The elliptic curve discrete logarithm problem and equivalent hard problems for elliptic divisibility sequences

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Elliptic Curves in Cryptography

- Suggested by Victor Miller and Neil Koblitz in 1985
- Now implemented many places; part of NSA's Suite B
- Relies on Problem:
  - Let E be an elliptic curve over a finite field $K = \mathbb{F}_q$. Suppose one is given points $P, Q \in E(K)$ such that $Q \in \langle P \rangle$. Determine k such that $Q = [k]P$.
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- So... it is in the interests of world security that we keep failing to solve this problem in new and creative ways.

Generic attacks

Attacks which work in any group where group operation is easy to compute. This relies on the 'birthday paradox': selecting elements of a set of size n randomly, we expect to see a repeat after $O(\sqrt{n})$ selections.

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Shanks baby-step-giant-step:

- Let $N = \lceil \sqrt{n} \rceil$.
- Create a list of elements $P, [2]P, \ldots, [N]P$.
- Create a list of elements $Q + R, Q + [2]R, \ldots, Q + [N]R$ where $R = [-N]P$.
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Pollard rho:

- Iterate a sufficiently 'mixing' function $f : G \to G$ (whose definition depends on Q) and wait for $f^{(i)}(P) = f^{(2i)}(P)$

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- Weil and Tate pairing attacks: more on this later.
- Many (mostly) failed attempts to do an index calculus for elliptic curves.
- Recent success by Claus Diem: curves over a family of finite fields $\mathbb{F}_{q^n}$ where $n = O(\sqrt{\log q})$.

Division polynomials

Consider a point $P = (x, y)$ and its multiples on an elliptic curve $E : y^2 = x^3 + Ax + B$. Then

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&\Psi_1 = 1, \qquad \Psi_2 = 2y, \\
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&\Psi_4 = 4y(x^6 + 5Ax^4 + 20Bx^3 - 5A^2x^2 - 4ABx - 8B^2 - A^3), \\
&\quad \Psi_{m+n}\Psi_{m-n}\Psi_1^2 = \Psi_{m+1}\Psi_{m-1}\Psi_n^2 - \Psi_{n+1}\Psi_{n-1}\Psi_m^2 \ .
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Translation properties

Theorem (Ward / Swart / Ayad)

Let W be an elliptic divisibility sequence such that $W(1) = 1, W(2)W(3) \neq 0$. Let $r \in \mathbb{Z}$ be such that $W(r) = 0$. Then there exist a, b such that

$$W(n + kr) = W(n)a^{nk}b^{k^2}.$$

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$k = 2 : W(18 + n) \equiv W(n)4^{2n}2^4 \equiv W(n) \mod 5$

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Theorem (Lauter,S.)

Suppose $(q-1, \mathrm{ord}(P)) = 1$. *Define* $\phi : E \to \mathbb{F}_q$ *by*

$$\phi(P) = \left(\frac{W_{E,P}(q-1)}{W_{E,P}(q-1+\mathrm{ord}(P))} \right)^{\frac{1}{\mathrm{ord}(P)^2}} .$$

Then $W(n) = \phi([n]P)$ *is a perfectly periodic elliptic divisibility sequence, and furthermore,*

$$\phi([n]P) = \phi(P)^{n^2} W_{E,P}(n).$$

Example of perfect periodicity

$E : y^2 + y = x^3 + x^2 - 2x, P = (0, 0)$ over $\mathbb{F}_5$
The usual elliptic divisibility sequence $W_{E,P}(n)$ is...

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From the theorem,

$$\phi(P) = \left(\frac{1}{2}\right)^1 = 3.$$

Then the sequence $\phi([n]P)$ is... $(\phi([n]P) = 3^{n^2} W_{E,P}(n))$

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Problem

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Problem (Width s EDS Discrete Log)

Given an elliptic divisibility sequence W *and terms* $W(k)$, $W(k+1)$, $\ldots$, $W(k+s-1)$, *determine* k.

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- Gave algorithms for computing a block of seven terms at position $a + b$ from blocks at position a and b.
- One might attempt generic discrete log attacks for groups, e.g. Pollard ρ.

Hard problems for EDS

Let E be an elliptic curve over a finite field $K = \mathbb{F}_q$. Suppose one is given points $P, Q \in E(K)$ such that $Q \in \langle P \rangle$, $Q \neq \mathcal{O}$, and $\operatorname{ord}(P) \geq 4$.

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Determine $W_{E,P}(k)$ *for the value of* $0 < k < \operatorname{ord}(P)$ *such that* $Q = [k]P$*.*

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Determine the quadratic residuosity of $W_{E,P}(k)$ *for the value of* $0 < k < \mathrm{ord}(P)$ *such that* $Q = [k]P$.

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- The smallest positive value of k such that $[k]P = Q$ will be called the *minimal multiplier*.

Relating hard problems

$$
\begin{tikzcd}
\text{perfectly periodic} & {[k]P} \arrow[dl, shift right, "(\log q)^3"'] \arrow[dd, dashed, "ECDLP"] \arrow[dr, dashed, "\begin{matrix} EDS \\ Association \end{matrix}"] & \text{not perfectly periodic} \\
\{\phi([i]P)\}_{i=k}^{k+2} \arrow[ur, shift right, "(\log q)^2"'] \arrow[dr, dashed, "\begin{matrix} Width\ 3 \\ EDS\ Discrete\ Log \end{matrix}"'] & & \{W_{E,P}(i)\}_{i=k}^{k+2} \arrow[dl, shift left, "\mathbb{F}_q^* DLP"] \\
& k \arrow[ur, shift left, "(\log q)^3"] &
\end{tikzcd}
$$

$[k]P \to \{\phi([i]P)\}_{i=k}^{k+2}$

- Perfectly periodic case.

$$\begin{tikzcd}
& {[k]P} \arrow[dl, squiggly, shift right] \arrow[dd, dashed] \arrow[dr, dashed] & \\
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- Use
$$\phi(P) = \left(\frac{W_{E,P}(q-1)}{W_{E,P}(q-1+\mathrm{ord}(P))} \right)^{\frac{1}{\mathrm{ord}(P)^2}} .$$
- Use Shipsey algorithms to calculate W to distance q.
- $(\log q)^3$ time.

$\{\phi([i]P)\}_{i=k}^{k+2} \to [k]P$

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- Use
$$x(P) - x([k]P) = \frac{\phi([k+1]P)\phi([k-1]P)}{\phi([k]P)^2}$$

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$k \to \{W_{E,P}(i)\}_{i=k}^{k+2}$

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- Non-perfectly periodic case.

- Use Shipsey algorithms to calculate terms.
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(which is from $\phi([k]P) = \phi(P)^{k^2} W_{E,P}(k)$ with $k, k+1$).

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$$\frac{\phi([k+1]P)}{\phi([k]P)} = \phi(P)^{2k+1}\frac{W_{E,P}(k+1)}{W_{E,P}(k)}.$$
(which is from $\phi([k]P) = \phi(P)^{k^2} W_{E,P}(k)$ with $k, k+1$).
- This $\mathbb{F}_q^*$ discrete log can be solved in sub-exponential time.

$\{W_{E,P}(i)\}_{i=k}^{k+2} \to k$

$$\begin{tikzcd}
& {[k]P} \arrow[dl, shift right] \arrow[dr, dashed] \arrow[dd, dashed] & \\
\{\phi([i]P)\}_{i=k}^{k+2} \arrow[ur, shift right] \arrow[dr, dashed] & & \{W_{E,P}(i)\}_{i=k}^{k+2} \arrow[dl, Rightarrow, squiggly] \\
& k \arrow[ur] &
\end{tikzcd}$$

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- This $\mathbb{F}_q^*$ discrete log can be solved in sub-exponential time.
- (This is a different method than Shipsey; similar result.)

Relating hard problems II

perfectly periodic

not perfectly periodic

$$
\begin{tikzcd}
 & {[k]P} \arrow[dl, shift right, "(\log q)^3"'] \arrow[dd, dashed, "ECDLP"] \arrow[dr, dashed, "\substack{EDS\\ Association}"] & \\
\{\phi([i]P)\}_{i=k}^{k+2} \arrow[ur, shift right, "(\log q)^2"'] \arrow[dr, dashed, "\substack{Width\ 3\\ EDS\ Discrete\ Log}"'] & & \{W_{E,P}(i)\}_{i=k}^{k+2} \arrow[dl, shift left, "\mathbb{F}_q^* DLP"] \\
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\end{tikzcd}
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It is enough to know parity of k

Suppose $\mathrm{ord}(P)$ is odd.
An algorithm for finding k such that $[k]P = Q$ and $0 < k < \mathrm{ord}(P)$.

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1. If P = Q, stop.
2. Find parity of smallest positive k such that $[k]P = Q$.
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- When we return to step 1, the new k' is $k/2$ or $(k-1)/2$ depending on parity in step 2.
- To find k when the algorithm ends, count up the sequence of parities – gives binary expansion of k.

EDS Residue

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- We have the $\mathbb{F}_q^*$ discrete logarithm equation:

$$\phi([k]P) = \phi(P)^{k^2} W_{E,P}(k).$$

- The parity of k can be calculated from the residuosity in polynomial time.

Can we solve EDS Residue?

No. Interestingly, we can calculate the residuosity of ratios of terms

$$\frac{W_{E,P}(k+1)}{W_{E,P}(k)}$$

but this doesn't help.

Theorem (Lauter, S.)

Let E be an elliptic curve over a finite field $\mathbb{F}_q$. If any one of the following problems is solvable in sub-exponential time, then all of them are:

1. *ECDLP*
2. *EDS Association for non-perfectly periodic sequences*
3. *Width 3 EDS Discrete Log for perfectly periodic sequences*

If $|E(\mathbb{F}_q)|$ is odd and $\mathrm{char}(\mathbb{F}_q) \neq 2$, we can also include

4. *EDS Residue for non-perfectly periodic sequences*

Division polynomials of higher rank?

The n-th division polynomial is associated to the vanishing of $[n]P$ on the curve.

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Or even ...

$$\boxed{[n]P + [m]Q + [t]R \leftrightarrow \Psi_{n,m,t}}$$

etc.

Definition (S)

Let K be a field. An *elliptic net* is a map $W : A \to K$ such that the following recurrence holds for all p, q, r, $s \in \mathbb{Z}^n$.

$$\begin{aligned} &W(p+q+s)W(p-q)W(r+s)W(r) \\ &\qquad + W(q+r+s)W(q-r)W(p+s)W(p) \\ &\qquad\qquad + W(r+p+s)W(r-p)W(q+s)W(q) = 0 \end{aligned}$$

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Definition of an elliptic net

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- Elliptic divisibility sequences are a special case $(n = 1)$
- In this talk, we will mostly discuss rank $n = 2$.
- The recurrence generates the net from finitely many initial values.

Net polynomial examples

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Can calculate more via the recurrence...

$$\begin{aligned}\Psi_{3,1} = {} & (x_2 - x_1)^{-3}(4x_1^6 - 12x_2x_1^5 + 9x_2^2x_1^4 + 4x_2^3x_1^3 \\ & - 4y_2^2x_1^3 + 8y_1^2x_1^3 - 6x_2^4x_1^2 + 6y_2^2x_2x_1^2 - 18y_1^2x_2x_1^2 \\ & + 12y_1^2x_2^2x_1 + x_2^6 - 2y_2^2x_2^3 - 2y_1^2x_2^3 + y_2^4 - 6y_1^2y_2^2 \\ & + 8y_1^3y_2 - 3y_1^4) \ .\end{aligned}$$

Curve-net bijection

Theorem (S.)

There is a bijection of partially ordered sets:

$$\left\{\begin{array}{c} \textit{elliptic net} \\ W : \mathbb{Z}^n \to K \\ \textit{modulo scale} \\ \textit{equivalence} \end{array}\right\} \quad \leftrightarrow \quad \left\{\begin{array}{c} \textit{cubic Weierstrass curve } C \textit{ over } K \\ \textit{together with } m \textit{ points in } C(K) \\ \textit{modulo change of variables} \\ x' = x + r, y' = y + sx + t \end{array}\right\}$$

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- $n=m$ and $W(\mathbf{v})=\Psi_{\mathbf{v}}(P_1,\ldots,P_m,C)$

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- $n=m$ and $W(\mathbf{v})=\Psi_{\mathbf{v}}(P_1,\dots,P_m,C)$
- explicit equations to go back and forth!
- singular cubics correspond to Lucas sequences or integers
- scale equivalence: $W\sim W' \iff W(\mathbf{v})=f(\mathbf{v})W'(\mathbf{v})$ for $f:\mathbb{Z}^n\to K^*$ quadratic
- on left, remove nets with zeroes too close to the origin
- on right, remove cases with small torsion points or pairs which are equal or inverses
- consider only nets with $W(\mathbf{v})=1$ for $\mathbf{v}=\mathbf{e}_i$ or $\mathbf{v}=\mathbf{e}_i+\mathbf{e}_j$

Example over $\mathbb{Q}$

$$E : y^2 + y = x^3 + x^2 - 2x; P = (0,0), Q = (1,0)$$

	-5	8	-19			
	1	3	-1			
$\uparrow$	1	1	2			
Q	0	1	1			
	$P \rightarrow$					

Example over $\mathbb{Q}$

$$E : y^2 + y = x^3 + x^2 - 2x; P = (0,0), Q = (1,0)$$

4335	5959	12016	−55287	23921	1587077
94	479	919	− 2591	13751	68428
− 31	53	−33	−350	493	6627
−5	8	−19	− 41	− 151	989
1	3	−1	− 13	−36	181
1	1	2	−5	7	89
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Example over $\mathbb{F}_5$

$$E : y^2 + y = x^3 + x^2 - 2x; P = (0,0), Q = (1,0)$$

0	4	4	3	1	2	4
4	4	4	4	1	3	0
4	3	2	0	3	2	1
0	3	1	4	4	4	4
1	3	4	2	4	1	0
1	1	2	0	2	4	1
0	1	1	2	1	3	4

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Example over $\mathbb{F}_5$

$$E : y^2 + y = x^3 + x^2 - 2x; P = (0,0), Q = (1,0)$$

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- The polynomial $\Psi_{\mathbf{v}}(\mathbf{P}) = 0$ if and only if $\mathbf{v} \cdot \mathbf{P} = 0$.

Example over $\mathbb{F}_5$

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$$\begin{array}{cc|c|c|c|c|c|c|c|}
\hline
 & & \color{red}{0} & 4 & 4 & 3 & 1 & 2 & 4 \\
\hline
 & & 4 & 4 & 4 & 4 & 1 & 3 & \color{red}{0} \\
\hline
 & & 4 & 3 & 2 & \color{red}{0} & 3 & 2 & 1 \\
\hline
 & & \color{red}{0} & 3 & 1 & 4 & 4 & 4 & 4 \\
\hline
 & & 1 & 3 & 4 & 2 & 4 & 1 & \color{red}{0} \\
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\hline
\multicolumn{2}{c}{} & \multicolumn{7}{l}{P \rightarrow}
\end{array}$$

- The polynomial $\Psi_{\mathbf{v}}(\mathbf{P}) = 0$ if and only if $\mathbf{v} \cdot \mathbf{P} = 0$.
- These zeroes lie in a lattice: the *lattice of apparition* associated to prime (here, 5).

Periodicity property with respect to lattice of apparition

	0	4	4	3	1	2	4
	4	4	4	4	1	3	0
	4	3	2	0	3	2	1
	0	3	1	4	4	4	4
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- The elliptic net is not periodic modulo the lattice of apparition.

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- The elliptic net is not periodic modulo the lattice of apparition.
- The appropriate translation property should tell how to obtain the green values from the blue values.

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$\uparrow Q$

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- The elliptic net is not periodic modulo the lattice of apparition.
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- There are such translation properties.

Translation properties

Let Γ be the lattice of apparition for an elliptic net W. Define $g : \Gamma \times \mathbb{Z}^n \to K^*$ by

$$g(\mathbf{r}, \mathbf{m}) = \frac{W(\mathbf{m} + \mathbf{r})}{W(\mathbf{m})}.$$

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Theorem (Ward $n = 1$; S., $n > 1$)

The function g is quadratic and affine linear in 2nd variable.

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$$g(\mathbf{r}, \mathbf{m}) = \frac{W(\mathbf{m} + \mathbf{r})}{W(\mathbf{m})}.$$

Theorem (Ward $n = 1$; S., $n > 1$)

The function g is quadratic and affine linear in 2nd variable.

Example

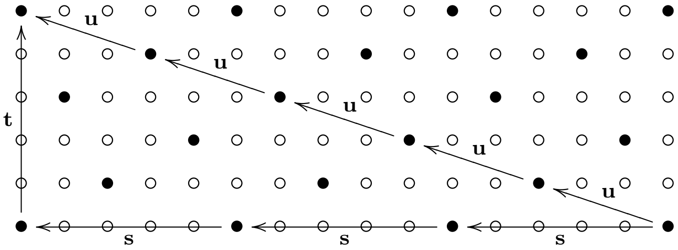
If $n = 1$, $W(r) = 0$, then

$$g(kr, m) = a^{mk} b^{k^2},$$

for all $k \in \mathbb{Z}$.

About $\mathbb{F}_q^*$ discrete logarithm equations

Other ways to find them: combine partial periodicity relations.



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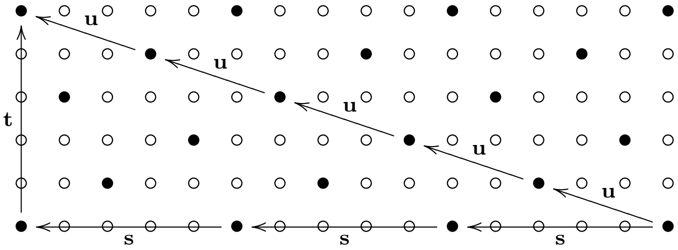

giving (where $m = \mathrm{ord}(P)$):

$$\left(\frac{W(m+1,0)W(2,0)}{W(m+2,0)}\right)^k = \left(\frac{W_{E,P}(k-1)}{W_{E,P}(k)}\right)^m \left(-\frac{W(1,m)W(2,0)}{W(2,m)W(1,-1)^m}\right).$$

This is similar to Shipsey's equation.

Shipsey's discrete logarithm

From previous slide:

$$\left(\frac{W(m+1,0)W(2,0)}{W(m+2,0)}\right)^k = \left(\frac{W_{E,P}(k-1)}{W_{E,P}(k)}\right)^m \left(-\frac{W(1,m)W(2,0)}{W(2,m)W(1,-1)^m}\right).$$

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Compare to Shipsey's:

$$\frac{W_{E,P,Q}(m+1,m+1)}{W_{E,P,Q}(0,m+1)}\left(\frac{W_{E,P}(k+1)}{W_{E,P}(k)}\right)^{m(m+2)} = W_{E,P}(m+1)^{2k+1}.$$

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Both can be explained as *Tate pairing values*.

Tate pairing

$m \geq 1$
E/K an elliptic curve

Tate pairing

$m \geq 1$	$P \in E(K)[m]$
E/K an elliptic curve	$Q \in E(K)/mE(K)$

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f_P with divisor $m(P) - m(\mathcal{O})$

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Define

$$\tau_m : E(K)[m] \times E(K)/mE(K) \to K^*/(K^*)^m$$

by

$$\tau_m(P, Q) = f_P(D_Q) \ .$$

It is well-defined, bilinear and Galois invariant.

Weil pairing

For $P, Q \in E(K)[m]$, the more well-known Weil pairing can be computed via two Tate pairings:

$$e_m(P,Q) = \tau_m(P,Q)\tau_m(Q,P)^{-1} \ .$$

It is bilinear, alternating, and non-degenerate.

Weil and Tate pairing attacks

These are isomorphism attacks:
Elliptic curve E defined over $\mathbb{F}_q$ (prime order, say), $Q = [k]P$.

- Menezes-Okamoto-Vanstone:
  For any auxiliary point T,

  $$e_m(Q,T) = e_m(P,T)^k$$

  and so we transfer the question of finding k to a discrete log in $\mathbb{F}_{q^t}$ for some t which is usually infeasibly large.
- Frey-Rück:

  $$\tau_m(P,Q) = \tau_m(P,P)^k$$

  is an equation in $\mathbb{F}_{q^t}$ where again t is usually infeasibly large.

Pairing from Elliptic Nets

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Theorem (S)

Choose $S \in E(K)$ such that $S \notin \{\mathcal{O}, -Q\}$. Let W be an elliptic net with basis $\mathbf{T}$ such that $p \cdot \mathbf{T} = P$, $q \cdot \mathbf{T} = Q$ and $s \cdot \mathbf{T} = S$. Then the quantity

$$\tau_m(P, Q) = \frac{W(s + mp + q)W(s)}{W(s + mp)W(s + q)}$$

is the Tate pairing.

The $\mathbb{F}_q^*$ DLP equation

From older slide:

$$\left(\frac{W(m+1,0)W(2,0)}{W(m+2,0)}\right)^k = \left(\frac{W_{E,P}(k-1)}{W_{E,P}(k)}\right)^m \left(-\frac{W(1,m)W(2,0)}{W(2,m)W(1,-1)^m}\right).$$

Becomes...

$$\tau_m(P,-P)^k = \tau_m(Q,-P).$$

Shipsey's $\mathbb{F}_q^*$ DLP equation

$$\frac{W_{E,P,Q}(m+1,m+1)}{W_{E,P,Q}(0,m+1)}\left(\frac{W_{E,P}(k+1)}{W_{E,P}(k)}\right)^{m(m+2)} = W_{E,P}(m+1)^{2k+1}.$$

Becomes...

$$\tau_m(P,Q)\tau_m(Q,P) = \tau_m(P,P)^{2k}.$$

For Further Reading

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American Journal of Mathematics, 70:13–74, 1948.

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Slides, Articles and Preprints at http://www.math.brown.edu/~stange/