A Multiname First Passage Model for Credit Risk

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Model Overview

- Model "credit quality" of an obligor
  - Continuous process
- Delineate systematic and idiosyncratic risk
  - Conditional independence structure
- Calibrates well across tranches and maturities simultaneously
  - CDX index tranches from 2006 and 2008

Challenges in Credit Risk Modeling

- Complex dependence
- Dynamics
- High dimension

Common Modeling Approaches

- Copula/Factor Models (Li, 2000)
- Intensity Models (Lando, 1998; Duffie and Singleton, 1999)
- Structural Models (Black and Cox, 1976)

The Black-Cox Model

- Firm value a geometric Brownian motion

$$S_t = S_0 \exp\left(\mu t + \sigma W_t\right)$$

- Default threshold deterministic

$$B_t = B_0 \exp\left(\lambda t\right)$$

- Default is first passage time of S_t to B_t

$$\tau = \inf\left\{t \geq 0 : S_t \leq B_t\right\}$$

Multivariate Version

Simply correlate the driving Brownian motions

- $S_t^i = S_0^i \exp\left(\mu_i t + \sigma_i W_t^i\right)$
  - $Corr\left(W_t^i, W_t^j\right) = \rho_{ij}$
- $B_t^i = B_0^i \exp\left(\lambda_i t\right)$
- $\tau_i = \inf\left\{t \geq 0 : S_t^i \leq B_t^i\right\}$

Issues with Black-Cox

- Implementation
  - Relevant mathematical structure is FPT of correlated BM
- Poor fit to market data
  - Predictions very similar to Gaussian copula
  - Hull et al. (2005); Overbeck and Schmidt (2005)

A Closer Look at Black-Cox

- Firms default at FPT of "credit quality" to zero
  - $X_t^i = \log\left(S_t^i / B_t^i\right) = x_i + \mu_i t + \sigma_i W_t^i$
  - W^i correlated BM
- μ_i,σ_i represent *trend* and *volatility* in credit quality
- Systematic risk - correlated "noise" about trend

Our Framework

- Model dynamics of credit quality as

$$dX_t^i = \mu_i(M_t)\,dt + \sigma_i(V_t)\,dW_t^i$$

- M_t, V_t correlated processes (unobserved)
- μ_i, σ_i deterministic functions
- W^i a BM independent of everything
- Default time τ_i is FPT of X^i to zero

Intuition (Heuristic)

- $X_{t+h}^i - X_t^i \overset{d}{\approx} N\left(h\mu_i\left(M_t\right), h\sigma_i^2\left(V_t\right)\right)$
  - Systematic factors "set the tone" for a day's operations
- $X_{t+h}^i - X_t^i$ and $X_{t+h}^j - X_t^j$ approximately independent
  - Once the tone has been set, obligors operate independently
- Continuous-time analogue of factor models

General Properties

$$X_t^i = X_0^i + \int_0^t \mu_i (M_s)\, ds + \int_0^t \sigma_i (V_s)\, dW_s^i$$

- In general credit qualities *not* Markovian
- Credit qualities are *continuous*
  - M_t, V_t may have jumps
- Credit qualities are *conditionally independent*

Default Process

- Define *default process*

$$D_N(t) := \frac{1}{N} \sum_{i=1}^{N} I\left(\tau_i \leq t\right)$$

- When it exists, call

$$D(t) := \lim_{N \to \infty} D_N(t)$$

the *asymptotic proportion of defaults*

  - Dimension reduction, insights/intuition

Large Portfolio Approximation

With probability one

$$\lim_{N \to \infty} \left(D_N(t) - E\left[D_N(t) \,|\mathcal{H}_t \right] \right) = 0$$

- $\mathcal{H}_t$ the filtration generated by $\{M_s, V_s : \ 0 \leq s \leq t\}$
- Intuition: can predict proportion of defaults in a large portfolio based solely on realized paths of M and V
  - "In a large portfolio, all risk is systematic"
- Whenever well-defined, $D(t)$ is $\mathcal{H}_t$-measurable
  - Path functional of systematic factors

Homogeneous Portfolios

- All obligors influenced by systematic factors in same way

$$dX_t^i = \mu\left(M_t\right)dt + \sigma\left(V_t\right)dW_t^i$$

- Asymptotic proportion of defaults is

$$D(t) = P\left(\tau_i \le t \left| \mathcal{H}_t \right.\right)$$

  - Conditional default probability of an arbitrary firm

Grouped Portfolios

- K homogeneous groups
  - P_k conditional default probability for group K
  - w_k the proportion of obligors in group K
- $D(t)$ a weighted average of conditional default probabilities

$$D(t) = \sum_{k=1}^{K} w_k \cdot P_k$$

A Linear Model

$$X_t^i = x_0 + Mt + \sqrt{V} W_t^i$$

- M, V random variables; x_0 constant
- $x_0 > 0$ a constant
- Closed-form for default rate
  - $D(t) = h(M, V, x_0, t)$

Calibrating the Model

- We assume
  - (M, V) have a Gaussian copula
  - M and $\log(V)$ have a Laplace distribution
- Assume constant interest and recovery rates
  - Minimize mean relative error (simulated annealing)
  - Use large portfolio approximation and Monte Carlo
  - CDS spreads *not* included in calibration

Calibration Results (2006)

	5Y		7Y		10Y	
	Market	Model	Market	Model	Market	Model
0-3%	24.38	24.43	40.44	40.61	51.25	49.1
3-7%	90	90.2	209	250.5	471	471.1
7-10%	19	17.5	46	45	112	112
10-15%	7	7	20	20	53	44
15-30%	3.5	2.5	5.75	9.3	14	19.8
30-100%	1.73	0.38	3.12	2	4	4
CDX	35	34.8	45	47.3	57	57.5

- 8 model parameters
- Data obtained from DiGraziano and Rogers (2006)

Interesting Observations

- $X_t^i = x_0 + Mt + \sqrt{V} W_t^i$
- Correlation between M and V exceeds 80% in both cases
  - 2006 and 2008
- Large portfolio losses (senior tranches impaired) characterized by
  - $M << 0$ and $V \approx 0$
  - "Low-volatility" market crashes

Understanding "Low-Volatility" Crashes

- Condition upon $(M, V) = (m, v)$

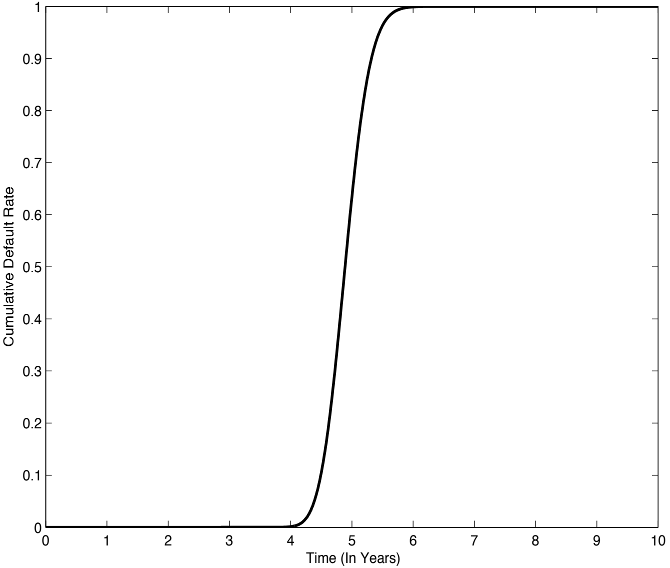
$$X_t^i = x_0 + mt + \sqrt{v} W_t^i$$

- Now send $v \to 0$

$$X_t^i \approx x_0 + mt$$

- If $m < 0$ default with near certainty at $t^* = -\frac{x_0}{m}$
  - $h(m, v, x_0, \cdot)$ converges to degenerate c.d.f. as $v \to 0$

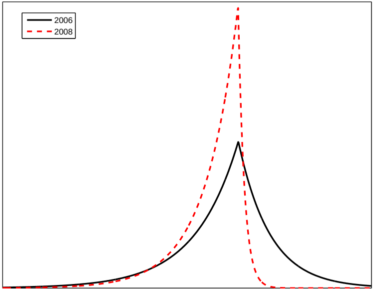
$h(-0.375, 0.003, 1.8, \cdot)$



Interpreting the Systematic Factors

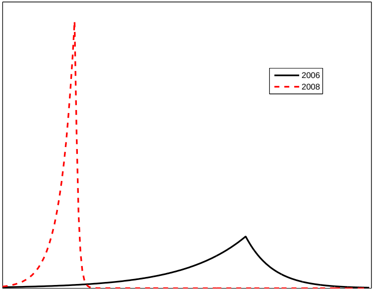
- $X_t^i = x_0 + Mt + \sqrt{V} W_t^i$
- $X_t^i \sim Mt$ as $t \to \infty$
  - M is the "dominant long-term" force
- V modulates influence of idiosyncratic component
  - Downgraded during "bad times"
  - Stochastic correlation factor?

Calibrated Densities - M



	5Y CDX				
	0-3%	3-7%	7-10%	10-15%	15-30%
2006	24.4	90	19	7	3.5
2008	67.4	727	403	204	115

Calibrated Densities - $\log(V)$



- Idiosyncratic risk has been "priced out"

The Model is Not Perfect

Major flaw - no time dynamics in systematic factors

- Economic environment “frozen in time”
  - No recovering from a recession
  - No cyclicality in default rate
- “Predictability” of default rate

"Predictability" of Default Rate

- $D(t) = h(M, V, x_0, t)$
- For fixed x_0
  - $(m, v) \mapsto h(m, v, x_0, \cdot)$ is one-to-one
- If we observe losses over any interval $[0, T]$ we can
  - Determine realized values of M, V
  - Predict future losses *with certainty*

Adding Time Dynamics

$$dX_t^i = M_t dt + \sqrt{V_t} dW_t^i \qquad X_0^i = x_0$$

- M_t,V_t processes with integrable sample paths
- $x_0 > 0$ constant
- $X_{t+h}^i - X_t^i \overset{d}{\approx} N\left(hM_t, hV_t\right)$

Conditional Default Probabilities

- Condition upon realized paths of (M, V), say (m_t, v_t).

$$
\begin{aligned}
X_t^i &= x_0 + \int_0^t m_s ds + \int_0^t \sqrt{v_s} dW_s^i \\
&\stackrel{\mathcal{L}}{=} x_0 + \int_0^t m_s ds + W^i \left(\int_0^t v_s ds \right) \\
&= a_t + W^i (b_t)
\end{aligned}
$$

- Default at first passage of TCBM to non-linear barrier

Simulating Portfolio Losses

- $D(t) = \Psi\left(A, B, t\right)$
  - $A_t = x_0 + \int_0^t M_s ds$
  - $B_t = \int_0^t V_s ds$
- We show that $\Psi\left(A^n, B^n, t\right) \implies \Psi\left(A, B, t\right)$ under very mild conditions
  - A^n, B^n piecewise linear approximations

Example

Model M_t, V_t as stationary mean-reverting diffusions

- $dM_t = \theta\left(\mu - M_t\right)dt + \nu\left(M_t\right)dZ_t^1$
- $dV_t = \alpha\left(\beta - V_t\right)dt + \xi\left(V_t\right)dZ_t^2$
- Z^1, Z^2 correlated Brownian motion
  - $\nu(\cdot)$ chosen so that M_t is Laplace
  - $\xi(\cdot)$ chosen so that V_t is log-Laplace

Calibration Results - Diffusion Model (2008)

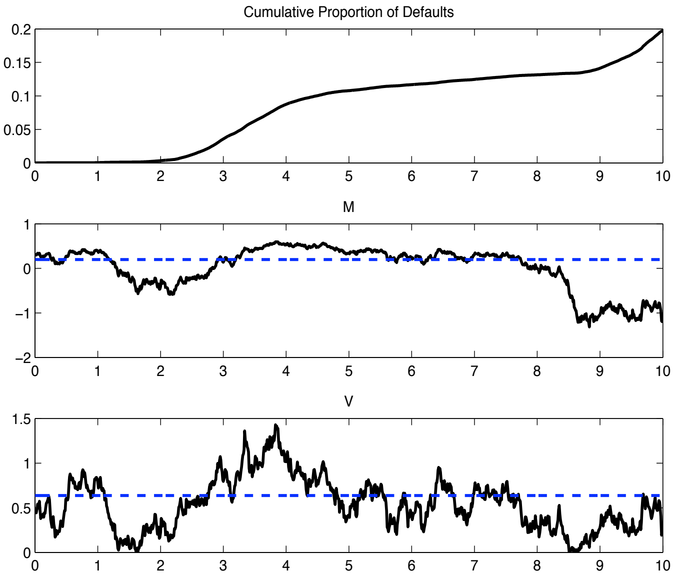
	5Y		7Y		10Y	
	Market	Model	Market	Model	Market	Model
0-3 %	67.38	64.71	70.5	70.46	73.5	71.89
3-7 %	727	727	780	842	895.5	899.6
7-10%	403	376	440	437	509	452
10-15%	204	223	248	263	282	282
15-30%	115	115	128.5	129.3	139.5	139.1

- 10 model parameters
- Data obtained from Krekel (2008)
- Implied CDS curve is hump-shaped

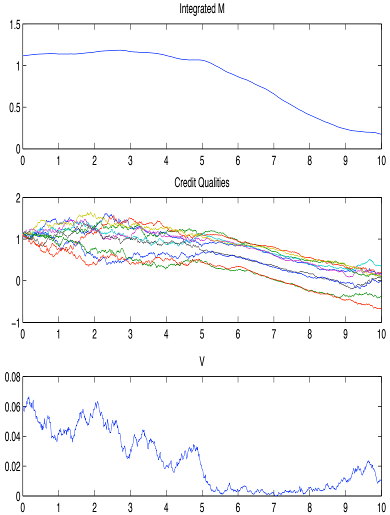
Comments

- M_t, V_t driven by correlated BM
  - Correlation exceeds 98% in both cases (2006 and 2008)
- Large portfolio losses (senior tranches impaired) characterized by periods where
  - $M_t << 0$ and $V_t \approx 0$ for prolonged periods of time
- Unlike linear model, economy *can* recover
  - Observe cyclical behaviour

The Importance of Time Dynamics



$$X_t^i = x_0 + \int_0^t M_s ds + \int_0^t \sqrt{V}_s dW_s^i$$



Conclusions/Future Work

- Continuous first-passage models *can* fit market data
- Relate credit quality to *observable* covariates
- Investigate dynamics of credit/CDS/tranche spreads
  - Impact of various investor information
- Importance sampling for diffusion processes

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10-15%	7	8	20	18.9	53	42
15-30%	3.5	3.5	5.75	6.21	14	14
30-100%	1.73	0.36	3.12	1.39	4	1.49
CDX	35	33.6	45	46	57	55.4

Calibration Results - Linear Model (2008)

	5Y		7Y		10Y	
	Market	Model	Market	Model	Market	Model
0-3 %	67.38	65.90	70.5	70.79	73.5	71.76
3-7 %	727	733	780	859	895.5	894.7
7-10%	403	355	440	417	509	430
10-15%	204	219	248	265	282	277
15-30%	115	100	128.5	128.1	139.5	141.2

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