Modeling Credit Exposure for Collateralized Counterparties

Michael Pykhtin

Credit Analytics & Methodology
Bank of America

Fields Institute Quantitative Finance Seminar

Toronto; February 25, 2009

Disclaimer

This document is NOT a research report under U.S. law and is NOT a product of a fixed income research department. Opinions expressed here do not necessarily represent opinions or practices of Bank of America N.A. The analyses and materials contained herein are being provided to you without regard to your particular circumstances, and any decision to purchase or sell a security is made by you independently without reliance on us. This material is provided for information purposes only and is not an offer or a solicitation for the purchase or sale of any financial instrument. Although this information has been obtained from and is based on sources believed to be reliable, we do not guarantee its accuracy. Neither Bank of America N.A., Banc Of America Securities LLC nor any officer or employee of Bank of America Corporation affiliate thereof accepts any liability whatsoever for any direct, indirect or consequential damages or losses arising from any use of this report or its contents.

Discussion Plan

- Margin agreements as a means of reducing counterparty credit exposure
- Collateralized exposure and the margin period of risk
- Semi-analytical method for collateralized EE

Margin agreements as a means of reducing counterparty credit exposure

Introduction

- ***Counterparty credit risk*** is the risk that a counterparty in an ***OTC*** derivative transaction will default prior to the expiration of the contract and will be unable to make all contractual payments.
  - ***Exchange-traded*** derivatives bear no counterparty risk.
- The primary feature that distinguishes counterparty risk from lending risk is the uncertainty of the exposure at any future date.
  - **Loan**: exposure at any future date is the outstanding balance, which is certain (not taking into account prepayments).
  - **Derivative**: exposure at any future date is the replacement cost, which is determined by the market value at that date and is, therefore, uncertain.
- For the derivatives whose value can be both positive and negative (e.g., swaps, forwards), counterparty risk is bilateral.

Exposure at Contract Level

- Market value of contract i with a counterparty is known only for current date $t = 0$. For any future date t, this value $V_i(t)$ is uncertain and should be assumed random.
- If a counterparty defaults at time τ prior to the contract maturity, economic loss equals the replacement cost of the contract
  - If $V_i(\tau) > 0$, we do not receive anything from defaulted counterparty, but have to pay $V_i(\tau)$ to another counterparty to replace the contract.
  - If $V_i(\tau) < 0$, we receive $V_i(\tau)$ from another counterparty, but have to forward this amount to the defaulted counterparty.
- Combining these two scenarios, we can specify ***contract-level exposure*** $E_i(t)$ at time t according to

$$E_i(\tau) = \max[V_i(\tau), 0]$$

Exposure at Counterparty Level

- ***Counterparty-level exposure*** at future time *t* can be defined as the loss experienced by the bank if the counterparty defaults at time *t* under the assumption of no recovery
- If counterparty risk is not mitigated in any way, ***counterparty-level*** exposure equals the sum of ***contract-level*** exposures

$$E(t) = \sum_i E_i(t) = \sum_i \max[V_i(t), 0]$$

- If there are ***netting agreements***, derivatives with positive value at the time of default offset the ones with negative value within each netting set NS_k, so that ***counterparty-level exposure*** is

$$E(t) = \sum_k E_{\mathrm{NS}_k}(t) = \sum_k \max\left[\sum_{i \in \mathrm{NS}_k} V_i(t),\ 0\right]$$

  – Each non-nettable trade represents a netting set

Margin Agreements

- ***Margin agreements*** allow for further reduction of counterparty-level exposure.
- Margin agreement is a legally binding contract between two counterparties that requires one or both counterparties to post collateral under certain conditions:
  - A threshold is defined for one (unilateral agreement) or both (bilateral agreement) counterparties.
  - If the difference between the net portfolio value and already posted collateral exceeds the threshold, the counterparty must provide collateral sufficient to cover this excess (subject to minimum transfer amount).
- The threshold value depends primarily on the credit quality of the counterparty.

Collateralized Exposure

- Assuming that every margin agreement requires a netting agreement, exposure to the counterparty is

$$E_C(t) = \sum_k \max\left\{ \sum_{i \in \mathrm{NS}_k} V_i(t) - C_k(t),\ 0 \right\}$$

where $C_k(t)$ is the market value of the collateral for netting set NS_k at time t.

  - If netting set NS_k is not covered by a margin agreement, then $C_k(t) \equiv 0$

- To simplify the notations, we will consider a single netting set:

$$E_C(t) = \max\{V_C(t),\ 0\}$$

where $V_C(t)$ is the collateralized portfolio value at time t given by

$$V_C(t) = V(t) - C(t) = \sum_i V_i(t) - C(t)$$

Collateralized exposure and the margin period of risk

Naive Approach

- Collateral covers excess of portfolio value $V(t)$ over threshold H:

$$C(t) = \max\{V(t) - H, 0\}$$

- Therefore, collateralized portfolio value is

$$V_C(t) = V(t) - C(t) = \min\{V(t), H\}$$

- Thus, ***any scenario*** of collateralized exposure

$$E_C(t) = \max\{V_C(t), 0\} = \begin{cases} 0 & \text{if } V(t) < 0 \\ V(t) & \text{if } 0 < V(t) < H \\ H & \text{if } V(t) > H \end{cases}$$

is limited by the threshold from above and by zero from below.

Margin Period of Risk

- Collateral is not delivered immediately – there is a lag δt_{col}.
- After a counterparty defaults, it takes time δt_{liq} to liquidate the portfolio.
- When loss on the defaulted counterparty is realized at time τ, the last time the collateral could have been received is $\tau - \delta t$, where $\delta t = \delta t_{\text{col}} + \delta t_{\text{liq}}$ is the ***margin period of risk*** (MPR).
- Thus, collateral at time t is determined by portfolio value at time $\tau - \delta t$.
- While δt is not known with certainty, it is usually assumed to be a fixed number.
  - Assumed value of δt depends on the portfolio liquidity
  - Typical assumption for liquid trades is $\delta t = 2$ weeks

Including MPR in the Model

- Suppose that at time $t-\delta t$ we have collateral collateral $C(t-\delta t)$ and portfolio value is $V(t-\delta t)$
- Then, the amount $\Delta C(t)$ that should be posted by time t is

$$\Delta C(t) = \max\{V(t-\delta t) - C(t-\delta t) - H, -C(t-\delta t)\}$$

  – Negative $\Delta C(t)$ means that collateral will be returned

- Collateral $C(t)$ available at time t is

$$C(t) = C(t-\delta t) + \Delta C(t) = \max\{V(t-\delta t) - H, 0\}$$

- Collateralized portfolio value is

$$V_C(t) = V(t) - C(t) = \min\{V(t), H + \delta V(t)\}$$

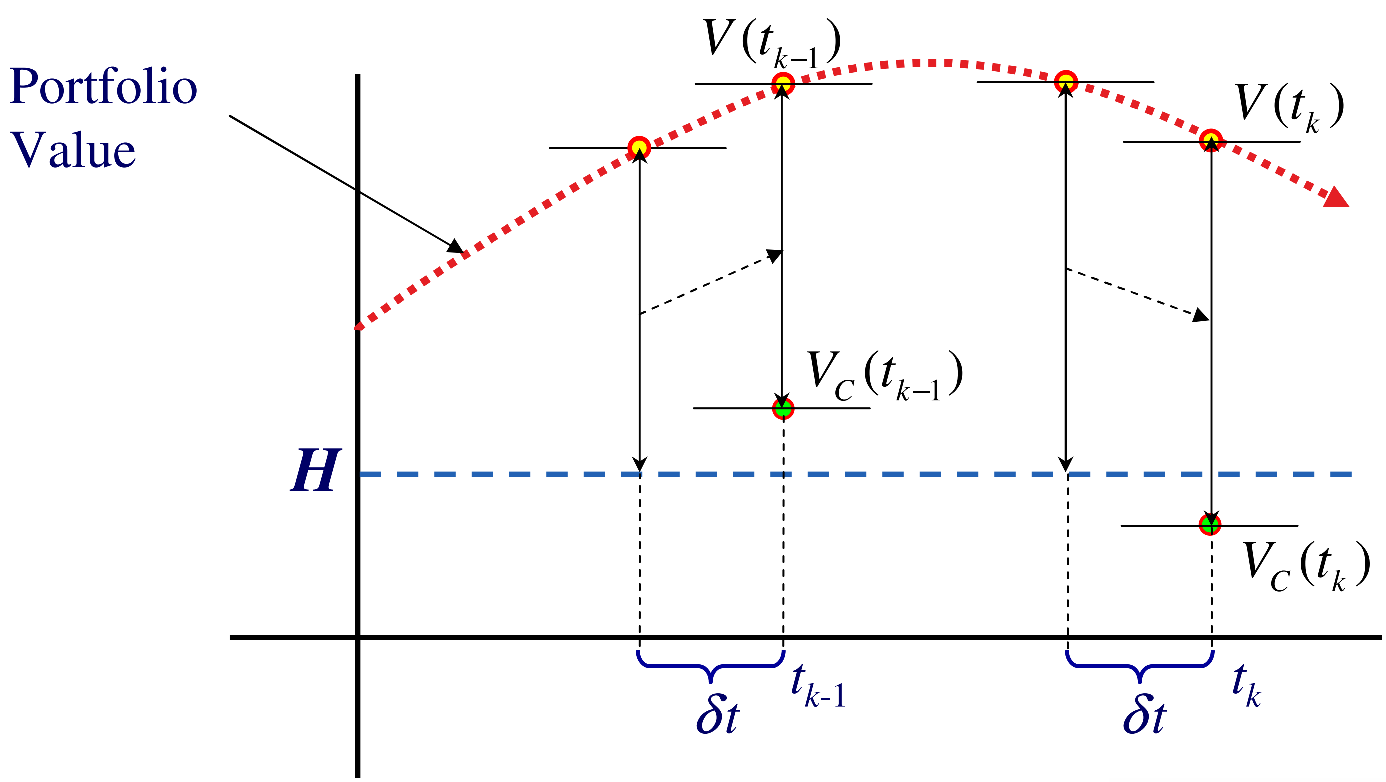
$$\delta V(t) = V(t) - V(t-\delta t)$$

Full Monte Carlo Algorithm

- Suppose we have a set of ***primary*** simulation time points $\{t_k\}$ for modeling non-collateralized exposure
- For each $t_k > \delta t$, define a ***look-back*** time point $t_k - \delta t$
- Simulate non-collateralized portfolio value along the path that includes both ***primary*** and ***look-back*** simulation times
- Given $V(t_{k-1})$ and $C(t_{k-1})$, we calculate
  - Uncollateralized portfolio value $V(t_k - \delta t)$ at next look-back time $t_k - \delta t$
  - Uncollateralized portfolio value $V(t_k)$ at next primary time t_k
  - Collateral at t_k: $C(t_k) = \max\{V(t_k - \delta t) - H, 0\}$
  - Collateralized value at t_k: $V_C(t_k) = V(t_k) - C(t_k)$
  - Collateralized exposure at t_k: $E_C(t_k) = \max\{V_C(t_k), 0\}$

Illustration of Full Monte Carlo Method

- Simulating collateralized portfolio value
  - Collateralized exposure can go above the threshold due to MPR and MTA



Semi-analytical method for collateralized EE

Portfolio Value at Primary Time Points

- Let us assume that we have run simulation ***only*** for primary time points t and obtained portfolio value distribution in the form of M quantities $V^{(j)}(t)$, where j (from 1 to M) designates different scenarios

- From the set $\{V^{(j)}(t)\}$ we can estimate the unconditional expectation $\mu(t)$ and standard deviation $\sigma(t)$ of the portfolio value, as well as any other distributional parameter

- Can we estimate collateralized EE profile ***without*** simulating portfolio value at the look-back time points $\{V^{(j)}(t-\delta t)\}$?

Collateralized EE Conditional on Path

- Collateralized EE can be represented as

$$\mathrm{EE}_C(t) = \mathrm{E}[\mathrm{EE}_C^{(j)}(t)]$$

where $\mathrm{EE}_C^{(j)}(t)$ is the collateralized EE ***conditional*** on $V^{(j)}(t)$:

$$\mathrm{EE}_C^{(j)}(t) = \mathrm{E}\left[\max\{V_C^{(j)}(t), 0\} \middle| V^{(j)}(t)\right]$$

- Collateralized portfolio value $V_C^{(j)}(t)$ is

$$V_C^{(j)}(t) = \min\left\{V^{(j)}(t), H + V^{(j)}(t) - V^{(j)}(t-\delta t)\right\}$$

- If we can calculate $\mathrm{EE}_C^{(j)}(t)$ analytically, the ***unconditional collateralized EE*** can be obtained as the simple average of $\mathrm{EE}_C^{(j)}(t)$ over all scenarios *j*

If Portfolio Value Were Normal…

- Let us assume that portfolio value $V(t)$ at time t is normally distributed with expectation $\mu(t)$ and standard deviation $\sigma(t)$.
- Then, we can construct ***Brownian bridge*** from $V(0)$ to $V^{(j)}(t)$
- Conditionally on $V^{(j)}(t)$, $V^{(j)}(t-\delta t)$ has ***normal distribution*** with ***expectation***

$$\alpha^{(j)}(t) = \frac{\delta t}{t} V(0) + \frac{t-\delta t}{t} V^{(j)}(t)$$

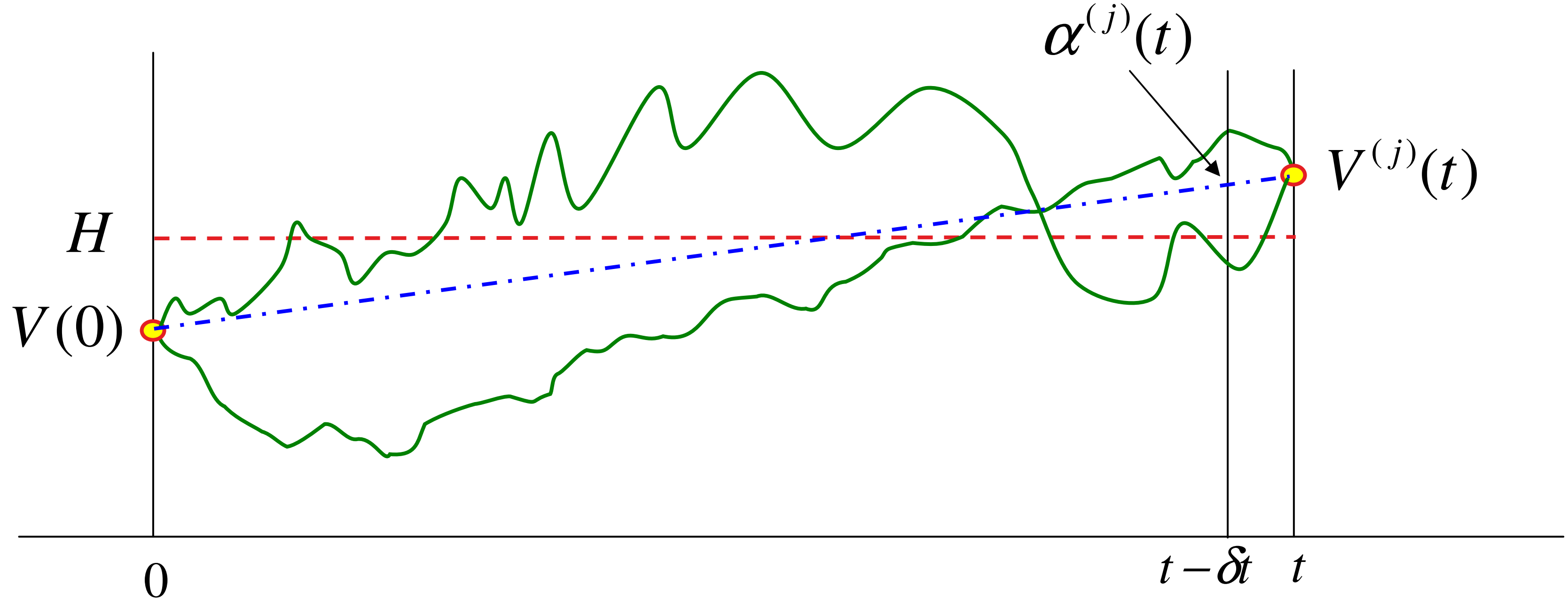
and ***standard deviation***

$$\beta^{(j)}(t) = \sigma(t) \sqrt{\frac{\delta t\,(t-\delta t)}{t^2}}$$

- ***Conditional collateralized EE*** can be obtained in closed form!

Illustration: Brownian Bridge

- Brownian bridge from $V(0)$ to $V^{(j)}(t)$



- Conditionally on $V^{(j)}(t)$, the distribution of $V^{(j)}(t-\delta t)$ is normal with mean $\alpha^{(j)}(t)$ and standard deviation $\beta^{(j)}(t)$

Arbitrary Portfolio Value Distribution

- We will keep the assumption that, conditionally on $V^{(j)}(t)$, the distribution of $V^{(j)}(t-\delta t)$ is normal, but will replace $\sigma(t)$ with the local quantity $\sigma_{\text{loc}}(t)$
- Let us describe portfolio value $V(t)$ at time t as

$$V(t) = v(t, Z)$$

where $v(t, Z)$ is a monotonically increasing function of a standard normal random variable Z.

- Let us also define a ***normal equivalent*** portfolio value as

$$W(t) = w(t, Z) = \mu(t) + \sigma(t) Z$$

- To obtain $\sigma_{\text{loc}}(t)$, we will scale $\sigma(t)$ by the ratio of probability densities of $W(t)$ and $V(t)$

Scaled Standard Deviation

- Let us denote probability density of quantity X via $f_X(\cdot)$ and scale the standard deviation according to

$$\sigma_{\text{loc}}(t,Z) = \frac{f_{W(t)}[w(t,Z)]}{f_{V(t)}[v(t,Z)]}\sigma(t)$$

- Changing variables from $W(t)$ and $V(t)$ to Z, we have

$$f_{V(t)}[v(t,Z)] = \frac{\phi(Z)}{\partial v(t,Z)/\partial Z} \qquad f_{W(t)}[w(t,Z)] = \frac{\phi(Z)}{\sigma(t)}$$

- Substitution to the definition of $\sigma_{\text{loc}}(t,Z)$ above gives

$$\sigma_{\text{loc}}(t,Z) = \frac{\partial v(t,Z)}{\partial Z}$$

Estimating CDF

- Value of $Z^{(j)}$ corresponding to $V^{(j)}(t)$ can be obtained from

$$Z^{(j)} = \Phi^{-1}\left(F_{V(t)}[V^{(j)}(t)]\right)$$

- Let us sort the array $V^{(j)}(t)$ in the increasing order so that

$$V^{[j(k)]}(t) = V_{\text{sorted}}^{(k)}(t)$$

where $j(k)$ is the sorting index

- From the sorted array we can build a piece-wise constant CDF that jumps by $1/M$ as $V(t)$ crosses any of the simulated values:

$$F_{V(t)}[V^{[j(k)]}(t)] \approx \frac{1}{2}\frac{k-1}{M} + \frac{1}{2}\frac{k}{M} = \frac{2k-1}{2M}$$

Bank of America Higher Standards

Estimating Derivative

- Now we can obtain $Z^{(j)}$ corresponding to $V^{(j)}(t)$ as

$$Z^{[j(k)]} = \Phi^{-1}\left(\frac{2k-1}{2M}\right)$$

- Local standard deviation $\sigma_{\text{loc}}^{(j)}(t)$ can be estimated as :

$$\sigma_{\text{loc}}^{[j(k)]}(t) \equiv \sigma_{\text{loc}}(t, Z^{[j(k)]}) \approx \frac{V^{[j(k+\Delta k)]}(t) - V^{[j(k-\Delta k)]}(t)}{Z^{[j(k+\Delta k)]} - Z^{[j(k-\Delta k)]}}$$

- Offset Δk should not be too small (too much noise) or too large (loss of resolution). This range works well:

$$20 \le \Delta k \le 0.05M$$

Back to the Bridge

- We assume that, conditionally on $V^{(j)}(t)$, $V^{(j)}(t-\delta t)$ has ***normal distribution*** with ***expectation***

$$\alpha^{(j)}(t) = \frac{\delta t}{t} V(0) + \frac{t-\delta t}{t} V^{(j)}(t)$$

and ***standard deviation***

$$\beta^{(j)}(t) = \sigma_{\mathrm{loc}}^{(j)}(t) \sqrt{\frac{\delta t\,(t-\delta t)}{t^2}}$$

- ***Collateralized exposure*** depends on $\delta V^{(j)}(t)$, which is also normal conditionally on $V^{(j)}(t)$ with the same standard deviation $\beta^{(j)}(t)$ and expectation $\delta\alpha^{(j)}(t)$ given by

$$\delta\alpha^{(j)}(t) = V^{(j)}(t) - \alpha^{(j)}(t) = \frac{\delta t}{t}\left[V^{(j)}(t) - V(0)\right]$$

Calculating Conditional Collateralized EE

- Collateralized EE conditional on scenario *j* at time *t* is

$$\mathrm{EE}_C^{(j)}(t) = \mathrm{E}\left[\max\left\{\min\left\{V^{(j)}(t), H + \delta V^{(j)}(t)\right\}, 0\right\} \middle| V^{(j)}(t)\right]$$

- $\mathrm{EE}_C^{(j)}(t)$ equals ***zero*** whenever $V^{(j)}(t) < 0$, so that

$$\mathrm{EE}_C^{(j)}(t) = 1_{\left\{V^{(j)}(t)>0\right\}}\,\mathrm{E}\left[\min\left\{V^{(j)}(t), H + \delta V^{(j)}(t)\right\} \middle| V^{(j)}(t)\right]$$

- Since $\delta V^{(j)}(t)$ has normal distribution, we can write

$$\mathrm{EE}_C^{(j)}(t) = 1_{\left\{V^{(j)}(t)>0\right\}} \int_{-\infty}^{\infty} \min\left\{V^{(j)}(t), H + \delta\alpha^{(j)}(t) + \beta^{(j)}(t) z\right\} \phi(z)\,dz$$

$$= 1_{\left\{V^{(j)}(t_k)>0\right\}} \left\{ \int_{-d_2}^{-d_1} \left[H + \delta\alpha^{(j)}(t) + \beta^{(j)}(t) z\right] \phi(z)\,dz + V^{(j)}(t) \int_{-d_1}^{\infty} \phi(z)\,dz \right\}$$

Conditional Collateralized EE Result

- Evaluating the integrals, we obtain:

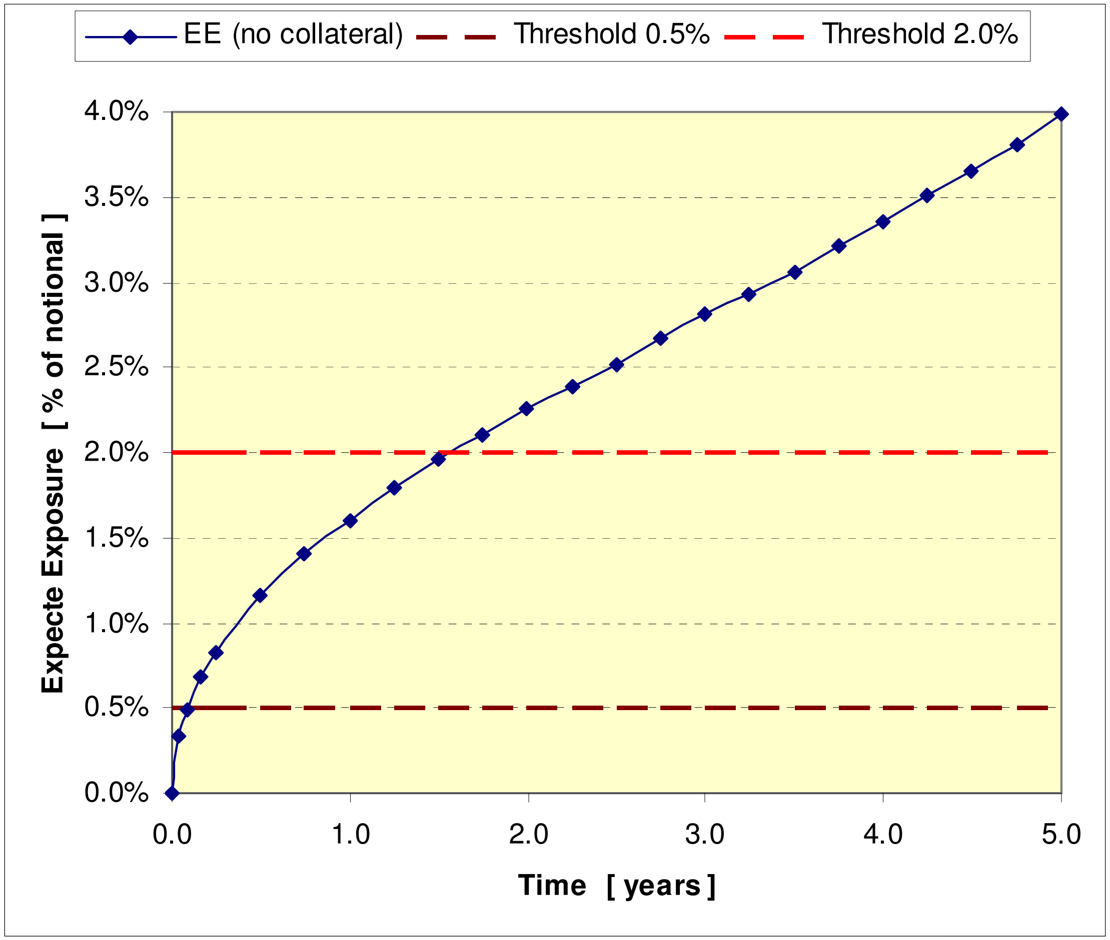
$$\mathrm{EE}_C^{(j)}(t) = 1_{\{V^{(j)}(t)>0\}}\Big\{\big[H + \delta\alpha^{(j)}(t)\big]\big[\Phi(d_2) - \Phi(d_1)\big]$$
$$+\beta^{(j)}(t)\big[\phi(d_2) - \phi(d_1)\big] + V^{(j)}(t)\Phi(d_1)\Big\}$$

where

$$d_1 = \frac{H + \delta\alpha^{(j)}(t) - V^{(j)}(t)}{\beta^{(j)}(t)} \qquad d_2 = \frac{H + \delta\alpha^{(j)}(t)}{\beta^{(j)}(t)}$$

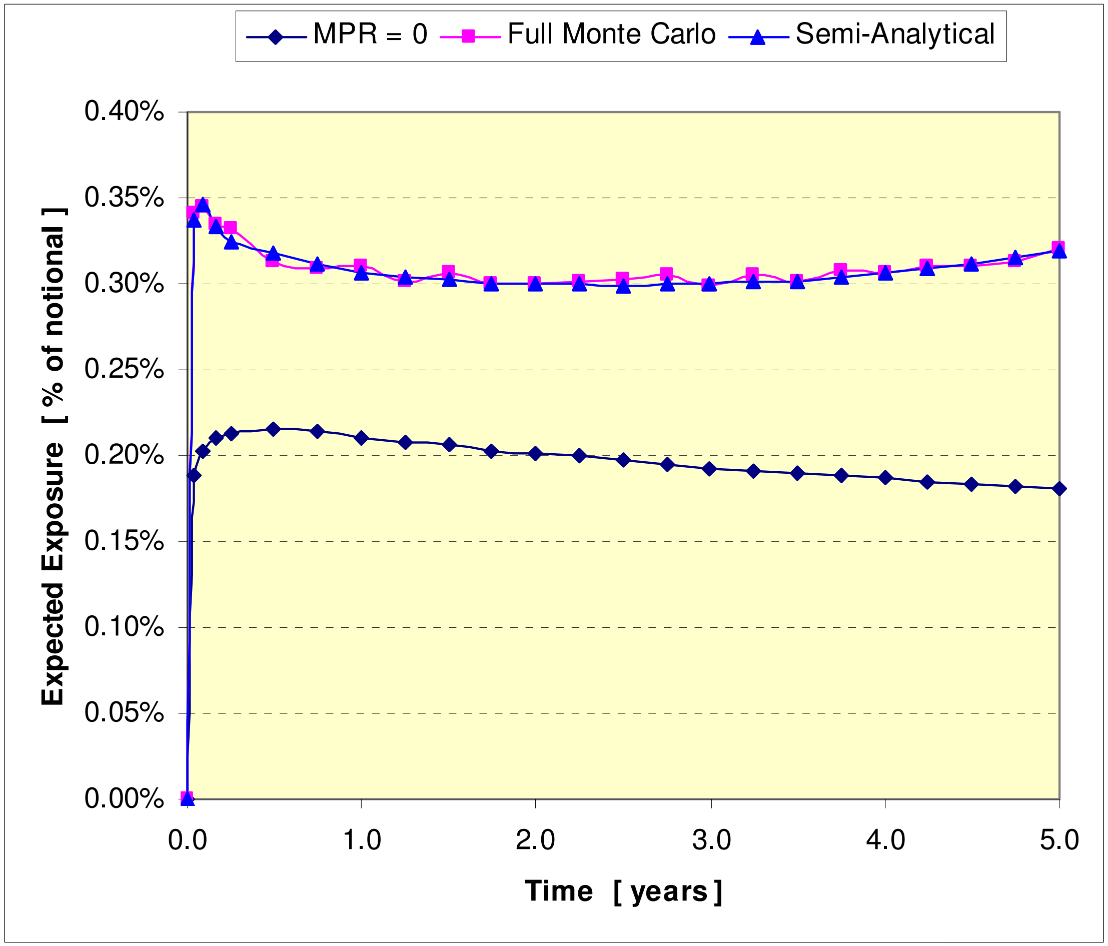
Example 1: 5-Year IR Swap Starting in 5 Years

- ***Uncollateralized EE*** and the ***two thresholds*** we will consider



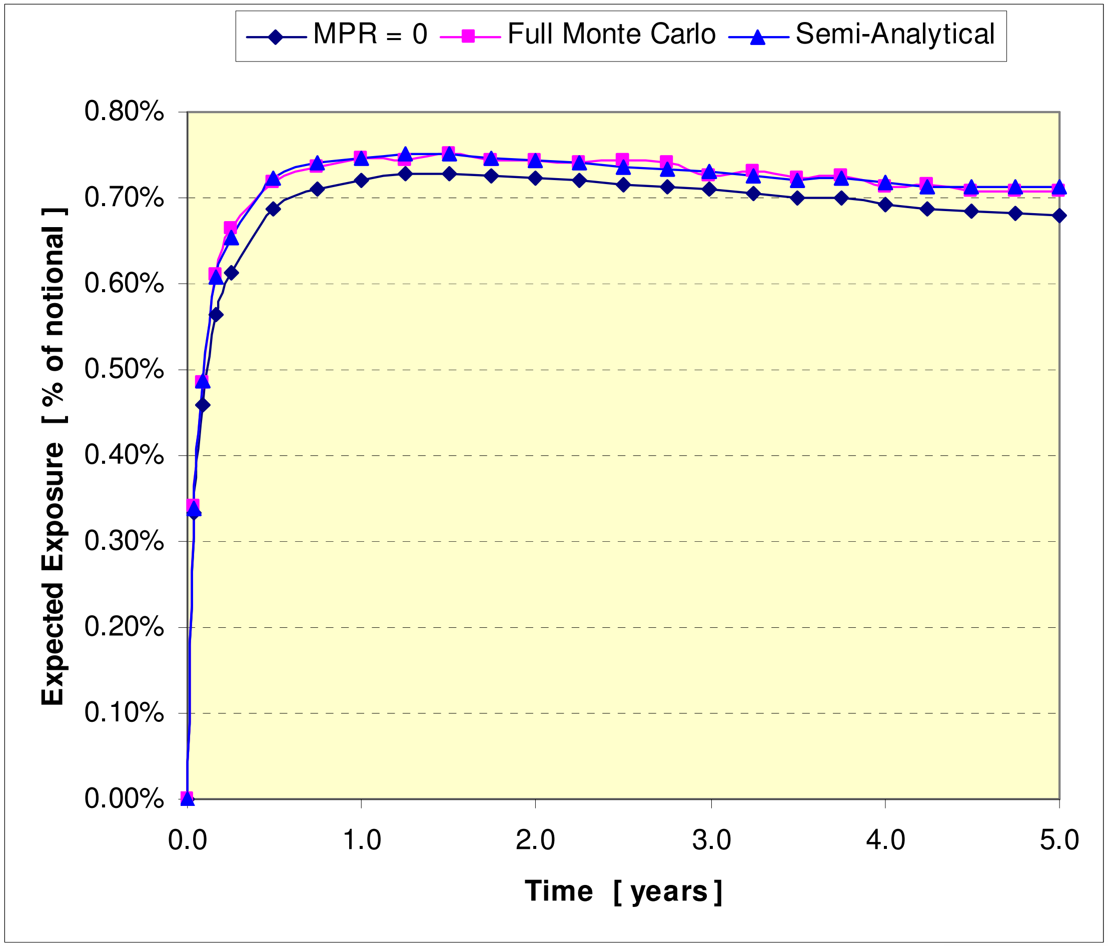
Forward Starting Swap and Small Threshold

- ***Collateralized EE*** when threshold is **0.5%**



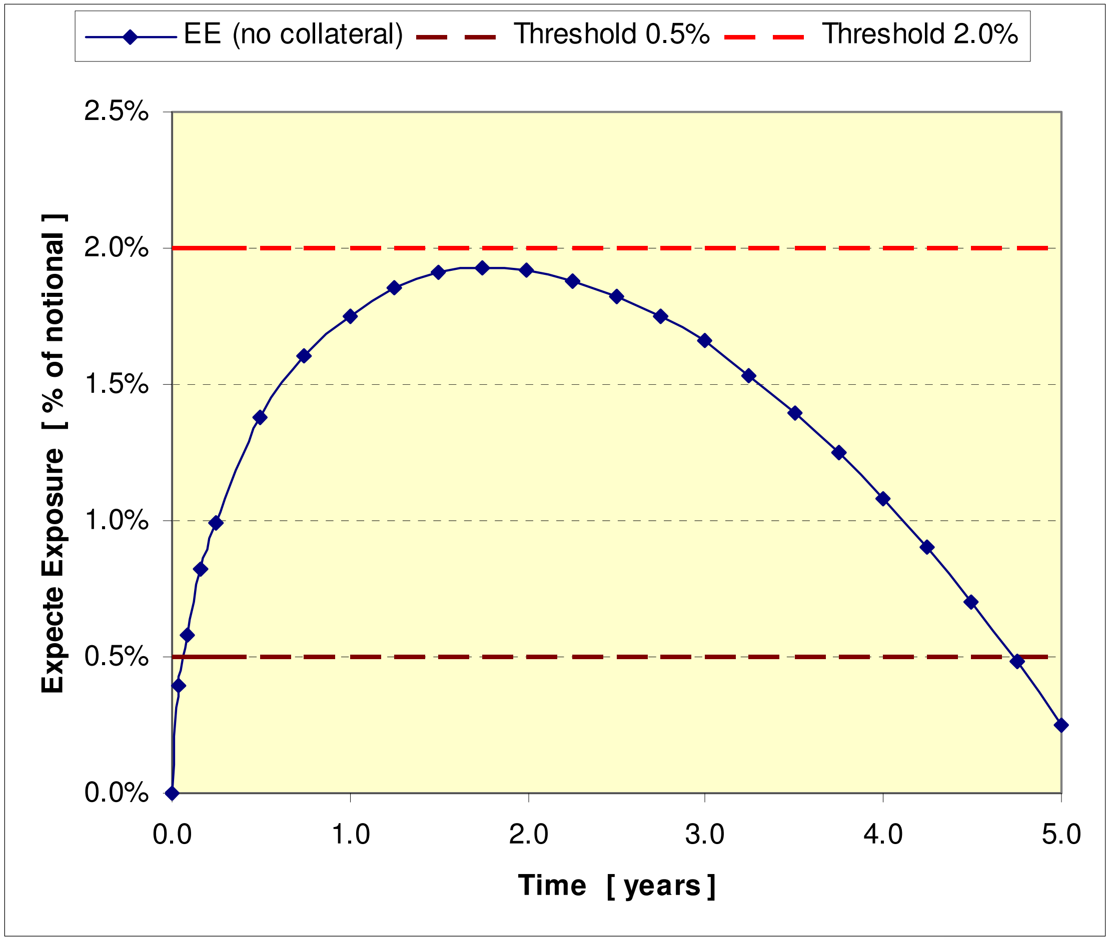
Forward Starting Swap and Large Threshold

- ***Collateralized EE*** when threshold is **2.0%**



Example 2: 5-Year IR Swap Starting Now

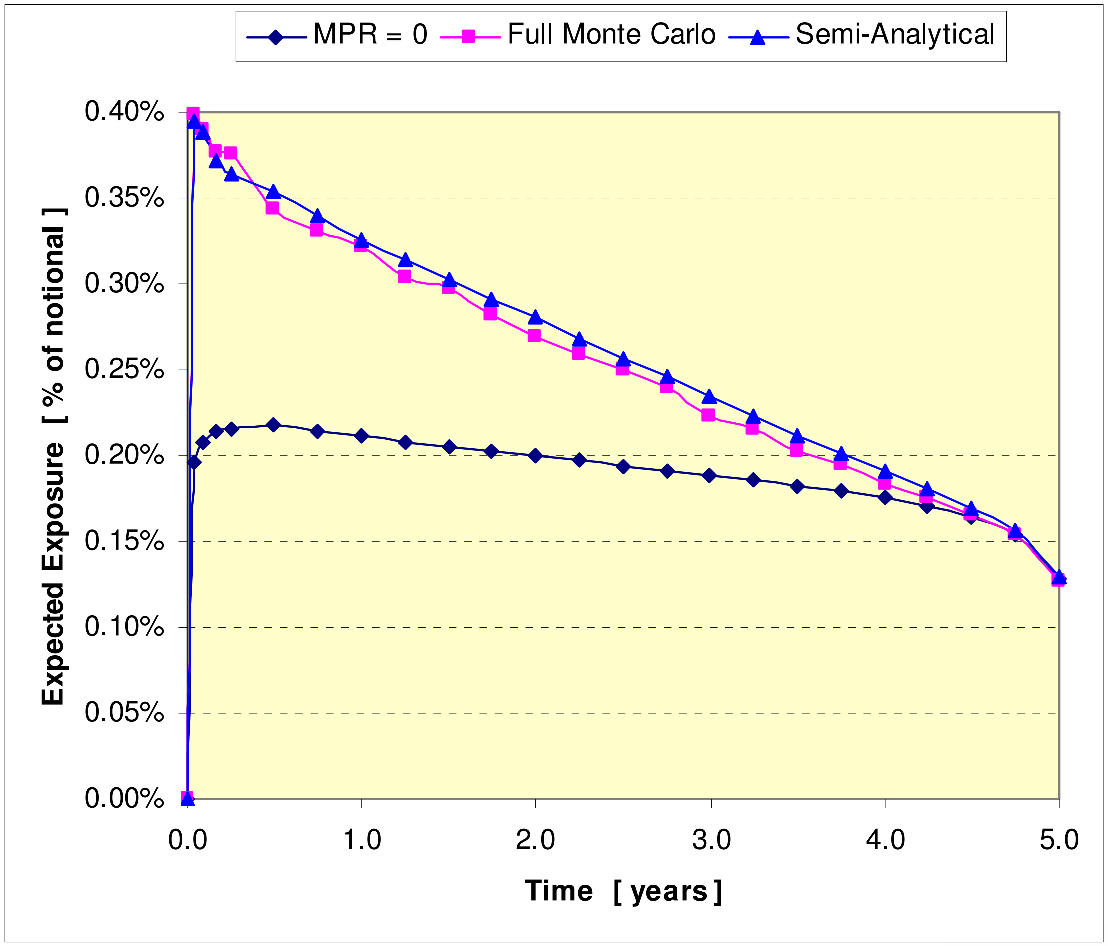
- ***Uncollateralized EE*** and the ***two thresholds*** we will consider



Bank of America Higher Standards

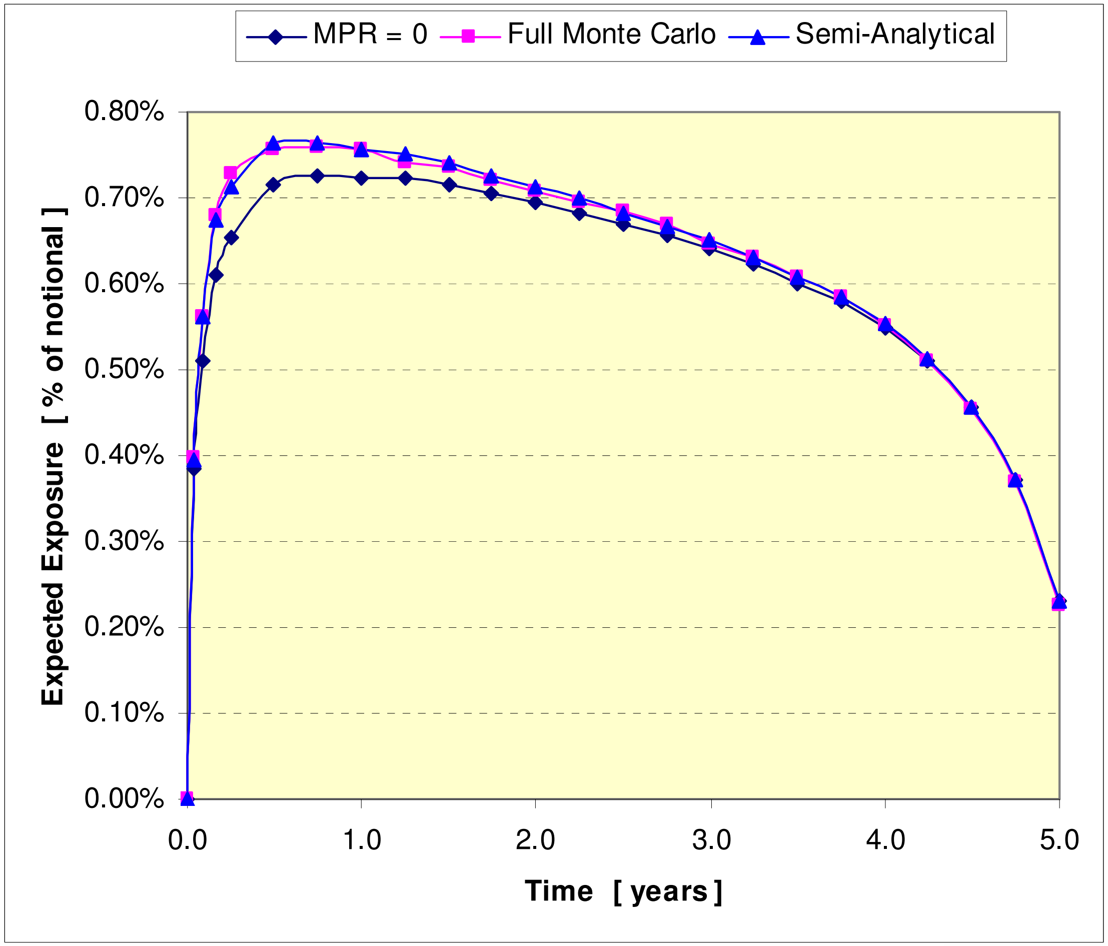
Swap Starting Now and Small Threshold

- ***Collateralized EE*** when threshold is **0.5%**



Swap Starting Now and Large Threshold

- ***Collateralized EE*** when threshold is **2.0%**



Conclusion

- ***Margin agreements*** are important risk mitigation tools that need to be modeled accurately
- ***Collateral*** available at a primary time point depends on the portfolio value at the corresponding look-back time point
- ***Full Monte Carlo*** method of simulating collateralized exposure is the most flexible approach, but requires simulating portfolio value at both primary and look-back time points
- We have developed a ***semi-analytical*** method of calculating collateralized EE that avoids doubling the simulation time
  - Portfolio value is simulated only at primary time points
  - For each portfolio value scenario at a primary time point, conditional collateralized EE is calculated in closed form
  - Unconditional collateralized EE at a primary time point is obtained by averaging the conditional collateralized EE over all scenarios