A Concise Interpretation of the Infrastructure of a Global Field

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Overview

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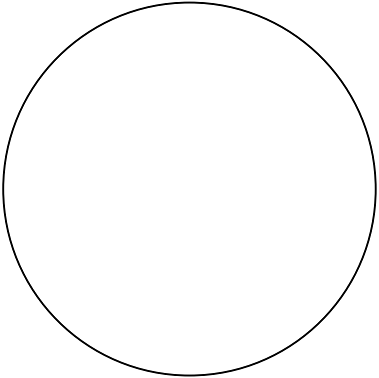

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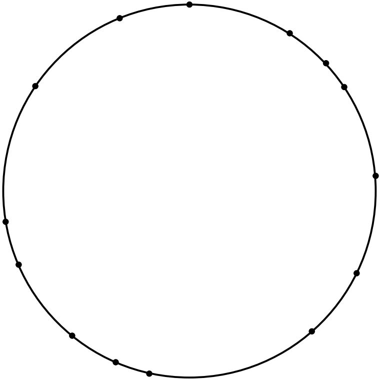

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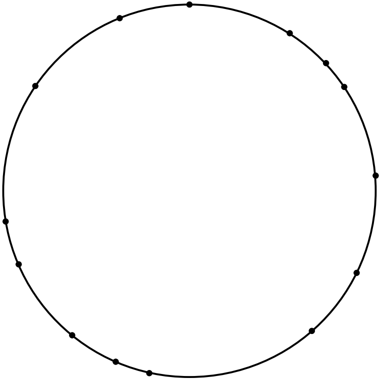

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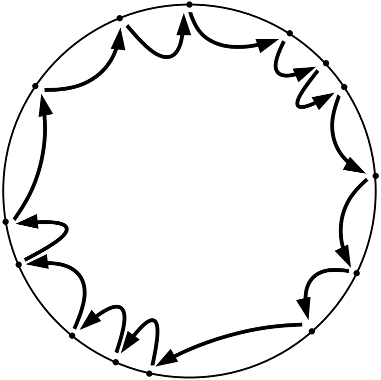

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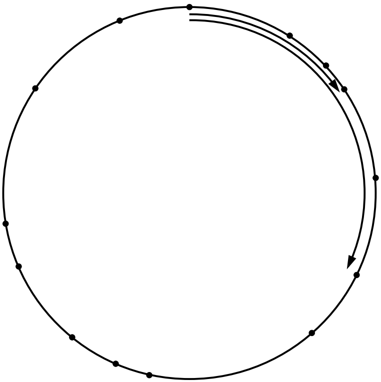

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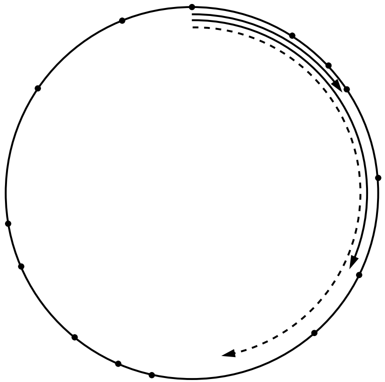

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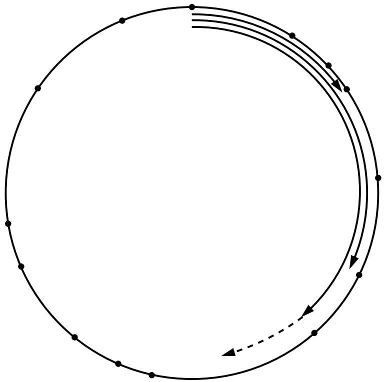

A Special Case: The Discrete Logarithm

We have:

- A finite cyclic group $X = \langle g \rangle$;
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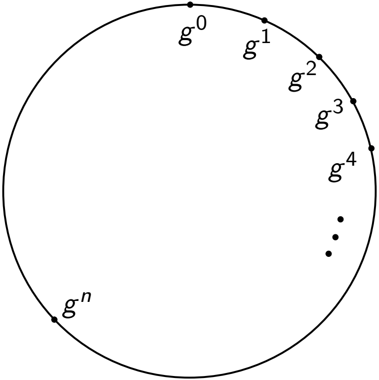

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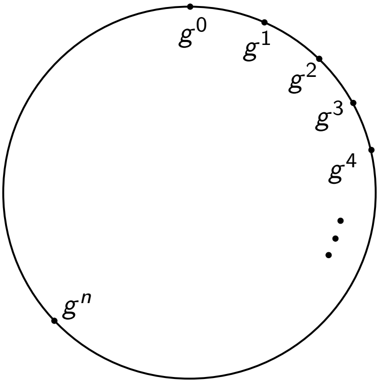
  - baby steps:
  $$h \mapsto gh;$$
  - giant steps:
  $$(h, h') \mapsto hh'.$$



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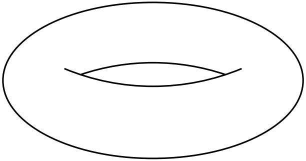

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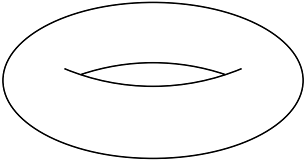

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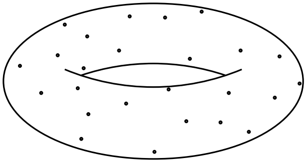

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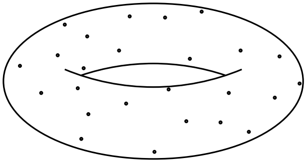

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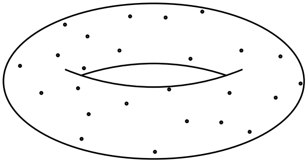

- This generalizes the Generalized Discrete Logarithm, i.e. writing group elements in terms of a fixed set of generators.
- But:
  - What should baby steps be?
  - Giant steps can be done, as we will see later...

Overview

f-Representations, Part One

Back to the one-dimensional case!

- The map
$$\begin{aligned} d : X \times \mathbb{R} &\to \mathbb{R}/R\mathbb{Z}, \\ (x, f) &\mapsto d(x) + f \end{aligned}$$
is surjective.
- Idea: Restrict to a subset of $X \times \mathbb{R}$ such that this map gets bijective!

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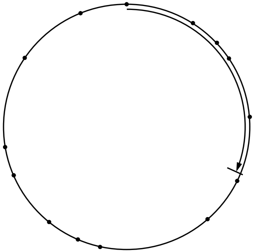

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- For example:

$$\begin{aligned} & (x, f) \in \mathrm{Rep}^f(X, d) \\ :\Longleftrightarrow {} & f \geq 0 \wedge \forall 0 < g \leq f : d(x) + g \notin d(X); \end{aligned}$$

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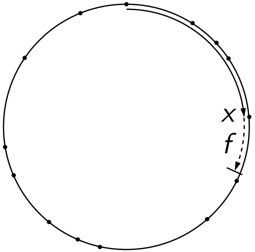

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- In general, the DP is at least as hard as the DLP.

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  - Assume that one has a global field[1] with two infinite places (infinite points, if you prefer curves).
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  - X is the set of principal reduced ideals.

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  - Hendrik Lenstra later gave a description by embedding the infrastructure into a "circular group", similar to our $\mathrm{Rep}^f(X, d)$.

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f-Representations, Part Three: More Dimensions

- How can f-representations be done in several dimensions?
- There is no longer an “obvious” way to write $t \in \mathbb{R}^n/\Lambda$ as $d(x) + f$ with $(x, f) \in X \times \mathbb{R}^n$.

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- An equivalent formulation: for $t \in \mathbb{R}^n/\Lambda$ we want to find some $x \in X$ with $d(x)$ “near to” t.
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- Note that f-representations give a giant step: if

$$(x, 0) + (x', 0) = (x'', f) \in \mathrm{Rep}^f(X, d),$$

then we can define $gs(x, x') := x''$.

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- We have to find a concrete instance of such a map in the interesting cases.
  - We are interested in infrastructures obtained from global fields.

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$$\mathcal{O} = \{f \in K \mid \nu_{\mathfrak{p}}(f) \geq 0 \text{ for all } \mathfrak{p} \notin S\}.$$

- Then $\Lambda := \Psi(\mathcal{O}^*)$ is a lattice in $\mathbb{R}^n$.

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- Then one has finitely many reduced divisors.
- Take those as X whose finite part equals the finite part of a principal divisor.
  - If $D = -(\mu)_{finite}$, define $\Psi(D) = \Psi(\mu) + \Lambda$.
  - This gives an injective map $\Psi : X \to \mathbb{R}^n/\Lambda$.

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  - Here, g is the genus of K;
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- Denote this map $t + \Lambda \mapsto D$ by *red*; this is a reduction map!

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Infrastructure

- This allows to define

$$\mathrm{Rep}^f(X, \Psi) := \{(D, f) \mid red(\Psi(D) + f) = D\}.$$

This gives a bijection

$$\Psi : \mathrm{Rep}^f(X, \Psi) \to \mathbb{R}^n/\Lambda, \qquad (D, f) \mapsto \Psi(D) + f$$

whose inverse is the map $t + \Lambda \mapsto (D, f)$ from above.

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- One can pull the addition of $\mathbb{R}^n/\Lambda$ over to $\mathrm{Rep}^f(X, \Psi)$ using this bijection.
  - One can describe this induced operation on $\mathrm{Rep}^f(X, \Psi)$ without using the map Ψ.
  - This allows effective computation of this group operation without the need to evaluate Ψ or Ψ^{-1}.

Relation to the Divisor Class Group

- The map
$$\Phi : \mathrm{Rep}^f(X, \Psi) \to \mathrm{Pic}^0(K),$$
$$(D, f) \mapsto D + \sum_{i=1}^{n} f_i \mathfrak{p}_i - (\dots)\, \mathfrak{p}_{n+1}$$
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- If combined with the bijection $\mathrm{Rep}^f(X, \Psi) \to \mathbb{R}^n/\Lambda$, one obtains a group isomorphism
$$\mathbb{R}^n/\Lambda \to \mathrm{img}(\Phi) \quad \text{resp.} \quad \mathbb{Z}^n/\Lambda \to \mathrm{img}(\Phi) \subseteq \mathrm{Pic}^0(K).$$

Overview

Conclusion

- Infrastructures can be seen as a generalization of abelian groups.
- We can obtain n-dimensional infrastructures from global fields, together with a reduction map.
- The obtained f-representations allow effective arithmetic in the infrastructure.
- The infrastructure can be seen as lying in the divisor class group;
  - by considering all reduced divisors and another formulation of the definition of $\mathrm{Rep}^f(X, \Psi)$, one obtains the whole divisor class group!
- The currently known algorithms for solving the DLP in $\mathrm{Pic}^0(K)$ also solve the Distance Problem in the infrastructure.

Research Problems

- Find more efficient algorithms to compute in the infrastructure.
  - Current ones are very general methods and *very slow*.
  - This question is related to finding efficient arithmetic in $\mathrm{Pic}^0(K)$.
- Find good generalization of baby steps.
  - In particular, in context of "abstract" n-dimensional infrastructures.
- Find more information on the distribution of reduced divisors in the function field case.
- How hard is computing Ψ, i.e. how hard is the Distance Problem?

Thank you
for your patience!