Proving the Regularity of the Minimal Probability of Ruin via a Game of Stopping and Control

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joint work with
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Probability of Lifetime Ruin with Stochastic Consumption

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- The wealth dynamics

$$dW_t = (r\,W_t + (\mu - r)\,\pi_t - c_t)\,dt + \sigma\,\pi_t\,dB_t, \quad W_0 = w > 0.$$

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- $\tau_0 = \inf\{t \geq 0 : W_t \leq 0\}$.
- τ_d is exponentially distributed with parameter λ (Time of death).

Our Goal

ψ given is decreasing and convex with respect to w, increasing with respect to c and is the unique classical solution of the following HJB equation

$$
\begin{aligned}
\lambda v &= (rw - c)\, v_w + a\, c\, v_c + \frac{1}{2}\, b^2\, c^2\, v_{cc} \\
&+ \min_{\pi} \left[(\mu - r)\, \pi\, v_w + \frac{1}{2}\, \sigma^2\, \pi^2\, v_{ww} + \sigma\, \pi\, b\, c\, \rho\, v_{wc} \right], \qquad (1) \\
v(0, c) &= 1 \text{ and } v(w, 0) = 0.
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$$+ \min_{\pi} \left[(\mu - r)\, \pi\, v_w + \frac{1}{2}\, \sigma^2\, \pi^2\, v_{ww} + \sigma\, \pi\, b\, c\, \rho\, v_{wc} \right], \qquad (1)$$
$$v(0, c) = 1 \text{ and } v(w, 0) = 0.$$

The optimal investment strategy π^* is given in feedback form by

$$\pi_t^* = -\frac{(\mu - r)\, \psi_w\, (W_t^*, c_t) + \sigma\, b\, \rho\, c_t\, \psi_{wc}(W_t^*, c_t)}{\sigma^2\, \psi_{ww}(W_t^*, c_t)},$$

in which W^* is the optimally controlled wealth process.

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- **Analysis of the Controller and Stopper Problem.**

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ϕ is the unique classical solution of the following HJB equation on
$\mathbb{R}_+$:

$$\lambda f = (\tilde{r} z - 1) f' + \frac{1}{2} b^2 (1 - \rho^2) z^2 f'' + \min_{\tilde{\pi}} \left[(\mu - r - \sigma b \rho) \tilde{\pi} f' + \frac{1}{2} \sigma^2 \tilde{\pi}^2 f'' \right], \quad (2)$$

$$f(0) = 1 \text{ and } \lim_{z \to \infty} f(z) = 0,$$

in which $\tilde{r} = r - a + b^2 + (\mu - r - \sigma b \rho) \rho b / \sigma$.

Interpretation of the Reduced Problem

Consider two (risky) assets with prices $\tilde{S}^{(1)}$ and $\tilde{S}^{(2)}$ following the diffusions

$$d\tilde{S}_t^{(1)} = \tilde{S}_t^{(1)} \left(\tilde{r}\, dt + b\sqrt{1-\rho^2}\, d\tilde{B}_t^{(1)} \right),$$

and

$$d\tilde{S}_t^{(2)} = \tilde{S}_t^{(2)} \left(\tilde{\mu}\, dt + \sqrt{b^2(1-\rho^2)+\sigma^2}\, d\tilde{B}_t^{(2)} \right),$$

in which $\tilde{\mu} = \mu - r + \sigma b \rho + \tilde{r}$.

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in which $\tilde{\mu} = \mu - r + \sigma b \rho + \tilde{r}$. Also, $\tilde{B}^{(1)}$ and $\tilde{B}^{(2)}$ are correlated standard Brownian motions with correlation coefficient

$$\tilde{\rho} = \frac{b\sqrt{1-\rho^2}}{\sqrt{b^2(1-\rho^2)+\sigma^2}}.$$

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- With a slight abuse of notation, let $\tilde{\pi}_t$ be the dollar amount that the individual invests in the second asset at time t; then, $Z_t - \tilde{\pi}_t$ is the amount invested in the first asset at time t.

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- With a slight abuse of notation, let $\tilde{\pi}_t$ be the dollar amount that the individual invests in the second asset at time t; then, $Z_t - \tilde{\pi}_t$ is the amount invested in the first asset at time t.
- The function ϕ is again a minimum probability of lifetime ruin!

$$\phi(z) = \inf_{\tilde{\pi} \in \tilde{\mathcal{A}}} \tilde{\mathbb{P}}^z \left(\tilde{\tau}_0 < \tau_d \right)$$

An Approximating Sequence

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- Let us define the auxiliary problem

$$\phi_M(z) = \inf_{\tilde{\pi} \in \tilde{\mathcal{A}}} \tilde{\mathbb{P}}^z \left(\tilde{\tau}_0 < (\tilde{\tau}_M \wedge \tau_d)\right),$$

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- The modified minimum probability of lifetime ruin ϕ_M is continuous on $\mathbb{R}_+$ and is decreasing, convex, and $\mathcal{C}^2$ on $(0, M)$.

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$$\lambda f = (\tilde{r}z - 1)\, f' + \frac{1}{2}\, b^2\, (1 - \rho^2)\, z^2\, f'' + \min_{\tilde{\pi}} \left[(\mu - r - \sigma\, b\rho)\, \tilde{\pi}\, f' + \frac{1}{2}\, \sigma^2\, \tilde{\pi}^2\, f'' \right],$$

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- Additionally, ϕ_M is the unique solution of the following HJB equation on $[0, M]$:

$$\begin{aligned} \lambda f &= (\tilde{r} z - 1)\, f' + \frac{1}{2}\, b^2\, (1-\rho^2)\, z^2\, f'' + \\ &\qquad \min_{\tilde{\pi}} \left[(\mu - r - \sigma\, b \rho)\, \tilde{\pi}\, f' + \frac{1}{2}\, \sigma^2\, \tilde{\pi}^2\, f'' \right], \\ f(0) &= 1, \quad f(M) = 0. \end{aligned}$$

- Furthermore, on $\mathbb{R}_+$, we have

$$\lim_{M \to \infty} \phi_M(z) = \phi(z).$$

A Controller and Stopper Problem

A Controller and Stopper Problem

Define a controlled stochastic process Y^{α} by

$$dY_t^{\alpha} = Y_t^{\alpha}\left[(\lambda - \tilde{r})\,dt + \frac{\mu - r - \sigma b \rho}{\sigma} d\hat{B}_t^{(1)}\right] + \alpha_t \left[b\sqrt{1-\rho^2}\,dt + d\hat{B}_t^{(2)}\right].$$

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Admissible strategies, $\mathcal{A}(y)$: $(\alpha_t)_{t \geq 0}$ that satisfy the integrability condition $\int_0^t \alpha_s^2\, ds < \infty$, and $Y_t^\alpha \geq 0$ almost surely, for all $t \geq 0$.

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The controller-and-stopper problem

$$\hat{\phi}_M(y) = \inf_{\tau} \sup_{\alpha \in \mathcal{A}(y)} \hat{\mathbb{E}}^y \left[\int_0^\tau e^{-\lambda t}\, Y_t^\alpha \, dt + e^{-\lambda \tau}\, u_M(Y_\tau^\alpha) \right],$$

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Here "payoff function" u_M for $y \geq 0$ is given by

$$u_M(y) := \min(My, 1).$$

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  - Suppose that $y_1 > 0$ is such that $\hat{\phi}_M(y_1) = u_M(y_1)$. First, suppose that $y_1 \leq 1/M$; then, because $\hat{\phi}_M(0) = 0$ and because $\hat{\phi}_M$ is non-decreasing, concave, and bounded above by the line My it must be that $\hat{\phi}_M(y) = My$ for all $0 \leq y \leq y_1$. Thus, if $y_1 \leq 1/M$ is not in D, then the same is true for $y \in [0, y_1]$.

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  - Finally, suppose that $y_1 \geq 1/M$; then, because $\hat{\phi}_M$ is non-decreasing, concave, and bounded above by the horizontal line 1 it must be that $\hat{\phi}_M(y) = 1$ for all $y \geq y_1$. Thus, if $y_1 \geq 1/M$ is not in D, then the same is true for $y \in [y_1, \infty)$.

Viscosity Solutions

$g \in \mathcal{C}^0(\mathbb{R}_+)$ is a viscosity supersolution (respectively, subsolution) if

$$\max\Bigg[\lambda g(y_1) - y_1 - (\lambda - \tilde{r}) y_1 f'(y_1) - m y_1^2 f''(y_1) - \max_{\alpha}\left[b\sqrt{1-\rho^2}\,\alpha f'(y_1) + \frac{1}{2}\alpha^2 f''(y_1)\right], \; g(y_1) - u_M(y_1)\Bigg] \geq 0$$

(respectively, ≤ 0) whenever $f \in \mathcal{C}^2(\mathbb{R}_+)$ and $g - f$ has a global minimum (respectively, maximum) at $y = y_1 \geq 0$. (*ii*) g is a viscosity solution of if it is both a viscosity super- and subsolution.

Back to the Continuation Region

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If $M > 1/\lambda$, then $D = (y_M, y_0)$ is non-empty. In particular, $y_M < 1/M < \lambda \leq y_0$. Also, $y_0 \geq \lambda$.

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- Suppose $M > 1/\lambda$, and suppose that D is empty. Then, for all $y \geq 0$, we have $\hat{\phi}_M(y) = u_M(y) = \min(My, 1)$. $\hat{\phi}_M = u_M$ is a viscosity solution. Because $M > 1/\lambda$, there exists $y_1 \in (1/M, \lambda)$. The value function is identically 1 in a neighborhood of y_1, the QVI evaluated at $y = y_1$ becomes $\max[\lambda - y_1, 0] = 0$, which contradicts $y_1 < \lambda$. Thus, the region D is non-empty.

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- Suppose $M > 1/\lambda$, and suppose that D is empty. Then, for all $y \geq 0$, we have $\hat{\phi}_M(y) = u_M(y) = \min(My, 1)$. $\hat{\phi}_M = u_M$ is a viscosity solution. Because $M > 1/\lambda$, there exists $y_1 \in (1/M, \lambda)$. The value function is identically 1 in a neighborhood of y_1, the QVI evaluated at $y = y_1$ becomes $\max[\lambda - y_1, 0] = 0$, which contradicts $y_1 < \lambda$. Thus, the region D is non-empty.
- On the other hand, for any $y > y_0$, since u_M is a viscosity solution, we have that $\max[\lambda - y, 0] = 0$, i.e., $\lambda \leq y$. This implies that $\lambda \leq y_0$.

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Assume that

$$D_+\hat{\phi}_M(y_0) < D_-\hat{\phi}_M(y_0).$$

Let

$$\delta \in (D_+(y_0)\hat{\phi}_M, D_-\hat{\phi}_M(y_0)).$$

. Then the function

$$\psi_\varepsilon(y) = 1 + \delta(y - y_0) - \frac{(y - y_0)^2}{2\varepsilon},$$

dominates $\hat{\phi}_M$ locally at y_0. Since $\hat{\phi}_M$ is a viscosity subsolution of we have that

$$\lambda - y_0 - (\lambda - \tilde{r})\lambda\delta + \frac{m\lambda^2}{\varepsilon} + \frac{1}{2}b^2(1 - \rho^2)\frac{\delta^2}{\varepsilon} \le 0.$$

Regularity of the Controller-and-Stopper Problem

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- $\hat{\phi}_M$ is the unique classical solution of the following free-boundary problem:

$$\lambda g = y + (\lambda - \tilde{r}) y g' + m y^2 g'' + \max_{\alpha} \left[b\sqrt{1-\rho^2}\alpha g' + \frac{1}{2}\alpha^2 g'' \right] \quad \text{on} \quad D,$$

$g(y_M) = M y_M$ and $g(y_0) = 1$.

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$g(y_M) = M y_M$ and $g(y_0) = 1$.

- The value function for this problem, namely $\hat{\phi}_M$, is non-decreasing (strictly increasing on D), concave (strictly concave on D), and $\mathcal{C}^2$ on $\mathbb{R}_+$ (except for possibly at y_M where it is $\mathcal{C}^1$).

Fenchel-Legendre Duality

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- Define the convex dual

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- If $z \geq M$, then $\Phi_M(z) = 0$ because $\hat{\phi}_M(y) \leq u_M(y) \leq My \leq zy$, from which it follows that the maximum on the right-hand side of (**) is achieved at $y^* = y_M$.

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- If $z \geq M$, then $\Phi_M(z) = 0$ because $\hat{\phi}_M(y) \leq u_M(y) \leq My \leq zy$, from which it follows that the maximum on the right-hand side of (**) is achieved at $y^* = y_M$.
- When $z < M$, $y^* = I_M(z)$ maximizes (**), in which I_M is the inverse of $\hat{\phi}'_M$ on $(y_M, y_0]$.

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- Which implies

$$\begin{aligned}\Phi_M'(z) &= \hat{\phi}_M'\left[I_M(z)\right] I_M'(z) - I_M(z) - zI_M'(z) \\ &= zI_M'(z) - I_M(z) - zI_M'(z) = -I_M(z).\end{aligned}$$

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- For $z < M$ we have

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- Which implies

$$\begin{aligned} \Phi'_M(z) &= \hat{\phi}'_M \left[I_M(z) \right] I'_M(z) - I_M(z) - z I'_M(z) \\ &= z I'_M(z) - I_M(z) - z I'_M(z) = -I_M(z). \end{aligned}$$

- Taking one more derivative

$$\Phi''_M(z) = -I'_M(z) = -1/\hat{\phi}''_M \left[I_M(z) \right].$$

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Rewrite this equation in terms of Φ_M to get

$$\lambda\Phi_M(z) = (\tilde{r}z - 1)\Phi'_M(z) - m\frac{(\Phi'_M(z))^2}{\Phi''_M(z)} + \frac{1}{2}b^2(1-\rho^2)z^2\Phi''_M(z).$$

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Also obtain the boundary conditions $\Phi_M(M) = 0$ and $\Phi_M(0) = 1$.

Fenchel-Legendre Duality

Letting $y = I_M(z) = -\Phi'_M(z)$ in the partial differential equation for $\hat{\phi}_M$ we get

$$\lambda\hat{\phi}_M\left[I_M(z)\right] = I_M(z) + (\lambda - \tilde{r})I_M(z)\hat{\phi}'_M\left[I_M(z)\right] + mI_M^2(z)\hat{\phi}''_M\left[I_M(z)\right] - \frac{1}{2}b^2(1-\rho^2)\frac{\left(\hat{\phi}'_M\left[I_M(z)\right]\right)^2}{\hat{\phi}''_M\left[I_M(z)\right]}.$$

Rewrite this equation in terms of Φ_M to get

$$\lambda\Phi_M(z) = (\tilde{r}z - 1)\Phi'_M(z) - m\frac{(\Phi'_M(z))^2}{\Phi''_M(z)} + \frac{1}{2}b^2(1-\rho^2)z^2\Phi''_M(z).$$

Also obtain the boundary conditions $\Phi_M(M) = 0$ and $\Phi_M(0) = 1$. Thanks to a verification theorem $\Phi_M = \phi_M$.

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- Show that smooth pasting holds for the controller-and-stopper problem.

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- A verification theorem shows that $\psi(w, c) = \phi(w/c)$.

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Thanks for your attention!