A Unified Framework for Pricing Credit and Equity Derivatives

Erhan Bayraktar

University of Michigan

joint work with
Bo Yang, Morgan Stanley

Outline

- Motivation
- Model
- Equity and Credit Derivatives
- Asymptotic Expansions
- Calibration

Motivation

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- Until recently the affects of default on the implied volatility surface has been ignored.
- We will build an intensity based model that is able to explicitly price to credit and equity derivatives $\longrightarrow$ Cross market calibration.

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- We will estimate the recovery rate and the default intensity jointly from the implied volatility surface and the bond yield.
- Predict the credit default swap + A much better fit to the implied volatility surface
- Implied Vol is composed of Stochastic Vol. (e.g. Index Options)+ Premium for Default Risk (way out of the money put options on individual stocks).

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\begin{aligned}
\lambda_t &= f(Y_t, Z_t), \\
dY_t &= \frac{1}{\epsilon}(m - Y_t)dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} dW_t^2, \quad Y_0 = y, \\
dZ_t &= \delta c(Z_t)dt + \sqrt{\delta} g(Z_t) dW_t^3, \quad Z_0 = z,
\end{aligned}
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Interest rate:

$$dr_t = (\alpha - \beta r_t)dt + \eta dW_t^1, \quad r_0 = r.$$

Stock price model

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Stock price:

$$d\bar{X}_t = \bar{X}_t \left(r_t dt + \sigma_t dW_t^0 - d\left(\tilde{N}_t - \int_0^{t\wedge\tau} \lambda_u du \right) \right),$$

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$\bar{X}_0 = x$, where the volatility is stochastic and is defined through

$$\begin{aligned} \sigma_t &= \sigma(\tilde{Y}_t), \\ d\tilde{Y}_t &= \left(\frac{1}{\epsilon}(\tilde{m} - \tilde{Y}_t) - \frac{\tilde{\nu}\sqrt{2}}{\sqrt{\epsilon}} \Lambda(\tilde{Y}_t) \right) dt + \frac{\tilde{\nu}\sqrt{2}}{\sqrt{\epsilon}} dW_t^4, \end{aligned}$$

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$\tilde{Y}_0 = \tilde{y}$. The pre-banktruptcy stock price coincides with the solution of

$$dX_t = (r_t + \lambda_t) X_t dt + \sigma_t X_t dW_t^0, \; X_0 = x.$$

Derivatives

Derivatives

Call Option:

$$C(t;T,K)=\mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)(\bar{X}_T-K)^+ 1_{\{\tau>T\}}\middle|\mathcal{G}_t\right]$$
$$=1_{\{\tau>t\}}\mathbb{E}\left[\exp\left(-\int_t^T (r_s+\lambda_s)ds\right)(X_T-K)^+\middle|\mathcal{F}_t\right].$$

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$$= 1_{\{\tau > t\}}\mathbb{E}\left[\exp\left(-\int_t^T (r_s + \lambda_s) ds\right)(X_T - K)^+\middle|\mathcal{F}_t\right].$$

Bond price (the holder of the bond recovers a constant fraction $1 - l$ of the pre-default value):

$$B^c(t;T) = \mathbb{E}\Bigg[\exp\left(-\int_t^T r_s ds\right) 1_{\{\tau > T\}}$$
$$+ \exp\left(-\int_t^\tau r_s ds\right) 1_{\{\tau \le T\}}(1-l)B^c(\tau-;T)\bigg|\mathcal{G}_t\Bigg]$$
$$= \mathbb{E}\left[\exp\left(-\int_t^T (r_s + l\,\lambda_s) ds\right)\middle|\mathcal{F}_t\right],$$

on $\{\tau > t\}$.

Credit Default Swap

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The present value of the premium leg of the contract:

$$
\begin{aligned}
&\text{Premium}(t;\mathcal{T}) \\
&= c^{ds}(t;\mathcal{T})\,\mathbb{E}\left[\sum_{m=1}^{M}\exp\left(-\int_t^{T_m} r_s ds\right)1_{\{\tau>T_m\}}\middle|\mathcal{G}_t\right] \\
&= 1_{\{\tau>t\}}c^{ds}(t;\mathcal{T})\sum_{m=1}^{M}\mathbb{E}\left[\exp\left(-\int_t^{T_m}(r_s+\lambda_s)ds\right)\middle|\mathcal{F}_t\right].
\end{aligned}
$$

Credit Default Swap

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&= 1_{\{\tau > t\}} c^{ds}(t;\mathcal{T}) \sum_{m=1}^{M} \mathbb{E}\left[\exp\left(-\int_t^{T_m} (r_s + \lambda_s) ds\right) \middle| \mathcal{F}_t\right].
\end{aligned}
$$

The present value of the protection leg:

$$
\begin{aligned}
&\text{Protection}(t;\mathcal{T}) \\
&= 1_{\{\tau > t\}} \mathbb{E}\left[\exp\left(-\int_t^{\tau} r_s ds\right) 1_{\{\tau \le T_M\}} l\, B^c(\tau-; T_M) \middle| \mathcal{G}_t\right]
\end{aligned}
$$

Determining the Premium

$$
\begin{aligned}
&\text{Protection}(t;\mathcal{T}) \\
&= 1_{\{\tau>t\}}\left(\frac{l}{1-l}\right)\left(B^c(t;T_M) - \mathbb{E}\left[\exp\left(-\int_t^{T_M} r_s ds\right) 1_{\{\tau>T_M\}}\middle|\mathcal{G}_t\right]\right) \\
&= 1_{\{\tau>t\}}\left(\frac{l}{1-l}\right)\left(B^c(t;T_M) - \mathbb{E}\left[\exp\left(-\int_t^{T_M} (r_s+\lambda_s) ds\right)\middle|\mathcal{F}_t\right]\right),
\end{aligned}
\tag{1}
$$

Determining the Premium

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&= 1_{\{\tau > t\}} \left(\frac{l}{1-l} \right) \left(B^c(t; T_M) - \mathbb{E}\left[\exp\left(-\int_t^{T_M} (r_s + \lambda_s) ds \right) \middle| \mathcal{F}_t \right] \right),
\end{aligned} \tag{1}
$$

By setting the protection leg=premium leg:

$$
c^{ds}(t; \mathcal{T}) = 1_{\{\tau > t\}} \frac{l}{1-l} \frac{B^c(t; T_M) - \mathbb{E}\left[\exp\left(-\int_t^{T_M} (r_s + \lambda_s) ds \right) \middle| \mathcal{F}_t \right]}{\sum_{m=1}^{M} \mathbb{E}\left[\exp\left(-\int_t^{T_m} (r_s + \lambda_s) ds \right) \middle| \mathcal{F}_t \right]}. \tag{2}
$$

Pricing Equation

$$P^{\epsilon,\delta}(t, X_t, r_t, Y_t, \tilde{Y}_t, Z_t) = \mathbb{E}\left[\exp\left(-\int_t^T (r_s + l\lambda_s)ds\right) h(X_T)\middle|\mathcal{F}_t\right].$$

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$P^{\varepsilon,\delta}$ is the solution of

$$\mathcal{L}^{\epsilon,\delta} P^{\epsilon,\delta}(t, x, r, y, \tilde{y}, z) = 0,$$
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where the partial differential operator $\mathcal{L}^{\varepsilon,\delta}$ is defined as

$$\mathcal{L}^{\epsilon,\delta} \triangleq \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\epsilon}}\mathcal{M}_3.$$

Differential Operators

$$
\begin{aligned}
\mathcal{L}_0 &\triangleq \nu^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y} + \tilde{\nu}^2 \frac{\partial^2}{\partial \tilde{y}^2} + (\tilde{m}-\tilde{y})\frac{\partial}{\partial \tilde{y}} + 2\rho_{24} v \tilde{v} \frac{\partial^2}{\partial y \partial \tilde{y}}, \\
\mathcal{L}_1 &\triangleq \rho_2 \sigma(\tilde{y}) \nu \sqrt{2} x \frac{\partial^2}{\partial x \partial y} + \rho_{12} \eta \nu \sqrt{2} \frac{\partial^2}{\partial r \partial y} + \rho_4 \sigma(\tilde{y}) \tilde{\nu} \sqrt{2} x \frac{\partial^2}{\partial x \partial \tilde{y}} + \rho_{14} \eta \tilde{\nu} \sqrt{2} \frac{\partial^2}{\partial r \partial \tilde{y}} \\
&\quad - \Lambda(\tilde{y}) \tilde{\nu} \sqrt{2} \frac{\partial}{\partial \tilde{y}}, \\
\mathcal{L}_2 &\triangleq \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2(\tilde{y}) x^2 \frac{\partial^2}{\partial x^2} + (r + f(y,z)) x \frac{\partial}{\partial x} + (\alpha - \beta r)\frac{\partial}{\partial r} + \sigma(\tilde{y}) \eta \rho_1 x \frac{\partial^2}{\partial x \partial r} \\
&\quad + \frac{1}{2}\eta^2 \frac{\partial^2}{\partial r^2} - (r + l\, f(y,z))\cdot, \\
\mathcal{M}_1 &\triangleq \sigma(\tilde{y}) \rho_3 g(z) x \frac{\partial^2}{\partial x \partial z} + \eta \rho_{13} g(z) \frac{\partial^2}{\partial r \partial z}, \quad \mathcal{M}_2 \triangleq c(z)\frac{\partial}{\partial z} + \frac{1}{2} g^2(z) \frac{\partial^2}{\partial z^2}, \\
\mathcal{M}_3 &\triangleq \rho_{23} \nu \sqrt{2} g(z) \frac{\partial^2}{\partial y \partial z} + \rho_{34} \tilde{\nu} \sqrt{2} g(z) \frac{\partial^2}{\partial \tilde{y} \partial z}.
\end{aligned}
$$

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Matching powers of δ

$$\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right) P_0^\epsilon = 0,$$

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$$P_0^{\epsilon}(T, x, r, y, \tilde{y}, z) = h(x),$$

and that P_1^{ε} satisfies

$$\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right) P_1^{\epsilon} = -\left(\mathcal{M}_1 + \frac{1}{\sqrt{\epsilon}}\mathcal{M}_3\right) P_0^{\epsilon},$$
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$$P_1^\epsilon(T, x, y, \tilde{y}, z, r) = 0.$$

Next, we expand in powers of $\sqrt{\epsilon}$

$$P_0^\epsilon = P_0 + \sqrt{\epsilon} P_{1,0} + \epsilon P_{2,0} + \epsilon^{3/2} P_{3,0} + \cdots$$
$$P_1^\epsilon = P_{0,1} + \sqrt{\epsilon} P_{1,1} + \epsilon P_{2,1} + \epsilon^{3/2} P_{3,1} + \cdots$$

Approximate Prices

$$\widetilde{P}^{\varepsilon,\delta} = P_0 + \sqrt{\varepsilon} P_{1,0} + \sqrt{\delta} P_{0,1},$$

$$\begin{cases} \langle \mathcal{L}_2 \rangle P_0 = 0 \\ P_0(T, x, r; z) = h(x). \end{cases} \tag{3}$$

$$\begin{cases} \langle \mathcal{L}_2 \rangle P_{1,0} = \langle \mathcal{L}_1 \mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle P_0, \\ P_{1,0}(T, x, r; z) = 0. \end{cases} \tag{4}$$

$$\begin{cases} \langle \mathcal{L}_2 \rangle P_{0,1} = -\langle \mathcal{M}_1 \rangle P_0, \\ P_{0,1}(T, x, r; z) = 0. \end{cases} \tag{5}$$

Driving Terms

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&\langle \mathcal{L}_1 \mathcal{L}_0^{-1} (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle P_0 \\
&= I \rho_2 \nu \sqrt{2} \langle \sigma \phi_y \rangle (z) x^2 \frac{\partial P_0}{\partial x^2} + I \rho_{12} \eta \nu \sqrt{2} \langle \phi_y \rangle (z) \frac{\partial}{\partial r} \left(x \frac{\partial P_0}{\partial x} - P_0 \right) \\
&+ \rho_4 \tilde{\nu} \sqrt{2} \left(\frac{1}{2} \langle \sigma \kappa_{\tilde{y}} \rangle x \frac{\partial}{\partial x} \left(x^2 \frac{\partial^2 P_0}{\partial x^2} \right) + \langle \sigma \psi_{\tilde{y}} \rangle \eta \rho_1 x \frac{\partial}{\partial x} \left(x \frac{\partial^2 P_0}{\partial x \partial r} \right) \right) \\
&+ \rho_{14} \eta \tilde{\nu} \sqrt{2} \left(\frac{1}{2} \langle \kappa_{\tilde{y}} \rangle x^2 \frac{\partial^3 P_0}{\partial x^2 \partial r} + \langle \psi_{\tilde{y}} \rangle \eta \rho_1 \left(x \frac{\partial^3 P_0}{\partial x \partial r^2} \right) \right) \\
&- \tilde{\nu} \sqrt{2} \left(\frac{1}{2} \langle \Lambda \kappa_{\tilde{y}} \rangle x^2 \frac{\partial P_0}{\partial x^2} + \langle \Lambda \psi_{\tilde{y}} \rangle \eta \rho_1 x \frac{\partial^2 P_0}{\partial x \partial r} \right).
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&+ \rho_4 \tilde{\nu} \sqrt{2} \left(\frac{1}{2} \langle \sigma \kappa_{\tilde{y}} \rangle x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P_0}{\partial x^2} \right) + \langle \sigma \psi_{\tilde{y}} \rangle \eta \rho_1 x \frac{\partial}{\partial x}\left(x \frac{\partial^2 P_0}{\partial x \partial r} \right) \right) \\
&+ \rho_{14} \eta \tilde{\nu} \sqrt{2} \left(\frac{1}{2} \langle \kappa_{\tilde{y}} \rangle x^2 \frac{\partial^3 P_0}{\partial x^2 \partial r} + \langle \psi_{\tilde{y}} \rangle \eta \rho_1 \left(x \frac{\partial^3 P_0}{\partial x \partial r^2} \right) \right) \\
&- \tilde{\nu} \sqrt{2} \left(\frac{1}{2} \langle \Lambda \kappa_{\tilde{y}} \rangle x^2 \frac{\partial P_0}{\partial x^2} + \langle \Lambda \psi_{\tilde{y}} \rangle \eta \rho_1 x \frac{\partial^2 P_0}{\partial x \partial r} \right).
\end{aligned}
$$

Also

$$
\mathcal{M}_1 = \sigma(\tilde{y}) \rho_3 g(z) x \frac{\partial^2}{\partial x \partial z} + \eta \rho_{13} g(z) \frac{\partial^2}{\partial r \partial z}.
$$

Explicit Expression for P_0

The leading order term P_0 is given by:

$$P_0(t,x,r;z) = B_0^c(t,r;z,T,l)\int_{-\infty}^{\infty} h(\exp(u))\frac{1}{\sqrt{2\pi v(t,T)}}$$
$$\exp\left(-\frac{(u-m(t,T))^2}{2v(t,T)}\right)du,$$

where

$$B_0^c(t,r;z,T,l) \triangleq \exp\left(-l\bar{\lambda}(z)(T-t)+a(T-t)-b(T-t)r\right). \quad (6)$$

Proof

Applying Feynman-Kac theorem

$$
\begin{aligned}
& P_0(t,x,r;z) \\
& = \mathbb{E}\left[\exp\left(-\int_t^T (r_s + l\bar{\lambda}(z))ds\right) h(S_T) \middle| S_t = x, r_t = r\right].
\end{aligned}
$$

Proof

Applying Feynman-Kac theorem

$$P_0(t,x,r;z)$$
$$= \mathbb{E}\left[\exp\left(-\int_t^T (r_s + l\bar{\lambda}(z))ds\right) h(S_T)\middle| S_t = x, r_t = r\right].$$

where the dynamics of S is given by

$$dS_t = (r_t + \bar{\lambda}(z))S_t dt + \bar{\sigma}_2 S_t d\widetilde{W}_t^0,$$

in which $\widetilde{W}^0$ is a Wiener process whose correlation with W^1 is $\bar{\rho}_1 = \frac{\bar{\sigma}_1}{\bar{\sigma}_2}\rho_1$.

Let us define

$$\tilde{P}_0(t, x, r; z) = \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) h(\widetilde{S}_T) \middle| \widetilde{S}_t = x, r_t = r\right],$$

in which

$$d\widetilde{S}_t = r_t \widetilde{S}_t dt + \bar{\sigma}_2 \widetilde{S}_t d\widetilde{W}_t^0.$$

Let us define

$$\tilde{P}_0(t,x,r;z) = \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) h(\widetilde{S}_T)\middle|\widetilde{S}_t = x, r_t = r\right],$$

in which

$$d\widetilde{S}_t = r_t\widetilde{S}_t dt + \bar{\sigma}_2\widetilde{S}_t d\widetilde{W}_t^0.$$

Then

$$P_0(t,x,r;z) = e^{-l\bar{\lambda}(T-t)}\tilde{P}_0(t, x\exp(\bar{\lambda}(z)(T-t)), z, r).$$

Let us define

$$\tilde{P}_0(t,x,r;z) = \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) h(\widetilde{S}_T)\middle|\widetilde{S}_t = x, r_t = r\right],$$

in which

$$d\widetilde{S}_t = r_t \widetilde{S}_t dt + \bar{\sigma}_2 \widetilde{S}_t d\widetilde{W}_t^0.$$

Then

$$P_0(t,x,r;z) = e^{-l\bar{\lambda}(T-t)} \tilde{P}_0(t, x\exp(\bar{\lambda}(z)(T-t)), z, r).$$

$$\frac{d\mathbb{P}^T}{d\mathbb{P}} = \frac{\exp\left(-\int_0^T r_s ds\right)}{B(0,T)},$$

where

$$B(t,T) = \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)\middle|\mathcal{F}_t\right].$$

We can obtain the following representation of $\tilde{P}_0$ using the T forward measure

$$\begin{aligned}
&\tilde{P}_0(t, \widetilde{S}_t, r_t; z) \\
&= B(t,T)\mathbb{E}^T\left[h(\widetilde{S}_T)|\mathcal{F}_t\right] = B(t,T)\mathbb{E}^T\left[h(F_T)|\mathcal{F}_t\right],
\end{aligned}$$

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\begin{aligned}
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\end{aligned}
$$

in which

$$
F_t \triangleq \frac{\widetilde{S}_t}{B(t,T)},
$$

which is a $\mathbb{P}^T$ martingale.

We can obtain the following representation of $\tilde{P}_0$ using the T forward measure

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\begin{aligned}
&\tilde{P}_0(t, \widetilde{S}_t, r_t; z) \\
&= B(t,T)\mathbb{E}^T\left[h(\widetilde{S}_T)|\mathcal{F}_t\right] = B(t,T)\mathbb{E}^T\left[h(F_T)|\mathcal{F}_t\right],
\end{aligned}
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in which

$$F_t \triangleq \frac{\widetilde{S}_t}{B(t,T)},$$

which is a $\mathbb{P}^T$ martingale.
Note that an explicit expression for $B(t,T)$ is available since r_t is a Vasicek model:

$$B(t,T) = \exp(a(T-t) - b(T-t)r_t).$$

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$$

Applying Itô's formula we observe that the dynamics of F is

$$
dF_t = F_t(\bar{\sigma}_1 d\widetilde{W}_t^0 + b(T-t)\eta d\widetilde{W}_t^1).
$$

Correction Terms, $P_{1,0}$

The correction term $\sqrt{\epsilon}P_{1,0}$ is given by

$$\sqrt{\epsilon}P_{1,0} = -(T-t)\left(V_1^\epsilon x^2 \frac{\partial^2 P_0}{\partial x^2} + V_2^\epsilon x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P_0}{\partial x^2}\right)\right)$$
$$+ I\, V_3^\epsilon \left(-x\frac{\partial^2 P_0}{\partial x \partial \alpha} - \frac{\partial P_0}{\partial \alpha}\right) + V_4 x^2 \frac{\partial^3 P_0}{\partial x^2 \partial \alpha} + V_5^\epsilon x \frac{\partial^2 P_0}{\partial \eta \partial x} + V_6^\epsilon x \frac{\partial^2 P_0}{\partial x \partial \alpha},$$

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$$\langle L_2 \rangle u = \left(x^n \frac{\partial^n}{\partial x^n} \right) P_0, \quad u(T, x, r; z) = 0.$$

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3) Differentiating "BS PDE" with respect to α, we see that $-\frac{\partial P_0}{\partial \alpha}$ also solves

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4) Using 1) and 2) above and the equation we obtain differentiating "BS PDE" with respect to η, we can show that $1/\eta \cdot (\bar{\sigma}_1 \rho_1 x \frac{\partial^2 P_0}{\partial x \partial \alpha} - \frac{\partial P_0}{\partial \eta})$ solves

$$\langle L_2 \rangle u = \frac{\partial^2 P_0}{\partial r^2}, \quad u(T, x, r; z) = 0.$$

Correction Term $P_{0,1}$

The correction term $\sqrt{\delta}P_{0,1}$ is given by

$$\sqrt{\delta}P_{0,1} = V_1^{\delta}\frac{(T-t)^2}{2}\left(x^2\frac{\partial^2 P_0}{\partial x^2} + (1-l)x\frac{\partial P_0}{\partial x}\right) + V_2^{\delta}\frac{1}{\beta}\left[x\frac{\partial^2 P_0}{\partial\alpha\partial x} - l\,\frac{\partial P_0}{\partial\alpha}\right.$$
$$\left. + \frac{(T-t)^2}{2}\left(x^2\frac{\partial^2 P_0}{\partial x^2} - l\,x\frac{\partial P_0}{\partial x} + l\,P_0\right) - (T-t)\left(x\frac{\partial^2 P_0}{\partial r\partial x} - l\,\frac{\partial P_0}{\partial r}\right)\right].$$

Proof

Differentiating BS-PDE with respect to z we see that $\frac{\partial P_0}{\partial z}$ solves

$$\langle \mathcal{L}_2 \rangle u = -\bar{\lambda}'(z) x \frac{\partial P_0}{\partial x} + l\,\bar{\lambda}'(z) P_0, \quad u(T, x, r; z) = 0.$$

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$$\frac{\partial P_0}{\partial z} = (T - t)\bar{\lambda}'(z) \left(x \frac{\partial P_0}{\partial x} - l\, P_0 \right)$$

from which it follows that $-\langle \mathcal{M}_1 \rangle P_0$ can be represented as

$$\begin{aligned} -\langle \mathcal{M}_1 \rangle P_0 = &-(T - t)\bar{\lambda}'(z) \Bigg(\bar{\sigma}_1 \rho_3 g(z) \Bigg(x^2 \frac{\partial^2 P_0}{\partial x^2} \\ &+ (1 - l) x \frac{\partial P_0}{\partial x} \Bigg) + \eta \rho_{13} g(z) \left(x \frac{\partial^2 P_0}{\partial x \partial r} - l \frac{\partial P_0}{\partial r} \right) \Bigg). \end{aligned}$$

Proof cont.

1) We first observe that $\frac{(T-t)^2}{2}(x^n \frac{\partial^n}{\partial x^n})P_0$ solves

$$\langle \mathcal{L}_2 \rangle u = -(T-t)\left(x^n \frac{\partial^n}{\partial x^n}\right) P_0, \quad u(T,x,r;z) = 0.$$

2) Next, we apply $\langle \mathcal{L}_2 \rangle$ on $(T-t)\frac{\partial P_0}{\partial r}$ and obtain

$$\langle \mathcal{L}_2 \rangle \left((T-t)\frac{\partial P_0}{\partial r}\right) = -\frac{\partial P_0}{\partial r} \\ +(T-t)\left(-x\frac{\partial P_0}{\partial x} + \beta\frac{\partial P_0}{\partial r} + P_0\right),$$

as a result of which see that

$$\frac{1}{\beta}[-\frac{\partial P_0}{\partial \alpha} - \frac{(T-t)^2}{2}(x\frac{\partial P_0}{\partial x} - P_0) + (T-t)\frac{\partial P_0}{\partial r}]$$

solves

$$\langle \mathcal{L}_2 \rangle u = (T-t)\frac{\partial P_0}{\partial r}, \quad u(T,x,r;z) = 0.$$

Parameter Estimation - i

- The parameters of the interest rate model $\{\alpha, \beta, \eta\}$ are obtained by a least-square fitting to the Treasury yield curve.
- $\bar{\rho}_1 = \frac{\bar{\sigma}_1}{\bar{\sigma}_2}\rho_1$, the "effective" correlation between risk-free interest rate *r* and stock price is estimated from historical risk-free spot rate and stock price data.
- $\bar{\sigma}_2$, the "effective" stock price volatility is estimated from the historical stock price data.

Estimation of $I\bar{\lambda}$ and $\{IV_3^{\epsilon}, IV_2^{\delta}\}$ from the Corporate Bond Price Data.

The approximate price formula for a defaultable bond

$$\widetilde{B}^c = B_0^c + \sqrt{\varepsilon} B_{1,0}^c + \sqrt{\delta} B_{0,1}^c,$$

in which B_0^c is given by (6) and

$$\sqrt{\epsilon} B_{1,0}^c = IV_3^{\epsilon} \frac{\partial B_0^c}{\partial \alpha},$$

$$\sqrt{\delta} B_{0,1}^c = IV_2^{\delta} \frac{1}{\beta} \left[-\frac{\partial B_0^c}{\partial \alpha} + \frac{(T-t)^2}{2} B_0^c + (T-t) \frac{\partial B_0^c}{\partial r} \right].$$

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We obtain $\{I\bar{\lambda}(z), IV_3^{\varepsilon}, IV_2^{\delta}\}$ from least-squares fitting, i.e. by minimizing

$$\sum_{i=1}^{n} (B_{\text{obs}}^c(t, T_i) - B_{\text{model}}^c(t, T_i; I\bar{\lambda}, IV_3^{\epsilon}, IV_2^{\delta}))^2,$$

Estimation of $\{I, V_1^{\epsilon}, V_2^{\epsilon}, V_4^{\epsilon}, V_5^{\epsilon}, V_6^{\epsilon}, V_1^{\delta}\}$ from the Equity Option Data

These parameters are calibrated from the stock options data by a least squares fit to the observed implied volatility:

$$\sum_{i=1}^{n} (I_{\text{obs}}(t, T_i, K_i) - I_{\text{model}}(t, T_i, K_i; \text{model parameters}))^2$$

$$\approx \sum_{i=1}^{n} \frac{(P_{\text{obs}}(t, T_i, K_i) - P_{\text{model}}(t, T_i, K_i; \text{model parameters}))^2}{\text{vega}^2(T_i, K_i)}.$$

Estimation of $\{I, V_1^\epsilon, V_2^\epsilon, V_4^\epsilon, V_5^\epsilon, V_6^\epsilon, V_1^\delta\}$ from the Equity Option Data

These parameters are calibrated from the stock options data by a least squares fit to the observed implied volatility:

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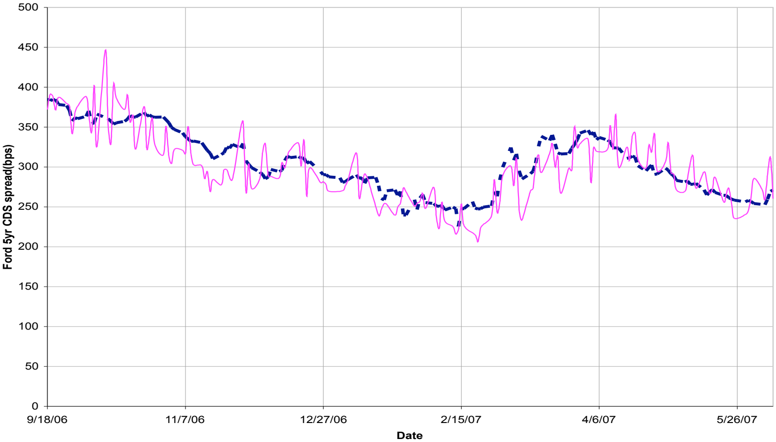
Recall that

$$\begin{aligned}
&P_{\text{model}}(t, T_i, K_i; \text{model parameters}) \\
&= P_0(t, T_i, K_i; \bar\lambda) + V_1^\epsilon g_1(T_i, K_i; \bar\lambda) + V_2^\epsilon g_2(T_i, K_i; \bar\lambda) \\
&+ V_3^\epsilon g_3(T_i, K_i; \bar\lambda) + V_4^\epsilon g_4(T_i, K_i; \bar\lambda) + V_5^\epsilon g_5(T_i, K_i; \bar\lambda) \\
&+ V_6^\epsilon g_6(T_i, K_i; \bar\lambda) + V_1^\delta g_7(T_i, K_i; \bar\lambda) + V_2^\delta g_8(T_i, K_i; \bar\lambda).
\end{aligned}$$

Model Implied CDS Premium

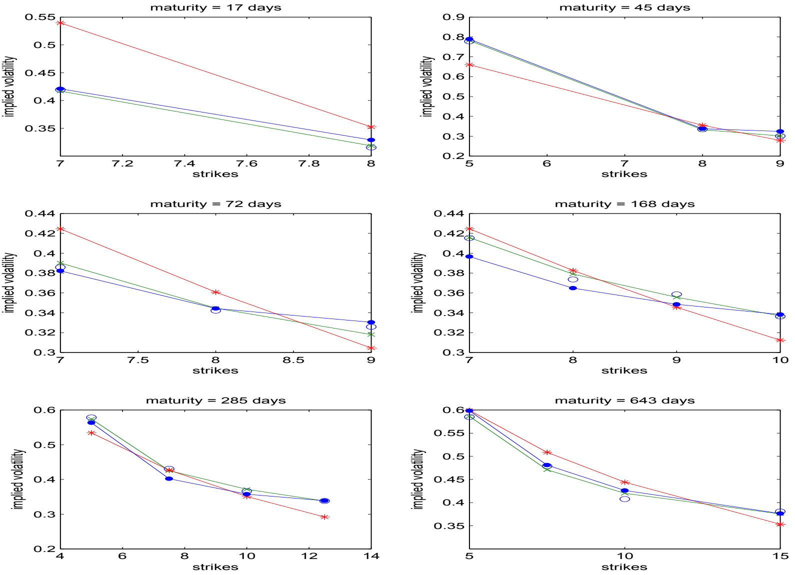
$$c^{ds}_{\text{model}}(t, T_M) = \frac{l}{1-l} \frac{\widetilde{B}^c(t, T_M; l) - \widetilde{B}^c(t, T_M; 1)}{\sum_{m=1}^{M} \widetilde{B}^c(t, T_m; 1)}. \tag{7}$$

Testing the Model

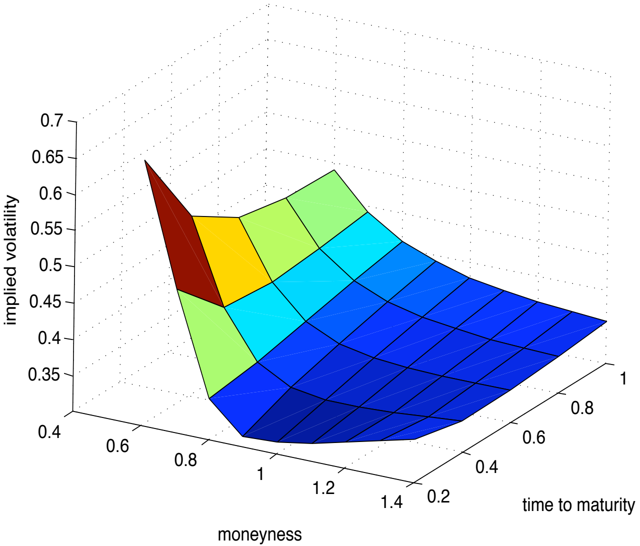


Ford 5 year CDS annual spread time series from 9/18/2006-6/8/2007.

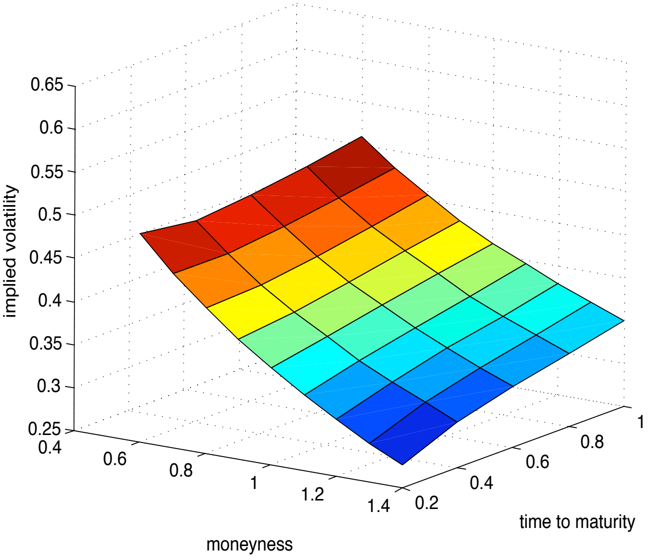
Implied Volatility on the 4th of April, 2007



Implied Volatility of our model, Ford June 8, 2007



Implied Volatility of Foque et al.'s model



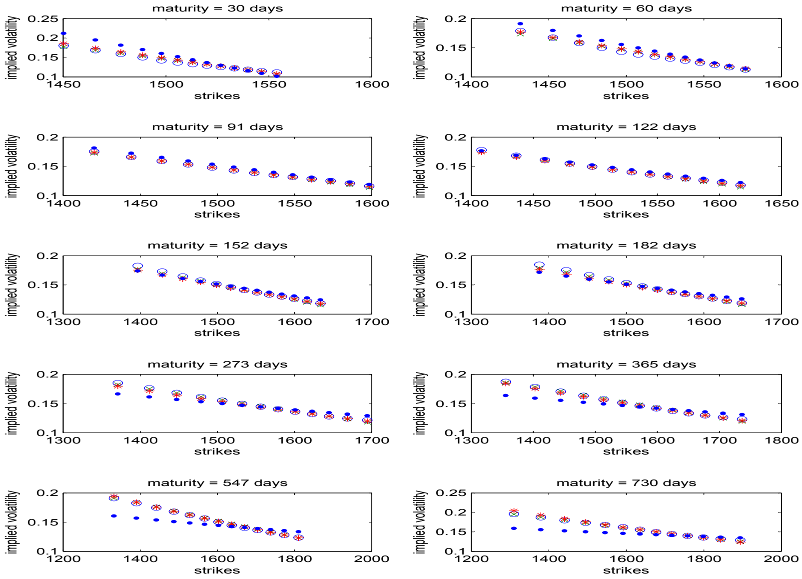


Figure: The fit to the Implied Volatility Surface of SPX on June 8, 2007

Thanks for your attention!