Modeling Credit Exposure for Collateralized Counterparties

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Discussion Plan

- Margin agreements as a means of reducing counterparty credit exposure
- Collateralized exposure and the margin period of risk
- Semi-analytical method for collateralized EE
Margin agreements as a means of reducing counterparty credit exposure
Introduction

- **Counterparty credit risk** is the risk that a counterparty in an *OTC* derivative transaction will default prior to the expiration of the contract and will be unable to make all contractual payments.
  - *Exchange-traded* derivatives bear no counterparty risk.

The primary feature that distinguishes counterparty risk from lending risk is the uncertainty of the exposure at any future date.

- **Loan**: exposure at any future date is the outstanding balance, which is certain (not taking into account prepayments).
- **Derivative**: exposure at any future date is the replacement cost, which is determined by the market value at that date and is, therefore, uncertain.

- For the derivatives whose value can be both positive and negative (e.g., swaps, forwards), counterparty risk is bilateral.
Exposure at Contract Level

- Market value of contract $i$ with a counterparty is known only for current date $t = 0$. For any future date $t$, this value $V_i(t)$ is uncertain and should be assumed random.

- If a counterparty defaults at time $\tau$ prior to the contract maturity, economic loss equals the replacement cost of the contract
  
  - If $V_i(\tau) > 0$, we do not receive anything from defaulted counterparty, but have to pay $V_i(\tau)$ to another counterparty to replace the contract.
  
  - If $V_i(\tau) < 0$, we receive $V_i(\tau)$ from another counterparty, but have to forward this amount to the defaulted counterparty.

- Combining these two scenarios, we can specify contract-level exposure $E_i(t)$ at time $t$ according to

$$E_i(\tau) = \max[V_i(\tau), 0]$$
Exposure at Counterparty Level

- **Counterparty-level exposure** at future time $t$ can be defined as the loss experienced by the bank if the counterparty defaults at time $t$ under the assumption of no recovery.

- If counterparty risk is not mitigated in any way, **counterparty-level exposure** equals the sum of **contract-level** exposures:

  $$E(t) = \sum_i E_i(t) = \sum_i \max[V_i(t), 0]$$

- If there are **netting agreements**, derivatives with positive value at the time of default offset the ones with negative value within each netting set $\text{NS}_k$, so that **counterparty-level exposure** is:

  $$E(t) = \sum_k E_{\text{NS}_k}(t) = \sum_k \max \left[ \sum_{i \in \text{NS}_k} V_i(t), 0 \right]$$

  - Each non-nettable trade represents a netting set.
Margin Agreements

- **Margin agreements** allow for further reduction of counterparty-level exposure.

- Margin agreement is a legally binding contract between two counterparties that requires one or both counterparties to post collateral under certain conditions:
  - A threshold is defined for one (unilateral agreement) or both (bilateral agreement) counterparties.
  - If the difference between the net portfolio value and already posted collateral exceeds the threshold, the counterparty must provide collateral sufficient to cover this excess (subject to minimum transfer amount).

- The threshold value depends primarily on the credit quality of the counterparty.
Collateralized Exposure

- Assuming that every margin agreement requires a netting agreement, exposure to the counterparty is

\[ E_C(t) = \sum_k \max \left\{ \sum_{i \in \text{NS}_k} V_i(t) - C_k(t), 0 \right\} \]

where \( C_k(t) \) is the market value of the collateral for netting set \( \text{NS}_k \) at time \( t \).

  - If netting set \( \text{NS}_k \) is not covered by a margin agreement, then \( C_k(t) \equiv 0 \)

- To simplify the notations, we will consider a single netting set:

\[ E_C(t) = \max \{ V_C(t), 0 \} \]

where \( V_C(t) \) is the collateralized portfolio value at time \( t \) given by

\[ V_C(t) = V(t) - C(t) = \sum_i V_i(t) - C(t) \]
Collateralized exposure and the margin period of risk
Naive Approach

- Collateral covers excess of portfolio value $V(t)$ over threshold $H$:
  \[ C(t) = \max\{V(t) - H, 0\} \]

- Therefore, collateralized portfolio value is
  \[ V_C(t) = V(t) - C(t) = \min\{V(t), H\} \]

- Thus, any scenario of collateralized exposure
  \[ E_C(t) = \max\{V_C(t), 0\} = \begin{cases} 
  0 & \text{if } V(t) < 0 \\
  V(t) & \text{if } 0 < V(t) < H \\
  H & \text{if } V(t) > H
  \end{cases} \]

  is limited by the threshold from above and by zero from below.
Margin Period of Risk

- Collateral is not delivered immediately – there is a lag $\delta_{t_{col}}$.
- After a counterparty defaults, it takes time $\delta_{t_{liq}}$ to liquidate the portfolio.
- When loss on the defaulted counterparty is realized at time $\tau$, the last time the collateral could have been received is $\tau - \delta t$, where $\delta t = \delta_{t_{col}} + \delta_{t_{liq}}$ is the margin period of risk (MPR).
- Thus, collateral at time $t$ is determined by portfolio value at time $\tau - \delta t$.
- While $\delta t$ is not known with certainty, it is usually assumed to be a fixed number.
  - Assumed value of $\delta t$ depends on the portfolio liquidity
  - Typical assumption for liquid trades is $\delta t = 2$ weeks
Including MPR in the Model

- Suppose that at time $t - \delta t$ we have collateral $C(t - \delta t)$ and portfolio value is $V(t - \delta t)$

- Then, the amount $\Delta C(t)$ that should be posted by time $t$ is
  \[
  \Delta C(t) = \max\{V(t - \delta t) - C(t - \delta t) - H, -C(t - \delta t)\}
  \]
  - Negative $\Delta C(t)$ means that collateral will be returned

- Collateral $C(t)$ available at time $t$ is
  \[
  C(t) = C(t - \delta t) + \Delta C(t) = \max\{V(t - \delta t) - H, 0\}
  \]

- Collateralized portfolio value is
  \[
  V_c(t) = V(t) - C(t) = \min\{V(t), H + \delta V(t)\}
  \]
  \[
  \delta V(t) = V(t) - V(t - \delta t)
  \]
Suppose we have a set of primary simulation time points \( \{t_k\} \) for modeling non-collateralized exposure.

For each \( t_k > \delta t \), define a look-back time point \( t_k - \delta t \).

Simulate non-collateralized portfolio value along the path that includes both primary and look-back simulation times.

Given \( V(t_{k-1}) \) and \( C(t_{k-1}) \), we calculate:

- Uncollateralized portfolio value \( V(t_k - \delta t) \) at next look-back time \( t_k - \delta t \)
- Uncollateralized portfolio value \( V(t_k) \) at next primary time \( t_k \)
- Collateral at \( t_k \) : \( C(t_k) = \max\{V(t_k - \delta t) - H, 0\} \)
- Collateralized value at \( t_k \) : \( V_C(t_k) = V(t_k) - C(t_k) \)
- Collateralized exposure at \( t_k \) : \( E_C(t_k) = \max\{V_C(t_k), 0\} \)
Simulating collateralized portfolio value
- Collateralized exposure can go above the threshold due to MPR and MTA
Semi-analytical method for collateralized EE
Portfolio Value at Primary Time Points

- Let us assume that we have run simulation only for primary time points $t$ and obtained portfolio value distribution in the form of $M$ quantities $V(j)(t)$, where $j$ (from 1 to $M$) designates different scenarios.

- From the set $\{V(j)(t)\}$ we can estimate the unconditional expectation $\mu(t)$ and standard deviation $\sigma(t)$ of the portfolio value, as well as any other distributional parameter.

- Can we estimate collateralized EE profile without simulating portfolio value at the look-back time points $\{V(j)(t - \delta t)\}$?
Collateralized EE Conditional on Path

- Collateralized EE can be represented as
  \[ EE_C(t) = E[EE_C^{(j)}(t)] \]
  where \( EE_C^{(j)}(t) \) is the collateralized EE conditional on \( V^{(j)}(t) \):
  \[ EE_C^{(j)}(t) = E\left[ \max\{V_C^{(j)}(t), 0\} \mid V^{(j)}(t) \right] \]

- Collateralized portfolio value \( V_C^{(j)}(t) \) is
  \[ V_C^{(j)}(t) = \min\{V^{(j)}(t), H + V^{(j)}(t) - V^{(j)}(t-\delta t)\} \]

- If we can calculate \( EE_C^{(j)}(t) \) analytically, the unconditional collateralized EE can be obtained as the simple average of \( EE_C^{(j)}(t) \) over all scenarios \( j \)
If Portfolio Value Were Normal…

- Let us assume that portfolio value $V(t)$ at time $t$ is normally distributed with expectation $\mu(t)$ and standard deviation $\sigma(t)$.
- Then, we can construct **Brownian bridge** from $V(0)$ to $V^{(j)}(t)$.
- Conditionally on $V^{(j)}(t)$, $V^{(j)}(t - \delta t)$ has **normal distribution** with expectation

$$\alpha^{(j)}(t) = \frac{\delta t}{t} V(0) + \frac{t - \delta t}{t} V^{(j)}(t)$$

and **standard deviation**

$$\beta^{(j)}(t) = \sigma(t) \sqrt{\frac{\delta t (t - \delta t)}{t^2}}$$

- **Conditional collateralized EE** can be obtained in closed form!
Illustration: Brownian Bridge

- Brownian bridge from $V(0)$ to $V^{(j)}(t)$

- Conditionally on $V^{(j)}(t)$, the distribution of $V^{(j)}(t - \delta t)$ is normal with mean $\alpha^{(j)}(t)$ and standard deviation $\beta^{(j)}(t)$
Arbitrary Portfolio Value Distribution

- We will keep the assumption that, conditionally on $V^{(j)}(t)$, the distribution of $V^{(j)}(t-\delta t)$ is normal, but will replace $\sigma(t)$ with the local quantity $\sigma_{\text{loc}}(t)$.

- Let us describe portfolio value $V(t)$ at time $t$ as
  \[ V(t) = \nu(t, Z) \]
  where $\nu(t, Z)$ is a monotonically increasing function of a standard normal random variable $Z$.

- Let us also define a normal equivalent portfolio value as
  \[ W(t) = w(t, Z) = \mu(t) + \sigma(t)Z \]

- To obtain $\sigma_{\text{loc}}(t)$, we will scale $\sigma(t)$ by the ratio of probability densities of $W(t)$ and $V(t)$.
Scaled Standard Deviation

- Let us denote probability density of quantity $X$ via $f_X(\cdot)$ and scale the standard deviation according to

$$\sigma_{\text{loc}}(t, Z) = \frac{f_{W(t)}[w(t, Z)]}{f_{V(t)}[v(t, Z)]} \sigma(t)$$

- Changing variables from $W(t)$ and $V(t)$ to $Z$, we have

$$f_{V(t)}[v(t, Z)] = \frac{\phi(Z)}{\partial v(t, Z)/\partial Z} \quad f_{W(t)}[w(t, Z)] = \frac{\phi(Z)}{\sigma(t)}$$

- Substitution to the definition of $\sigma_{\text{loc}}(t, Z)$ above gives

$$\sigma_{\text{loc}}(t, Z) = \frac{\partial v(t, Z)}{\partial Z}$$
Estimating CDF

- Value of $Z^{(j)}$ corresponding to $V^{(j)}(t)$ can be obtained from

$$Z^{(j)} = \Phi^{-1}\left(F_{V(t)}[V^{(j)}(t)]\right)$$

- Let us sort the array $V^{(j)}(t)$ in the increasing order so that

$$V^{[j(k)]}(t) = V^{(k)}_{\text{sorted}}(t)$$

where $j(k)$ is the sorting index

- From the sorted array we can build a piece-wise constant CDF that jumps by $1/M$ as $V(t)$ crosses any of the simulated values:

$$F_{V(t)}[V^{[j(k)]}(t)] \approx \frac{1}{2} \frac{k-1}{M} + \frac{1}{2} \frac{k}{M} = \frac{2k-1}{2M}$$
Estimating Derivative

Now we can obtain $Z^{(j)}$ corresponding to $V^{(j)}(t)$ as

$$Z^{[j(k)']} = \Phi^{-1}\left(\frac{2k - 1}{2M}\right)$$

Local standard deviation $\sigma_{\text{loc}}^{(j)}(t)$ can be estimated as:

$$\sigma_{\text{loc}}^{[j(k)']} (t) \equiv \sigma_{\text{loc}}(t, Z^{[j(k)']}) \approx \frac{V^{[j(k+\Delta k)]}(t) - V^{[j(k-\Delta k)]}(t)}{Z^{[j(k+\Delta k)]} - Z^{[j(k-\Delta k)]}}$$

Offset $\Delta k$ should not be too small (too much noise) or too large (loss of resolution). This range works well:

$$20 \leq \Delta k \leq 0.05M$$
We assume that, conditionally on $V^{(j)}(t)$, $V^{(j)}(t - \delta t)$ has normal distribution with expectation

$$\alpha^{(j)}(t) = \frac{\delta t}{t} V(0) + \frac{t - \delta t}{t} V^{(j)}(t)$$

and standard deviation

$$\beta^{(j)}(t) = \sigma^{(j)}_{loc}(t) \sqrt{\frac{\delta t (t - \delta t)}{t^2}}$$

Collateralized exposure depends on $\delta V^{(j)}(t)$, which is also normal conditionally on $V^{(j)}(t)$ with the same standard deviation $\beta^{(j)}(t)$ and expectation $\delta \alpha^{(j)}(t)$ given by

$$\delta \alpha^{(j)}(t) = V^{(j)}(t) - \alpha^{(j)}(t) = \frac{\delta t}{t} \left[ V^{(j)}(t) - V(0) \right]$$
Calculating Conditional Collateralized EE

- Collateralized EE conditional on scenario $j$ at time $t$ is
  \[ EE^C_j(t) = E \left[ \max \left\{ \min \left\{ V^{(j)}(t), H + \delta V^{(j)}(t) \right\}, 0 \right\} \right| V^{(j)}(t) \]  

- $EE^C_j(t)$ equals zero whenever $V^{(j)}(t) < 0$, so that
  \[ EE^C_j(t) = 1_{\{V^{(j)}(t)>0\}} E \left[ \min \left\{ V^{(j)}(t), H + \delta V^{(j)}(t) \right\} \right| V^{(j)}(t) \]  

- Since $\delta V^{(j)}(t)$ has normal distribution, we can write
  \[
  EE^C_j(t) = 1_{\{V^{(j)}(t)>0\}} \int_{-\infty}^{\infty} \min \left\{ V^{(j)}(t), H + \delta \alpha^{(j)}(t) + \beta^{(j)}(t) z \right\} \phi(z) dz \\
  = 1_{\{V^{(j)}(t_k)>0\}} \left\{ \int_{-d_2}^{-d_1} \left[ H + \delta \alpha^{(j)}(t) + \beta^{(j)}(t) z \right] \phi(z) dz + V^{(j)}(t) \int_{-d_1}^{\infty} \phi(z) dz \right\}
  \]
Conditional Collateralized EE Result

Evaluating the integrals, we obtain:

$$EE_{C}^{(j)}(t) = 1_{\{V^{(j)}(t)>0\}} \left\{ \left[ H + \delta \alpha^{(j)}(t) \right] \left[ \Phi(d_2) - \Phi(d_1) \right] \\
+ \beta^{(j)}(t) \left[ \phi(d_2) - \phi(d_1) \right] + V^{(j)}(t)\Phi(d_1) \right\}$$

where

$$d_1 = \frac{H + \delta \alpha^{(j)}(t) - V^{(j)}(t)}{\beta^{(j)}(t)}$$

$$d_2 = \frac{H + \delta \alpha^{(j)}(t)}{\beta^{(j)}(t)}$$
Example 1: 5-Year IR Swap Starting in 5 Years

- Uncollateralized EE and the two thresholds we will consider
Forward Starting Swap and Small Threshold

- Collateralized EE when threshold is 0.5%
Forward Starting Swap and Large Threshold

- **Collateralized EE** when threshold is 2.0%
Example 2: 5-Year IR Swap Starting Now

- Uncollateralized EE and the two thresholds we will consider

![Graph showing expected exposure over time for different thresholds. The graph includes a line for EE (no collateral) and two dashed lines for thresholds 0.5% and 2.0% of notional.](image)
Swap Starting Now and Small Threshold

- **Collateralized EE** when threshold is **0.5%**
Swap Starting Now and Large Threshold

- **Collateralized EE** when threshold is **2.0%**
Conclusion

- **Margin agreements** are important risk mitigation tools that need to be modeled accurately.

- **Collateral** available at a primary time point depends on the portfolio value at the corresponding look-back time point.

- **Full Monte Carlo** method of simulating collateralized exposure is the most flexible approach, but requires simulating portfolio value at both primary and look-back time points.

- We have developed a **semi-analytical** method of calculating collateralized EE that avoids doubling the simulation time:
  - Portfolio value is simulated only at primary time points.
  - For each portfolio value scenario at a primary time point, conditional collateralized EE is calculated in closed form.
  - Unconditional collateralized EE at a primary time point is obtained by averaging the conditional collateralized EE over all scenarios.