/\delta}$<br>$\pi_n^* = \delta\pi^*$ | Optimal Growth Policy: (Kelly)<br>$\pi^* = \arg\sup_\pi \mathrm{E}\ln(1+\pi \mathrm{Z}_1)$ |

* Optimal growth corresponds to $U(x) = \ln(x)$.

  Kelly (1957) treated case where $P(Z_i = 1) = \theta = 1 - P(Z_i = -1)$, with $\theta > 1/2$, so
$$\pi^* = 2\theta - 1$$

* Ferguson (1965) conjectured that just as $U(x) = \ln(x)$ corresponds to "growth", the exponential utility function, $U(x) = -e^{-\delta x}$ corresponds to "survival", for some $\delta$. Browne (95, 97, 99 for continuous-time relevance)

I. Single period problem:

$*$ Markowitz (1952-1959)

$$\max E\left(X_n \mid X_{n-1}\right) \quad \text{subject to} \quad Var\left(X_n \mid X_{n-1}\right) \leq v^*$$

$$\min Var\left(X_n \mid X_{n-1}\right) \quad \text{subject to} \quad E\left(X_n \mid X_{n-1}\right) \geq \mu^*$$

· Efficient frontier gives .... but which one ??

$*$ Economic theory (Tobin...): Utility function

$U\left(E\left(X_n \mid X_{n-1}\right), Var\left(X_n \mid X_{n-1}\right)\right)$ with $U_1 > 0$, and $U_2 < 0$

A utility maximizer will invest

$$\pi^* = -\frac{U_1}{2U_2}\Sigma^{-1}\left(\mu - R1\right)$$

· Portfolio seperation, Mutual fund theorem: individuals differ only according to their risk preference, $-\frac{U_1}{2U_2}$.

II. Multiperiod problem - dynamic portfolios

$$X_n = X_{n-1}\left(1 + \boldsymbol{\pi}_n' \boldsymbol{Z}_n + (1 - \boldsymbol{\pi}_n' \mathbf{1}) R\right) = X_0 \prod_{i=1}^{n} \left(1 + R + \boldsymbol{\pi}_i' \left[\boldsymbol{Z}_i - R\mathbf{1}\right]\right)$$

$*$ Utility theory: $U(X_N)$= utility from terminal wealth

Dynamic programming:

$$\begin{aligned} F(x,n) &= \max E\left[U(X_N) \,|\, X_n = x\right] \\ &= \max_{\boldsymbol{\pi}} E\left[F\left(x\left[1 + R + \boldsymbol{\pi}'\left(\boldsymbol{Z} - R\mathbf{1}\right)\right], n+1\right)\right] \\ F(x,N) &= U(x) \end{aligned}$$

$*$ THEOREM

(Bellman 1957, Hakansson 1970, Mossin 1968, Samuelson 1970)

$\pi_n^* = \pi$ (constant proportions) if and only if

$$U(x) = x^\alpha \, (\alpha < 1) \;\; \text{or} \;\; U(x) = \ln(x)$$

· Constant proportions is a *contrarian* policy

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$*$ Optimal Growth (Kelly 1956)

∘ Suppose that $\pi_n = \pi$, all $n$ (Constant Proportions policy)

$$\begin{aligned} X_n &= X_0 \exp\left\{ n \left[ \frac{1}{n} \sum_{i=1}^{n} \ln\left(1 + R + \pi' \left[ Z_i - R1 \right]\right) \right] \right\} \\ &= X_0 \exp\left\{ n \left[ E \ln\left(1 + R + \pi' \left[ Z_1 - R1 \right]\right) \right] + \tilde{o}\left(\frac{1}{n}\right) \right\} \end{aligned}$$

∘ to "grow" optimally, choose

$$\pi^* = \arg\sup_{\pi} \mathrm{E} \ln\left(1 + \mathrm{R} + \pi' \left[ Z_1 - \mathrm{R}1 \right]\right)$$



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- i. $\pi^*$ also optimal for $\sup_\pi E(\ln X_N)$.

- ii. $\pi^*$ is asymptotically optimal for $\min_\pi E(\text{time to } b)$, as $b \uparrow \infty$. (Breiman 1961)

- iii. $\pi^*$ is game theoretically optimal to maximize $P(\text{beat opponent})$, in one play (Bell & Cover 1980)

$*$ Special case: $k = 1$, simple random walk

$$Z_i = \begin{cases} +\delta & \text{w.p. } \theta \\ -\delta & \text{w.p. } 1-\theta \end{cases} \Rightarrow \pi^* = \frac{(1+R)\delta}{\delta^2 - R^2}\left[2\theta - 1 - R\right]$$

## Continuous-time

Sequence of random walks: $\{Z_i^{(1)}\}_{i\geq 1}, \{Z_i^{(2)}\}_{i\geq 1}, \ldots, \{Z_i^{(m)}\}_{i\geq 1}, \ldots$ where

$$Z_i^{(m)} = \begin{cases} +\delta_m & \text{w.p. } \theta_m \\ -\delta_m & \text{w.p. } 1-\theta_m \end{cases}$$

$$R_m = \frac{r}{m}, \;\; \delta_m := \frac{\sigma}{\sqrt{m}}, \;\; \theta_m := \frac{1}{2} + \frac{\mu}{2\sigma\sqrt{m}}$$

* i. $\sum_{i=1}^{[nt]} Z_i^n \xrightarrow{\text{w}} \mu t + \sigma W_t$, where $\{W_t, t \geq 0\}$ is standard Brownian motion.

* ii. For 'optimal growth',

$$\pi_n^* = \frac{(1+r/n)(\mu - r)}{\sigma^2 - r/n} \rightarrow \pi^* := \frac{\mu - r}{\sigma^2}$$

$$X_{[nt]}^* \xrightarrow{\text{w}} X_t^* := X_0 \exp\left\{\left(r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right)t + \left(\frac{\mu - r}{\sigma}\right)W_t\right\}$$

* iii. For any constant $\pi$,

$$X_{[nt]} \xrightarrow{\mathrm{w}} X_t := X_0 \cdot \exp\left\{\left(r + \pi(\mu - r) - \frac{\pi^2\sigma^2}{2}\right) t + \pi\sigma W_t\right\}$$

and by Ito,

$$dX_t = [r + \pi(\mu - r)]\, X_t dt + \pi\sigma X_t dW_t$$

(Ruin is impossible in finite time.)

The Basic Continuous Time Model:

* Stock Price (geometric Brownian motion): $dS_t = \mu S_t dt + \sigma S_t dW_t$
* Money Market (Riskless Bond): $dB_t = rB_t dt$
* Portfolio Allocation (Trading) Strategy:

  $\pi_t =$ fraction of wealth invested in risky stock at time $t$

* Wealth Process: $X_t^\pi =$ wealth associated with strategy $\pi = \{\pi_t, t \geq 0\}$

$$\begin{aligned} dX_t^\pi &= \pi_t X_t^\pi \frac{dS_t}{S_t} + (1 - \pi_t) X_t^\pi \frac{dB_t}{B_t} \\ &= [r + \pi_t (\mu - r)] X_t^\pi dt + \pi_t \sigma X_t^\pi dW_t \end{aligned}$$

- For $\pi_t = 0$ for all $t$, $X_t^0 = B_t$
- For $\pi_t = 1$ for all $t$, $X_t^1 = S_t$

* Constant proportions strategy: $\pi_t = \pi$ for all $t$

  ∘ For constant $\pi$, $X_t^\pi = X_0 \cdot \exp\left\{\left(r + \pi(\mu - r) - \frac{\pi^2\sigma^2}{2}\right)t + \pi\sigma W_t\right\}$

* When is constant proportions optimal ?

  ∘ Maximizing terminal utility of wealth, with $U(x) = x^\alpha$ for $\alpha < 1$,
  or
  $U(x) = \ln(x)$ [Bellman, Hakansson, Samuelson, Merton & others]

  ∘ For any $t$, "Growth rate" $\frac{1}{t} E \ln\left(X_t / X_0\right)$, is maximized by

$$\pi^* = \frac{\mu - r}{\sigma^2} \quad \text{[Optimal Growth (OG) strategy]}$$

OG Wealth: $X_t^* = X_0 \exp\left\{\left(r + \frac{(\sigma\pi^*)^2}{2}\right)t + \sigma\pi^* W_t\right\}$

  · $\pi^*$ optimal for log-utility

  · $\pi^*$ minimizes the expected time to reach any goal

* Supermartingale structure and asymptotic dominance of OG:

For any other strategy $\{\pi_t, t \geq 0\}$, the wealth process $X_t^\pi$ satisfies

$$E\left(\frac{X_{t+s}^\pi}{X_{t+s}^*}\,\middle|\,\mathcal{F}_t\right) \leq \frac{X_t^\pi}{X_t^*} \longrightarrow 0, \quad \text{as } t \to \infty$$

∘ For any constant $\pi$, $\dfrac{X_t^\pi}{X_t^*} = \exp\left\{-\dfrac{\sigma^2}{2}\left(\pi^* - \pi\right)^2 t - \sigma\left(\pi^* - \pi\right) W_t\right\}$

* Option Pricing & Optimal Growth:

∘ Derivative security: At $T$ will get payoff $g\left(X_T^1\right)$

$$\text{Time } t \text{ “fair” (Black \& Scholes) price } = X_t^0\, E\left(\frac{g(X_T^1)}{X_T^*}\,\middle|\,\mathcal{F}_t\right)$$

$*$ Some Portfolio Goal Problems

◦ Fixed Liability, must pay \$$c$ per unit time (Browne, MOR 1997)

NOTATION: $f_t$ = amount invested in risk stock(s) $= \pi_t X_t$

$$dX_t^f = \left\{ f_t \frac{dS_t}{S_t} + \left( X_t^f - f_t \right) \frac{dB_t}{B_t} \right\} - c\, dt = \left[ r X_t^f + f_t(\mu - r) - c \right] dt + f_t \sigma dW_t$$

$$X_t^f = e^{rt} \left[ X_0 - \frac{c}{r} \right] + \frac{c}{r} + \int_0^t e^{r(t-s)} f_s \left[ (\mu - r) ds + \sigma dW_s \right]$$

Wealth space breaks down into 2 regions:

· i. $X_0 < \frac{c}{r}$: Danger Zone (Survival important)

· ii. $X_0 > \frac{c}{r}$: Safe Region (Growth possible)

$*$ Survival in danger zone: $X_0 < \frac{c}{r}$.

· Objective is to min $P$(get to $b$ before $a$), for $b < c/r$.

$$f^*(X_t) = \frac{2r}{\mu - r}\left(\frac{c}{r} - X_t\right)$$

i. Invest a constant proportion of the shortfall to the safe region, independent of $\sigma^2, a, b$.

ii. Is conservative near $b$, but doesn't panic near $a$.

iii. Under $f^*$, the optimal wealth process, $X_t^*$ is

$$dX_t^* = \left(\frac{c}{r} - X_t\right) dt + \frac{2\sigma r}{\mu - r}\left(\frac{c}{r} - X_t\right) dW_t$$

PROBLEM: Let $b \to \frac{c}{r}$, then

1. $P_x\left(\tau_a^* > \tau_{c/r}^*\right) = 1 - \left(\frac{c-rx}{c-ra}\right)^{\frac{\gamma}{r}+1} < \infty$, so $c/r$ is an "attracting boundary"

2. $E_x\left(\min\left\{\tau_a^*, \tau_{c/r}^*\right\}\right) = \infty$, (in fact $\tau_{c/r}^* = \infty$ a.s.) so $c/r$ is an "inaccessible boundary"

Conclusion: No optimal policy for going from danger zone to the safe region.

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- Similar solution and issues if we want to maximize the expected NPV of the obligation (i.e., annuity)

$$E_x \int_0^\tau c e^{-rs} ds = \frac{c}{r}\left(1 - E_x\left(e^{-r\tau}\right)\right)$$

- Maximizing probability of survival until Exponentially distributed death

$*$ Survival: Dubins and Savage in continuous-time

$$dY_t^f = m(Y_t^f, f_t)dt + \nu(Y_t^f, f)dW_t$$

$$\Psi(y) = \sup_f P_y\left(\tau_a^f > \tau_b^f\right) \ , \ \ \Psi(a) = 0, \Psi(b) = 1$$

HJB: $\sup_f \left\{ m(y,f)\Psi_y + \frac{\nu^2(y,f)}{2}\Psi_{yy} \right\} = 0 = \sup_f \left\{ \left[ \frac{m(y,f)}{\nu^2(y,f)}\Psi_y + \frac{1}{2}\Psi_{yy} \right] \nu^2(y,f) \right\}$

For classical solution, $\Psi_y > 0, \Psi_{yy} < 0$, as such

$$f^* = \arg\sup_f \left\{ \frac{m(y,f)}{\nu^2(y,f)} \right\}$$

$$\rho(x) = \frac{m(y,f^*)}{\nu^2(y,f^*)}, \ \ s(x) = \exp\left\{ -2\int^x \rho(y)dy \right\} \ , \ \ \Psi(y) = \frac{\int_a^y s(x)dx}{\int_a^b s(x)dx}$$

- True only for $\nu^2 > 0$, if not ?
- Can develop $\epsilon$-optimal policy

- Growth ($X_0 \geq c/r$): How to invest in the safe region ?
  - Objective: get to $b$ as quickly as possible (minimize expected time)

$$f^*(X_t) = \frac{\mu - r}{\sigma^2}\left(X_t - \frac{c}{r}\right)$$

i. Invests a constant proportion ($\textit{ordinary optimal growth}$) of the surplus

ii. Independent of target $b$

iii. Makes danger zone inaccessible from above

iv. gives form for "Constant Proportion Portfolio Insurance"

– Survival: Exogenous risk/incomplete market (Browne, MOR 1995)

$Y_t$ is an exogenous risk (e.g. insurance claims)

$$\begin{aligned} dX_t^f &= f_t \frac{dS_t}{S_t} + \left(X_t^f - f_t\right) \frac{dB_t}{B_t} + dY_t \\ &\equiv \left[rX_t^f + f_t(\mu - r)\right] dt + f_t \sigma dW_t^{(1)} + \alpha dt + \beta dW_t^{(2)} \end{aligned}$$

I. By continuous-time Dubins and Savage, to $\max_f P\left(\tau_a^f > \tau_b^f\right)$

$f^*(x) =$

$$\frac{1}{\mu - r}\left(\sqrt{\left(rx + \alpha - \frac{\rho\beta}{\sigma}(\mu - r)\right)^2 + \beta^2(1-\rho^2)\left(\frac{\mu - r}{\sigma}\right)^2} - (rx + \alpha)\right)$$

i. $f^{*\prime}(x) < 0$, and $f^*(x) \to 0$, as $x \uparrow$.

ii. When $r = 0$,

$$f^*(x) = f^*(0) = \frac{\mu}{\sigma\eta^+} - \frac{\rho\beta}{\sigma}$$

where $\eta^+$ is poitive root to $\eta^2\left(\frac{\beta^2(1-\rho^2)}{2}\right) - \eta\left(\alpha - \frac{\rho\beta\mu}{\sigma}\right) - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 = 0$.

iii. If constrained by $f_t \le X_t$, then $f_t^* = \max\{X_t, f^*(0)\}$.

iv. Discounted time treated by HJB methods

II. Utility maximization: For $U(x) = \kappa - \theta e^{-\delta x}$, optimal policy to max $E_x\left(U\left(X_T^f\right)\right)$ is

$$f_t^* = \frac{\mu - r}{\delta\sigma^2}e^{-r(T-t)} - \frac{\rho\beta}{\sigma}$$

- For $r = 0$,

$$f_t^* = \frac{\mu - r}{\delta\sigma^2} - \frac{\rho\beta}{\sigma}$$

So, maximizing exponential utility with $\delta = \eta^+$ maximizes survival. (Verifies Ferguson's 1965 conjecture.)

– Active Portfolio Management (incomplete market case)

- Controlled wealth:

$$dX_t^\pi = X_t^\pi \left[(r + \pi_t(\mu - r))\, dt + \pi_t \sigma \, dW_t^{(1)}\right]$$

- Benchmark Target process:

$$dY_t = Y_t \left[\mu_Y dt + \sigma_Y \, dW_t^{(2)}\right], \text{ with } E(W_t^{(1)} W_t^{(2)}) = \rho t$$

  · If $\rho^2 < 1$, "incomplete" (no perfect min-var hedge exists).

  · If $\rho^2 = 1$, and $Y$ is traded, then "no arbitrage" $\Rightarrow \frac{\mu - r}{\sigma} = \frac{\mu_Y - r}{\sigma_Y}$

  $\mu_Y = r + \pi(\mu - r)$, and $\sigma_Y = \pi\sigma$, for some $\pi$

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- Investment goal $u$ reached at $t$ if $X_t^\pi = u \cdot Y_t$.
- Shortfall level $l$ reached at $t$ if $X_t^\pi = l \cdot Y_t$.
- Constant allocations optimal for many 'goal' objectives

  · Minimize shortfall probability (i.e. maximize probability of reaching goal before shortfall)

  · Minimize [maximize] expected time to reach goal [shortfall]

  · Max [Min] expected discounted reward [cost] of reaching goal [shortfall]

– Maximizing Probability of reaching goal before shortfall

- Ratio: $Z_t^\pi := X_t^\pi / Y_t \implies \left\{ \; dZ_t^\pi = Z_t^\pi \left( m(\pi_t)dt + v(\pi_t)d\tilde{W}_t \right) \; \right\}$
  - Drift: $m(\pi) = \pi(\mu - r - \sigma^2\beta) - \left(\mu_Y - r - \sigma_Y^2\right)$
  - Diffusion: $v^2(\pi) = \pi^2\sigma^2 + \sigma_Y^2 - 2\pi\sigma^2\beta$
  - $\beta = \rho\sigma_Y/\sigma$
- Minimum - Diffusion portfolio strategy: $\pi_t = \beta$
  - $v^2(\beta) = \sigma_Y^2\left(1 - \rho^2\right)$
- Maximum probability strategy:

$$\pi_t^{minprob} = \text{ the pointwise maximizer of } \frac{m(\pi_t)}{v^2(\pi_t)}$$

$$\pi^{minprob} = M - \sqrt{M^2 + S^2 - 2\rho MS}$$

$$\text{where} \quad M = \frac{\mu_Y - r - \sigma_Y^2}{\mu - r - \sigma^2\beta} \quad \text{and} \quad S = \frac{\sigma_Y}{\sigma}$$

- Who invests more ?

$$\pi^{minprob} < \beta$$

– Minimizing/maximizing expected time:

Depends on sign of "favorability" parameter: $\theta := \frac{1}{2}\left[\sigma_Y^2 + \left(\frac{\mu-r}{\sigma}\right)^2\right] - (\mu_Y - r)$

(independent of $\rho$ !)

Ordinary optimal growth strategy, $\pi^* \equiv (\mu - r)/\sigma^2$, is again optimal with benchmark

- If $\theta > 0$, then $\pi^*$ minimizes expected time to goal
- If $\theta < 0$, then $\pi^*$ maximizes expected time to shortfall

Active Portfolio Management: complete market case, $\rho^2 = 1$

$$\mu_Y = r + \pi(\mu - r) \quad \text{and} \quad \sigma_Y = \pi\sigma\,,$$

where $\pi$ is the "benchmark" strategy

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* If objective is to maximize probability of beating benchmark return by a fixed deadline $T$, then optimal strategy is to replicate a binary (digital) option.
  (Browne 1996 & 1999)
  - Risky, in that it can lead to substantial shortfalls

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* Infinite horizon probability maximizing problem becomes trivial (can reach goal with probability 1)
* Expected time to shortfall can be made infinite, and $\pi^*$ will minimize expected time to goal

So whats wrong with $\pi^*$ ?

## So whats wrong with $\pi^*$ ?

* Strategy is independent of the benchmark policy $\pi$.

* Probability of reaching goal before shortfall is independent of benchmark policy, as well as any other parameter.

  In particular, the ratio is the geometric BM

$$dZ_t(\pi^*, \pi) = Z_t(\pi^*, \pi)\left[\gamma dt + (\pi^* - \pi)\,\sigma dW_t\right], \quad \text{where} \quad \gamma = \sigma^2 (\pi^* - \pi)^2 / 2$$

  for which

  ◦ The probability of reaching goal $u$ before shortfall $l$, starting from $z$ is

$$\theta(z) = \frac{u}{z}\left(\frac{z-l}{u-l}\right)$$

  ◦ The expected time to exit the strip $(l, u)$ is

$$E_z\left(\tau(\pi^*, \pi)\right) = \gamma^{-1}\left[\theta(z)\ln\left(\frac{u}{l}\right) - \ln\left(\frac{z}{l}\right)\right]$$

– Linear tradeoff between shortfall probability and expected time to goal.

$$\sup_f \left\{ \alpha P_z \left( Z^f_{\tau^f} = u \right) - \beta E_z \left( \tau^f \right) \right\}$$

where $\tau^f$ is the first escape time from the strip $(l, u)$.

* Optimal portfolio strategy is no longer constant:

$$f^*(Z_t) = \pi^* + (\pi^* - \pi) \frac{b}{Z_t}, \quad \text{where} \quad b = \frac{u e^{-\gamma\alpha/\beta} - l}{1 - e^{-\gamma\alpha/\beta}}$$

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- Inversely modulated by the level of the ratio process $Z$.
- Depends on benchmark through $\gamma = \sigma^2 (\pi^* - \pi) / 2$
- The probability of reaching goal before shortfall is

  $$\frac{(z-l)(u+b)}{(z+b)(u-l)} \quad \text{(follows from } Z^*_t = \left(1 + \tfrac{b}{Z_0}\right) Z_t(\pi^*, \pi) - b\text{)}$$

- We always have $b \geq -l$

Risk-Constrained Minimal Time: (Gottlieb 1985)

* Initial shortfall probability prespecified: $P_{Z_0}(Z^f_{\tau^f} = u) \geq p$
* Risk-constrained problem: minimize expected time to beat benchmark subject to shortfall probability constraint
* The dual of this problem is

$$\sup_f \left[ P_z(Z^f_{\tau^f} = u) - \beta E_z \tau^f \right] \quad \text{where now } \beta \text{ is Lagrangian multiplier.}$$

* From risk-constraint (met at equality), we can determine $\beta$, or equivalently $b$

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Optimal strategy:

$$f^*(Z_t) = \pi^* + (\pi^* - \pi)\,\frac{b}{Z_t}, \quad \text{where} \quad b = \frac{pZ_0(u-l) - u(Z_0 - l)}{Z_0 - l - p(u-l)}$$

Optimal strategy:

$$f^*(Z_t) = \pi^* + (\pi^* - \pi)\,\frac{b}{Z_t}\,, \quad \text{where} \quad b = \frac{pZ_0(u-l) - u(Z_0 - l)}{Z_0 - l - p(u-l)}$$

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- Problem is feasible only for $p > (Z_0 - l)/(u - l)$
- For $p = 1$, $b = -l$, which makes the lower barrier $l$ unattainable as in many "portfolio insurance" models
- The insurance level $b$ is positive for values of $p$ satisfying

$$\frac{Z_0 - l}{u - l} < p < \frac{u}{Z_0}\left(\frac{Z_0 - l}{u - l}\right)$$

  and $b$ is negative for larger values in the region

$$p > \frac{u}{Z_0}\left(\frac{Z_0 - l}{u - l}\right) \equiv \theta(Z_0)\,,$$

  i.e., to have a higher "success" probability than the $\pi^*$, the active portfolio manager must take less risk and invest less (since $b < 0$) than the ordinary optimal growth investor
- Optimal value function provides another connection between utility (in this case HARA) and goal problems

2-Player game-theoretic goal problems:

* Two stocks, each investor restricted from one stock...

$$dS^{(i)} = \mu_i S^{(i)} dt + \sigma_i S^{(i)} dW^{(i)}, i = 1, 2, \quad E\left(W^{(1)}W^{(2)}\right) = \rho dt$$

$$\text{Investor A: } dX_t^f = f_t X_t^f \frac{dS^{(1)}}{S^{(1)}} + (1 - f_t) X_t^f \frac{dB_t}{B_t}$$

$$\text{Investor B: } dX_t^g = g_t X_t^g \frac{dS^{(2)}}{S^{(2)}} + (1 - g_t) X_t^g \frac{dB_t}{B_t}$$

* Games have nontrivial values IFF $\rho^2 < 1$ (contrary to the discrete case...)

* Constant proportions (i.e., $f = C_1, g = C_2$) are optimal for a variety of games:

  ◦ Degree of advantage parameter: $\kappa = \pi_1^*/\pi_2^* \quad (\pi_i^* = \frac{\mu_i - r}{\sigma_i^2})$

* Probability maximizing game: Solution exists if $\rho < \kappa < \frac{1}{\rho}$

$$f^* = \pi_1^* C \quad \text{and} \quad g^* = \pi_2^* \kappa^2 C,$$

$$\text{where} \quad C = \frac{\rho/\kappa - 1)\gamma - 1}{(1-\rho^2)\gamma^2 - 1} \quad \text{and} \quad \gamma = \frac{1-\kappa^2}{1+\kappa^2 - 2\rho\kappa}$$

◦ Who bets more ?

$$\frac{f^*}{g^*} = \frac{\mu_2 - r}{\mu_1 - r}$$

◦ Symmetric case: $f^* = g^* = \pi^*$

* Expected time minimizing (maximizing) game: require $\kappa > (<)1$

$$f^* = \pi_1^*\,, \quad g^* = \pi_2^* \quad \text{(optimal growth again)}$$