

Recurrent solutions of Alber's equation for random water-wave fields

by

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Alber's Equation (1978)

$$i \left(\frac{\partial \rho}{\partial t} + \frac{1}{2} \sqrt{\frac{g}{k_0}} \frac{\partial \rho}{\partial x} \right) - \frac{1}{4} \sqrt{\frac{g}{k_0^3}} \frac{\partial^2 \rho}{\partial r \partial x} =$$
$$\sqrt{gk_0^5} \rho(x, r, t) \left[\rho \left(x + \frac{r}{2}, 0, t \right) - \rho \left(x - \frac{r}{2}, 0, t \right) \right]$$

$$\rho(x, r, t) = \left\langle A \left(x + \frac{r}{2}, t \right) A^* \left(x - \frac{r}{2}, t \right) \right\rangle$$

$$2\eta(x, t) = A(x, t) e^{i \left(k_0 x - \sqrt{gk_0} t \right)}_{+*},$$

Correlation Function for Homogeneous Seas, Kinsman (1965), p 377

$$2 \eta(x, t) = e^{i(k_0 x - \sqrt{gk_0} t)} \int_{-\infty}^{\infty} e^{i[(k - k_0)x - (\sqrt{gk} - \sqrt{gk_0})t + \theta(k)]} \cdot \sqrt{S(k) dk} + *$$

$$A(x, t) = \int_{-\infty}^{\infty} e^{i[(k - k_0)x - (\sqrt{gk} - \sqrt{gk_0})t + \theta(k)]} \frac{dk}{\sqrt{S(k) dk}}$$

$$\rho(x, r, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\langle e^{i[(k_1 - k_0)(x + \frac{r}{2}) - (\sqrt{gk_1} - \sqrt{gk_0})t + \theta_1(k_1)]} \right. \\ \left. \cdot e^{-i[(k_2 - k_0)(x - \frac{r}{2}) - (\sqrt{gk_2} - \sqrt{gk_0})t + \theta_2(k_2)]} \right\rangle \frac{dk_1 dk_2}{\sqrt{S(k_1) S(k_2) dk_1 dk_2}}$$

$$\rho_h(r) \equiv \int_{-\infty}^{\infty} e^{i(k - k_0)r} S(k) dk$$

Instability of Inhomogeneous Disturbances

$$\rho(x, r, t) = \rho_h(r) + \delta \rho_1(x, r, t),$$

Assume a disturbance of the form:

$$\rho_1(x, r, t) = R(r) \left\{ e^{i \left[K \left(x - \frac{1}{2} \sqrt{\frac{g}{k_o}} t \right) - \Omega t \right]} + * \right\},$$

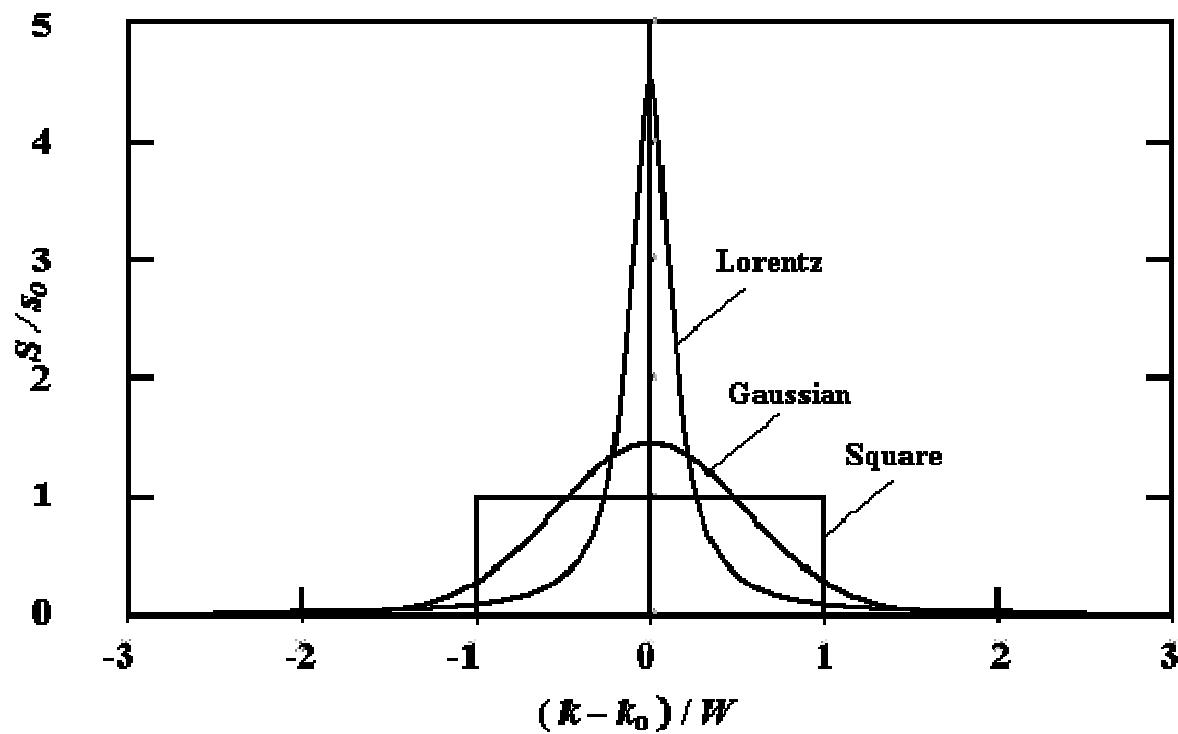
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$$1 = 4 k_0^4 \int_{-\infty}^{\infty} \frac{S(k) dk}{[i(k - k_0) + \gamma]^2 + K^2 / 4}$$

where

$$\gamma = 4 i \sqrt{\frac{k_o^3}{g}} \frac{\Omega}{K}$$

Three Narrow Spectra



Stability diagrams

Isolines of the non-dimensional growth-rate $\tilde{\Omega}_I$ for three spectra:

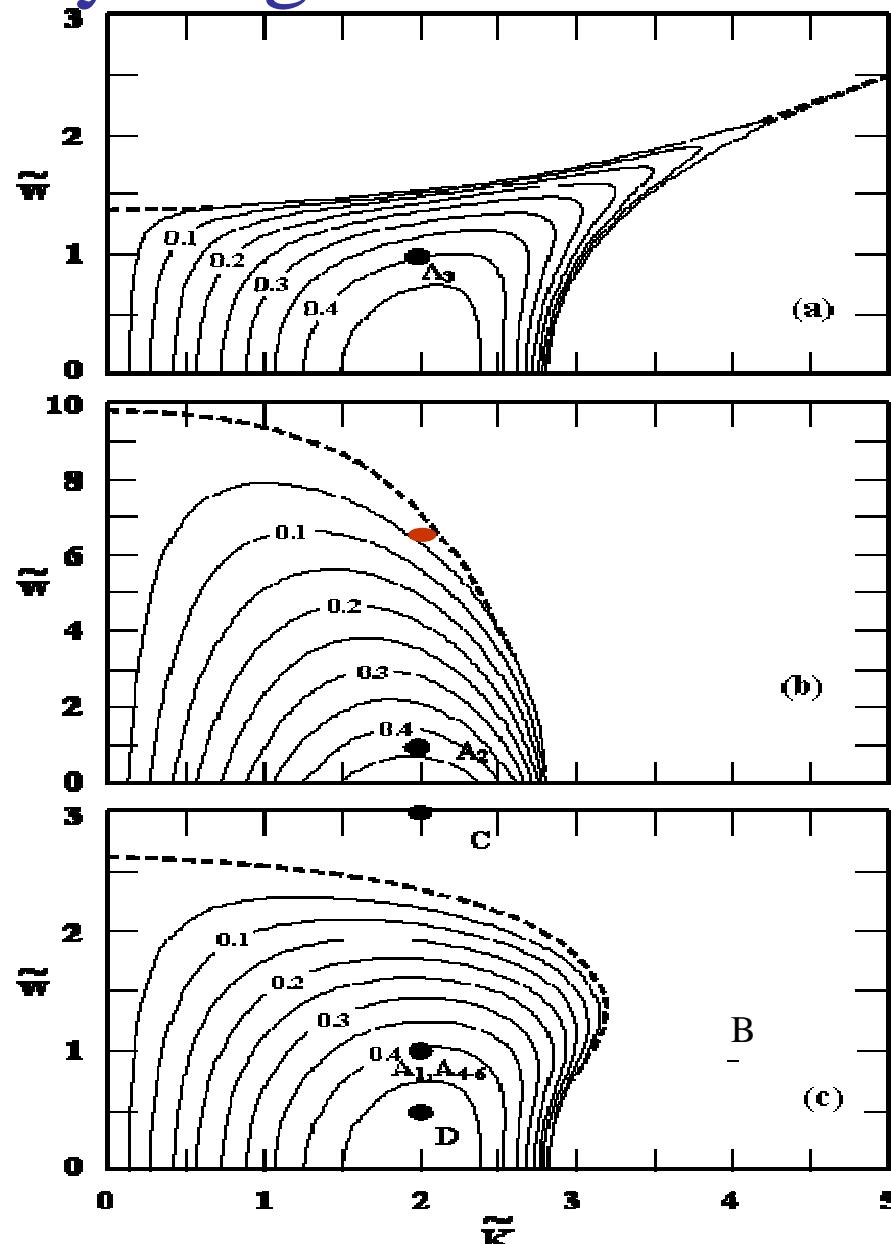
- a) Square spectrum;
- b) Lorentz spectrum;
- c) Gaussian spectrum.

Dots refer to the cases for which long-time evolution is studied.

$$\tilde{\Omega}_I = \Omega_I / \varepsilon^2 \sqrt{g k_0} ,$$

$$\tilde{K} = K / \varepsilon k_0 ,$$

$$\tilde{W} = W / \varepsilon k_0$$



Spectral interpretation of the initial conditions

$$S(x, k) = S_h(k) + \delta S_1(x, k),$$

$$\rho(x, r, t) = \int_{-\infty}^{\infty} e^{i(k - k_0)r} \sqrt{S(k, x + \frac{r}{2}, t) S(k, x - \frac{r}{2}, t)} dk$$

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$$S_1(x, k) = 2 s(k) \cos(Kx)$$

$$s(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(r) e^{-i(k - k_0)r} dr$$

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Interrelations

$$\rho_h(r) = \int_{-\infty}^{\infty} S_h(k) e^{i(k - k_0)r} dk$$

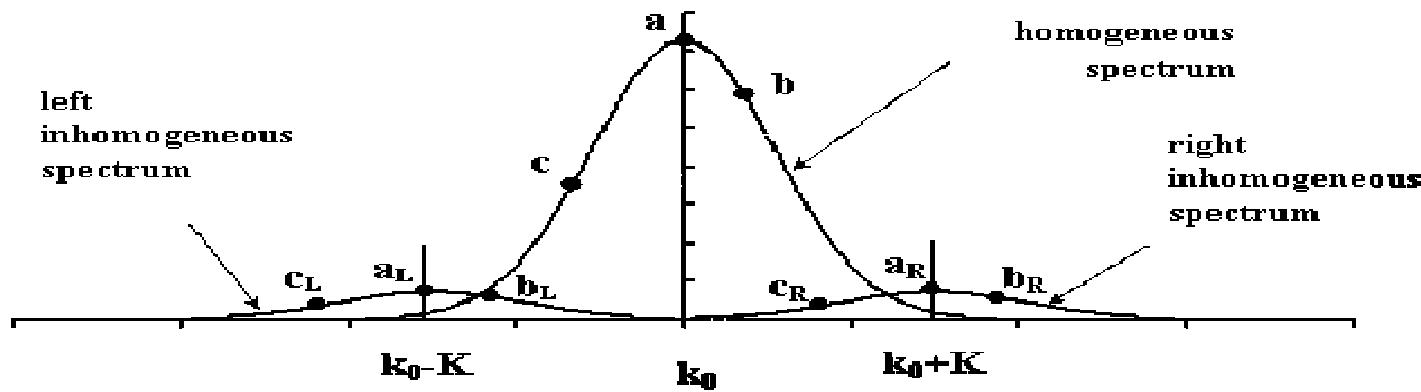
$$S_h(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho_h(r) e^{-i(k - k_0)r} dr$$

$$R(r) = \int_{-\infty}^{\infty} s(k) e^{i(k - k_0)r} dk$$

$$s(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(r) e^{-i(k - k_0)r} dr$$

Connection to the initial surface elevation η

$$2\eta = \int_{-\infty}^{\infty} \left[e^{i(kx + \theta(k))} \sqrt{S_h} + \frac{1}{2} e^{i(kx + \theta(k-K))} \sqrt{\frac{\delta^2 s^2(k-K)}{S_h(k-K)}} \right. \\ \left. + \frac{1}{2} e^{i(kx + \theta(k+K))} \sqrt{\frac{\delta^2 s^2(k+K)}{S_h(k+K)}} \right] \sqrt{dk} + *$$



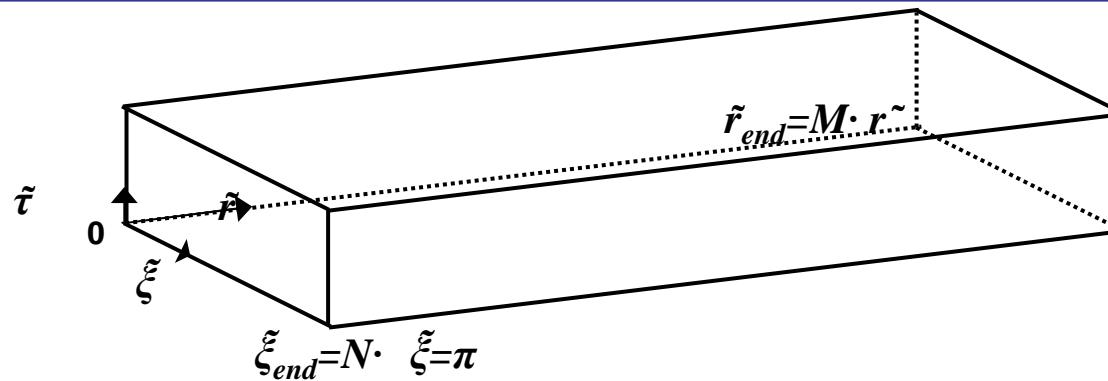
Schematic description of the main homogeneous spectrum and the inhomogeneous disturbance (here $S_h(k)$ and $s(k)$ are both Gaussian). As an example, the phases at (a_R, b_R, c_R) and at (a_L, b_L, c_L) are the same as at (a, b, c) , respectively.

Numerical solution of Alber's equation

$$\tilde{\rho} = \frac{k_o^2}{\varepsilon^2} \rho, \quad \tilde{\xi} = \varepsilon k_o \left(x - \frac{1}{2} \sqrt{\frac{g}{k_o}} t \right), \quad \tilde{\tau} = \left(\varepsilon^2 \sqrt{g k_o} \right) t, \quad \tilde{r} = \varepsilon k_o r$$

$$i \frac{\partial \tilde{\rho}}{\partial \tilde{\tau}} + 2\lambda \frac{\partial^2 \tilde{\rho}}{\partial \tilde{\xi} \partial \tilde{r}} - 2\nu \tilde{\rho} \left[\tilde{\rho} \left(\tilde{\xi} + \frac{\tilde{r}}{2}, 0 \right) - \tilde{\rho} \left(\tilde{\xi} - \frac{\tilde{r}}{2}, 0 \right) \right] = 0$$

where, $\nu = 1/2$ and $\lambda = -1/8$.



Boundary conditions

- Periodicity in ξ
- Condition at large \tilde{r}

Approximating $S(k,x)$ by a square spectrum in $(k-k_0) \in (-w,w)$, and integrating, leads to :

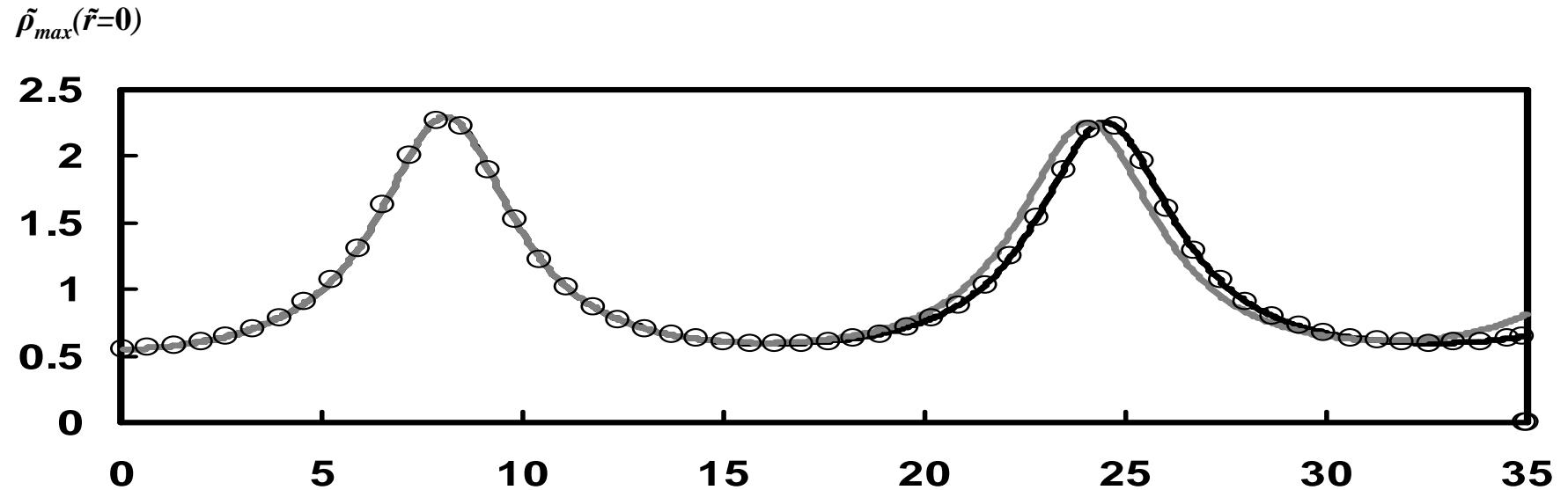
$$\rho(x, r, t) = 2 \sqrt{S(x + \frac{r}{2}, t) S(x - \frac{r}{2}, t)} \frac{\sin(wr)}{r}$$

For $r = 0$: $\rho(x, 0, t) = 2S(x, t)w$,
thus $S(x, t) = \rho(x, 0, t) / 2w$

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$$\tilde{\rho}(\xi, \tilde{r}, \tilde{\tau}) = \sqrt{\tilde{\rho}(\xi + \frac{\tilde{r}}{2}, 0, \tilde{\tau}) \tilde{\rho}(\xi - \frac{\tilde{r}}{2}, 0, \tilde{\tau})} \frac{\sin(\tilde{w}\tilde{r})}{\tilde{w}\tilde{r}}$$

The influence of the extent of \tilde{r}



The influence of the extent of the \tilde{r} domain on the maximum value of $\tilde{\rho}$ at $\tilde{r}=0$ as a function of non-dimensional time $\tilde{\tau}$, for $\tilde{r}_{end}=10\xi_{end}$ ——— ;
 $\tilde{r}_{end}=30\xi_{end}$ ——— , and $\tilde{r}_{end}=50\xi_{end}$ ○○○. These calculations are for a homogenous Gaussian spectrum and an inhomogeneous Gaussian disturbance with $\tilde{K}=2$, $\tilde{W}=1$, $\delta=0.1$. A₁

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Invariants of motion

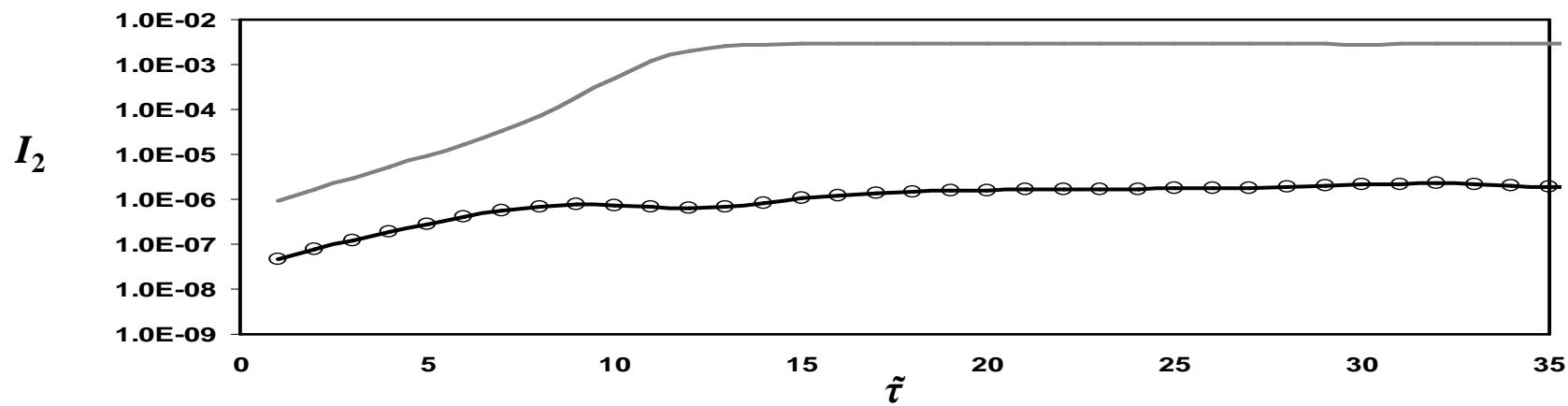
$$I_1 = \int_0^{\Pi'} \rho(x, 0, t) dx , \quad (10^{-7})$$

$$I_2 = \int_0^{\Pi'} \int_{-\infty}^{\infty} \kappa S' dx d\kappa$$

$$I_3 = \int_0^{\Pi'} \rho^2(x, 0, t) dx - \frac{1}{4} \int_0^{\Pi'} \int_{-\infty}^{\infty} \kappa^2 S' dx d\kappa , \quad (2\%)$$

$$S'(x, \kappa, t) = \int_{-\infty}^{\infty} \rho(x, r, t) e^{i\kappa r} dr$$

Second Invariant



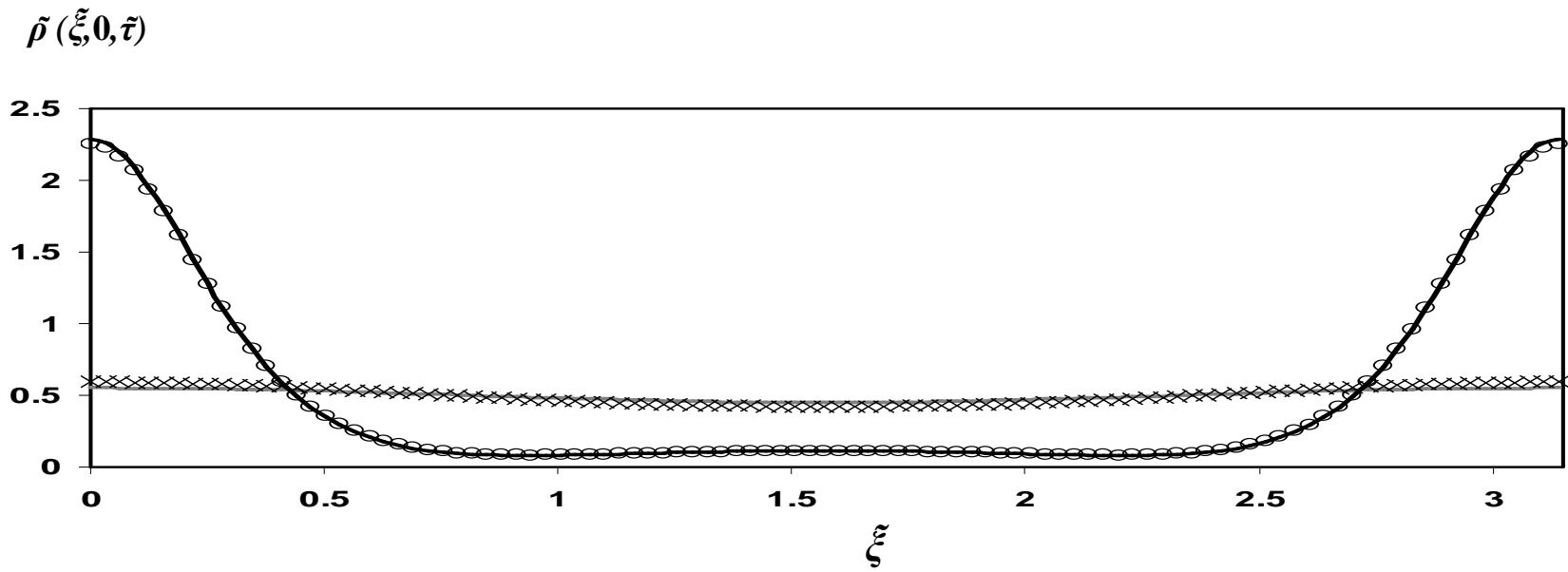
The influence of the extent of the r domain on the invariant I_2 as a function of the non-dimensional time $\tilde{\tau}$, for: $\tilde{r}_{end}=10\tilde{\xi}_{end}$ — ;
 $\tilde{r}=30\tilde{\xi}_{end}$ — ; and $\tilde{r}=50\tilde{\xi}_{end}$ ○ ○ ○ . A₁.

Description of the initial conditions

Summary of parameters chosen for the various simulations, see also [6]

Case	δ	\tilde{K}	\tilde{W}	$\tilde{\Omega}_I$	Initial homogeneous spectrum S_h	Inhomogeneous disturbance spectrum s
A ₁	0.1	2.0	1.0	0.405	Gaussian	Gaussian
A ₂	0.1	2.0	1.0	0.425	Lorentz	Lorentz
A ₃	0.1	2.0	1.0	0.4	Square	Square
A ₄	0.1	2.0	1.0	0.405	Gaussian	Square Width= 0.1 \tilde{W}
A ₅	1	2.0	1.0	0.405	Gaussian	Square Width= 0.1 \tilde{W}
A ₆	10	2.0	1.0	0.405	Gaussian	Square Width= 0.01 \tilde{W}
B	0.1	4.0	1.0	0	Gaussian	Gaussian
C	0.1	2.0	2.0	0	Gaussian	Gaussian
D	0.1	2.0	0.5	0.47	Gaussian	Gaussian

Results: Reference case (A₁)

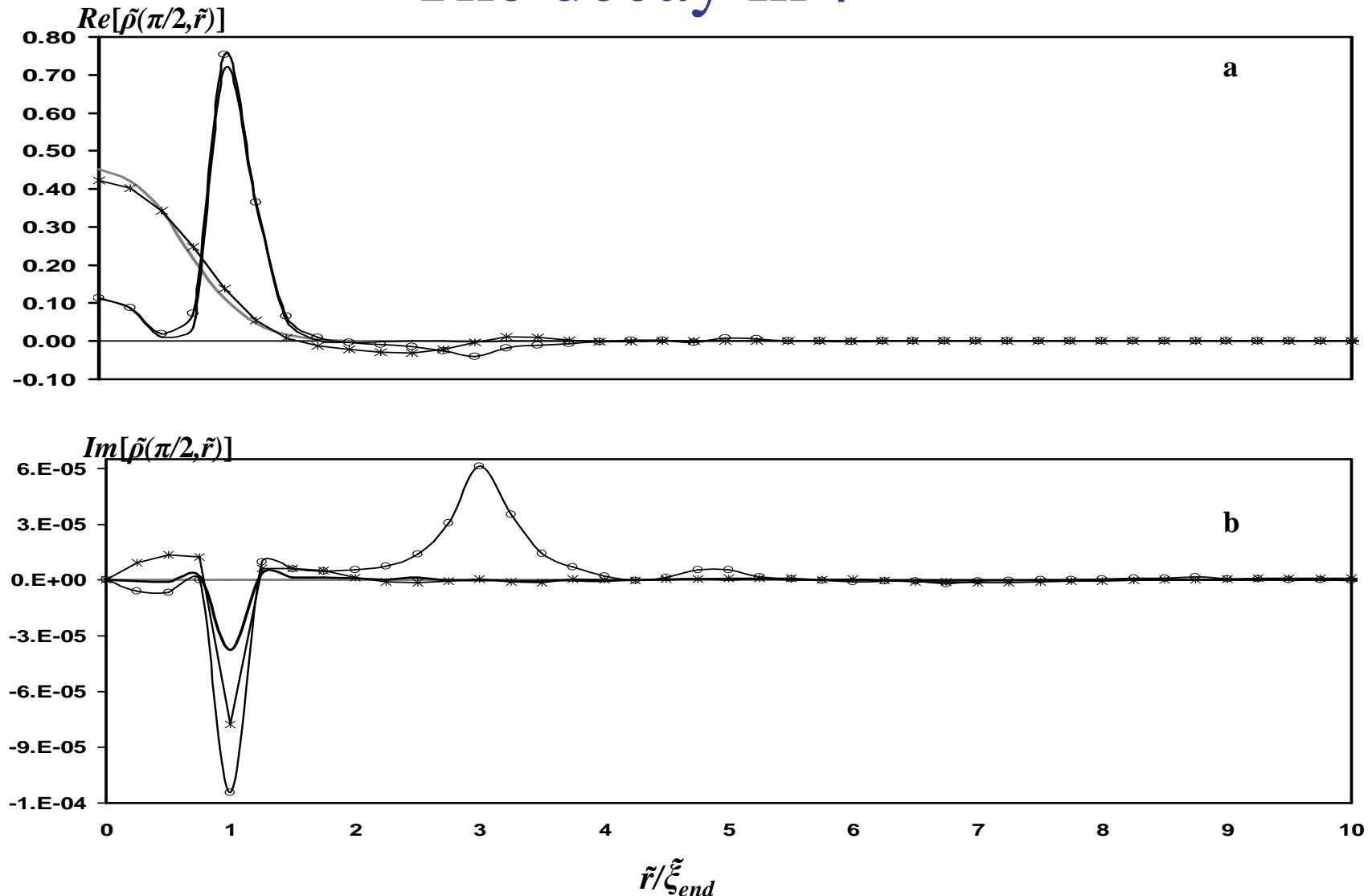


The value of $\tilde{\rho}$ as a function of ξ , at $\tilde{r}=0$ for different times:

$\tilde{t}=0$ — ; $\tilde{t}=8.25$ — ; $\tilde{t}=16.25$ \times ; $\tilde{t}=24.5$ \circ \circ \circ . Case A₁.

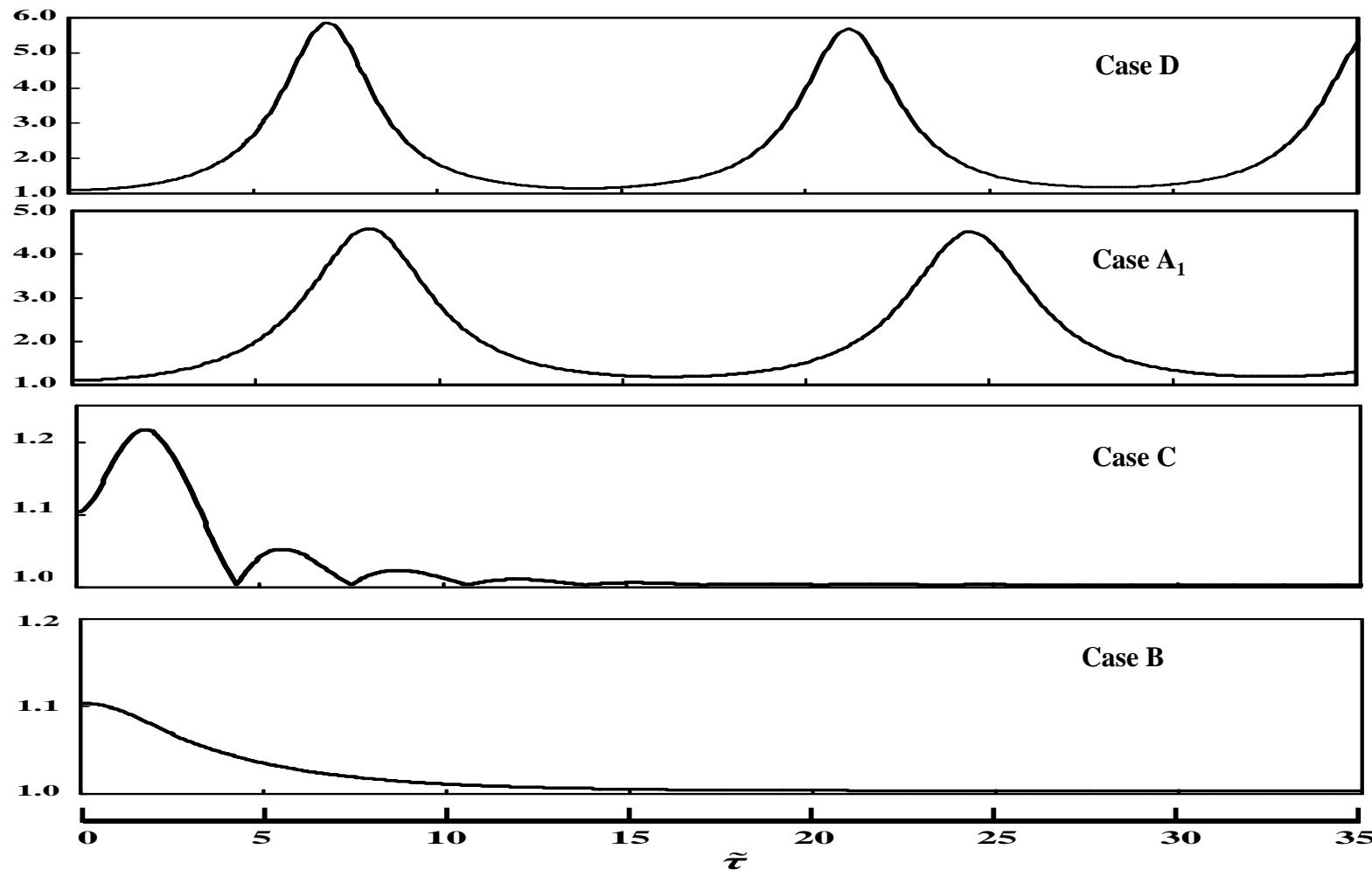
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The decay in \tilde{r}



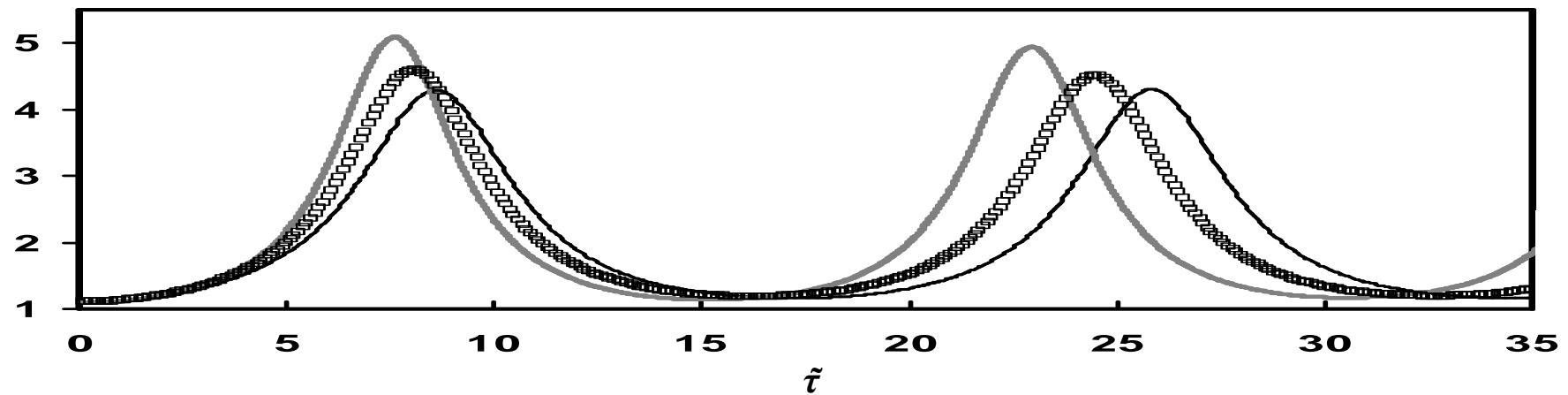
The values of the real part of $\tilde{\rho}$ (a) and the imaginary part (b) as functions of \tilde{r}/ξ_{end} , at the cross section $\xi=\xi_{end}/2$, for different times $\tilde{t}=0$ — ; $\tilde{t}=8.25$ — ; $\tilde{t}=16.25$ — \times — ; $\tilde{t}=24.5$ — \circ — . Case A₁.

Results: The influence of the initial linear growth rate on the long time evolution



The value of $\tilde{\rho}(0,0,\tilde{\tau})/\tilde{\rho}h(0,0)$ as a function of time, for 4 different initial growth rates. Cases D,A₁,C, and B

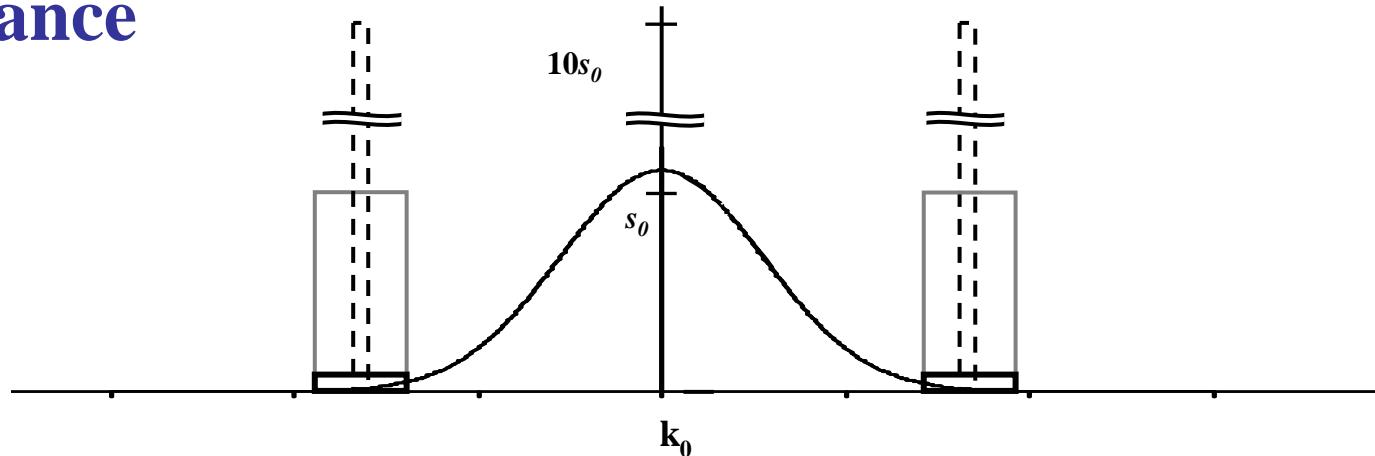
Results: Influence of the shape of the initial spectrum



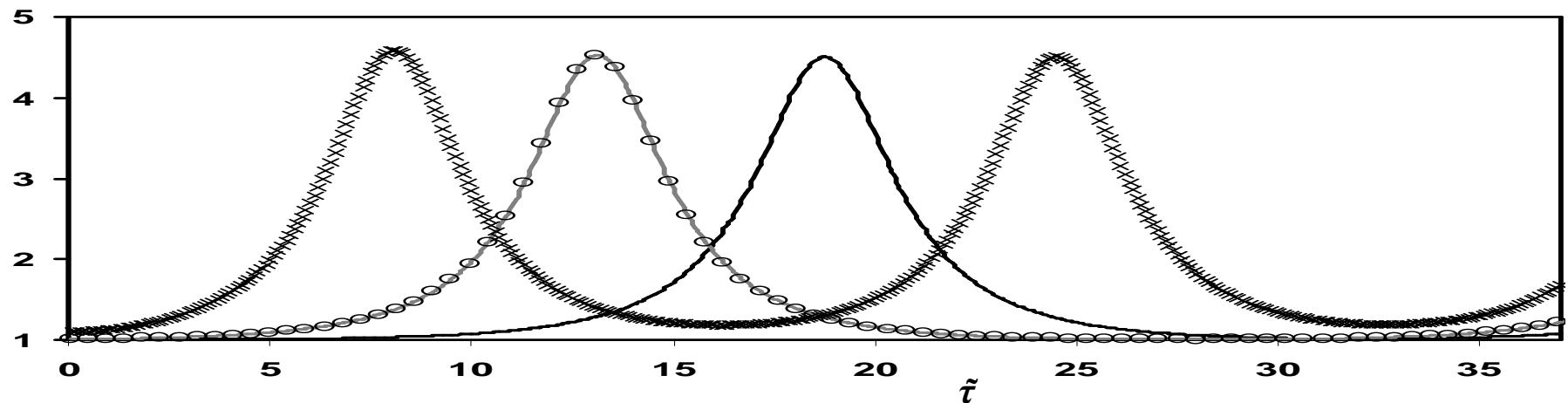
The values of $\tilde{\rho}(0,0,\tilde{\tau})/\tilde{\rho}_h(0,0)$ as a function of time for 3 different initial spectra:
Square (A_1) — ; Gaussian (A_3) — ; Lorentz (A_2) □ .

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Results: Influence of the shape of the inhomogeneous disturbance



Schematic description of the initial homogenous spectrum and the inhomogeneous disturbances,
 case A₄ — ; case A₅ — ; case A₆ - - - .



The values of $\tilde{\rho}(0,0,\tilde{t})/\tilde{\rho}_h(0,0)$ as functions of time, for 4 different initial inhomogeneous disturbances:

20 Case A₁ × ; Case A₄ — ; Case A₅ — ; Case A₆ ○ .

• **Conclusions**

- The most striking result of this work is the recurrent behaviour
- This finding is very different from the results for homogeneous seas
- Reason: in the present study, the initial disturbance spectra are profoundly inhomogeneous, through their phase relation to the homogenous spectrum, in contrast with previous studies for which all initial phases were independent and randomly chosen
- **Physical significance ?**

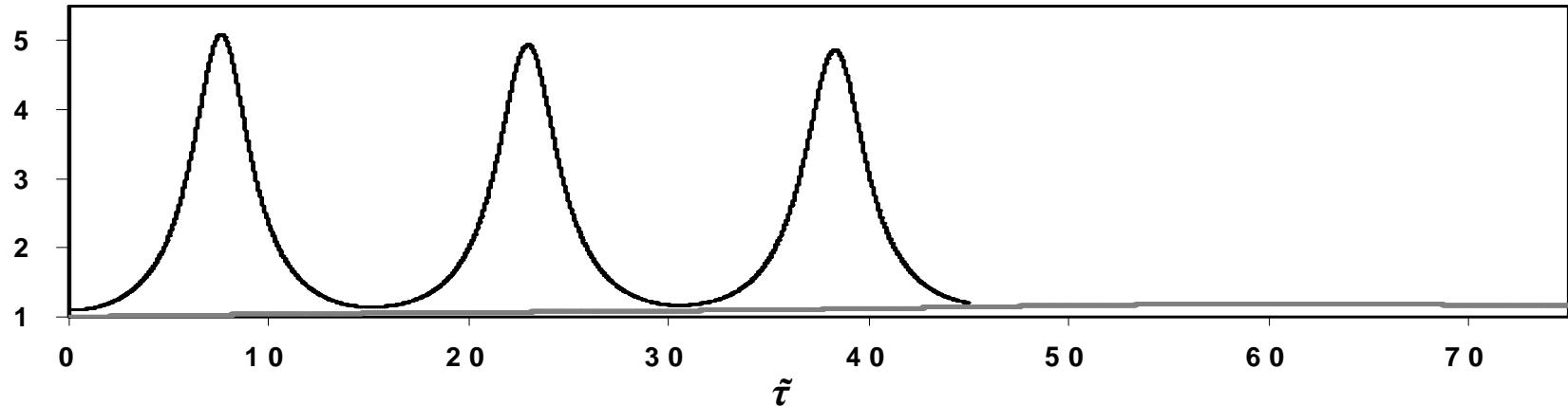
Acknowledgment

This research was supported by THE ISRAEL SCIENCE FOUNDATION (Grant No. 695/04).

References

- Alber, I.E., 1978. The effects of randomness on the stability of two-dimensional wavetrains. *Proc. R. Soc. London A.*, **363**, 525-546.
- Annenkov, S.Y. Shrira, V.I., 2006. Role of non-resonant interactions in evolution on nonlinear random water wave fields. *J. Fluid Mech.* (in press)
- Crawford, D.R., Saffman, P.G. and Yuen, H.C., 1980. Evolution of a random inhomogeneous field of nonlinear deep-water gravity waves, *Wave Motion*, **2**, 1-16.
- Dysthe K.B., Trulsen, K., Krogstad, H.E., and Socquet-Juglard, H., 2003. Evolution of a narrow-band spectrum of random surface gravity waves. *J. Fluid Mech.* **478**, 1–10,
- Janssen, P.A.E.M., 1983. Long-time behavior of a random inhomogeneous field of weekly nonlinear surface gravity waves, *J. Fluid Mech.* **133**, 113-132.
- Janssen P.A.E.M., 2003. Nonlinear Four-Wave Interaction and Freak Waves, *J. Phys. Oceanography.* **33** , 863-884.
- Onorato, M., Osborne, A.R, Serio, M., Resio, D., Pushkarev, A., Zakharov, V. E., Brandini, C., 2002 Freely decaying weak turbulence for sea surface gravity waves, *Phys. Rev. Lett.* **89** (14): art. no. 144501.

A Lorentzian Profile with a small disturbance,



The values of $\tilde{\rho}(0,0,\tilde{\tau})/\tilde{\rho}_h(0,0)$ as functions of time, for:

Case A₂ — ; Janssen (1983) — .

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Sea-Swell Interaction as a Mechanism for the Generation of Freak-Waves

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Theoretical background

Alber's Eq. for narrow banded random surface waves

$$i \left(\frac{\partial \rho}{\partial t} + \frac{1}{2} \sqrt{\frac{g}{k_o}} \frac{\partial \rho}{\partial x} \right) - \frac{1}{4} \sqrt{\frac{g}{k_o^3}} \frac{\partial^2 \rho}{\partial r \partial x} = \sqrt{g k_o^5} \rho(x, r, t) \left[\rho \left(x + \frac{r}{2}, 0, t \right) - \rho \left(x - \frac{r}{2}, 0, t \right) \right]$$

The definition of the two-point spatial correlation $\rho(x, r, t)$ is:

$$\rho(x, r, t) = \left\langle A \left(x + \frac{r}{2}, t \right) A^* \left(x - \frac{r}{2}, t \right) \right\rangle$$

$A(x, t)$ (the complex envelope) is related to the random free-surface elevation $\eta(x, t)$ by:

$$2\eta(x, t) = A(x, t) e^{i(k_o x - \sqrt{g k_o} t)} + * ,$$

Theoretical background

The correlation for a homogeneous ocean at $r=0$ is given by the integral of the energy spectrum:

$$\rho_h(r = 0) = \int_{-\infty}^{\infty} S(k) dk$$

And thus:

$$\rho_h(r = 0) \propto H_{rms}^2$$

where H_{rms} is the root mean square wave height of the homogeneous ocean.

One can assume that for an inhomogeneous ocean:

$$\rho(x, r = 0, t) \propto H_{rms}^2(x, t)$$

Theoretical background

The Rayleigh distribution:

$$P(H \geq \hat{H}) = e^{-\left(\frac{\hat{H}}{H_{rms}}\right)^2}$$

For a chosen value of ρ one can show

$$P\left(\frac{H}{H_{rms0}} > \frac{\hat{H}}{H_{rms0}}\right) = \exp\left\{-\left(\frac{\hat{H}}{H_{rms0}}\right)^2 \frac{\rho_h}{\rho}\right\}$$

The probability to obtain $H \geq \hat{H}$ throughout the spatial and temporal evolution of ρ

$$P\left(\frac{H}{H_{rms0}} \geq \frac{\hat{H}}{H_{rms0}}\right) = \int \text{pdf}(\rho) e^{-\left(\frac{\hat{H}}{H_{rms0}}\right)^2 \cdot \frac{\rho_h}{\rho}} d\rho$$

Seven Case Studies ,

$$S(k) = 1.45 s_o e^{-1.64 \left[(k - k_o)/W \right]^2}$$

Case			A ₁	A ₂	B	C	D	E	F
Swell conditions	T _i (s)	period	18	18	18	18	18	18	18
	a _i (m)	amplitude	1	2	1	1	1	1	1
	λ _i (m)	length	505	505	505	505	505	505	505
	K(m ⁻¹)	wave-number	0.0124	0.0124	0.0124	0.0124	0.0124	0.0124	0.0124
Initial Sea conditions	T _s (s)	peak period	10	10	10	10	10	8	12
	a _s (m)	amplitude	4	4	4	4	4	4	4
	H _s (m)	significant height	11.3	11.3	11.3	11.3	11.3	11.3	11.3
	λ _s (m)	length	156	156	156	156	156	100	225
	k ₀ (m ⁻¹)	wave-number	0.04	0.04	0.04	0.04	0.04	0.063	0.028
	W(m ⁻¹)	spectral width	0.0032	0.0032	0.0065	0.0097	0.013	0.0158	0.0032
	s ₀ (m ³)	Eq. 4.7	1234	1234	617	411	309	253	1280
Non-dimensional parameters	ε	wave steepness	0.16	0.16	0.16	0.16	0.16	0.25	0.11
	W̃	Eq. 4.8	0.5	0.5	1	1.5	2	1	1
	BFI	Benjamin-Feir index	1.4	1.4	0.7	0.47	0.35	0.7	0.7
	K̃	Eq. 4.8	1.9	1.9	1.9	1.9	1.9	0.8	4
	Q̃I	Growth rate	0.46	0.46	0.41	0.3	0.14	0.22	0
	δ	Eq. 4.5	0.08	0.165/15	0.08	0.08	0.08	0.126	0.056

Stability Diagram and Recurrent Solutions

The initial conditions for ρ , are defined as

$$\rho(x, r, t=0) = \rho_h(r) + \delta \rho(x, r, t=0),$$

Where the small disturbance is given by:

$$\rho_l(x, r, t) = R(r) \cos(Kx),$$

For a sea-swell interaction :

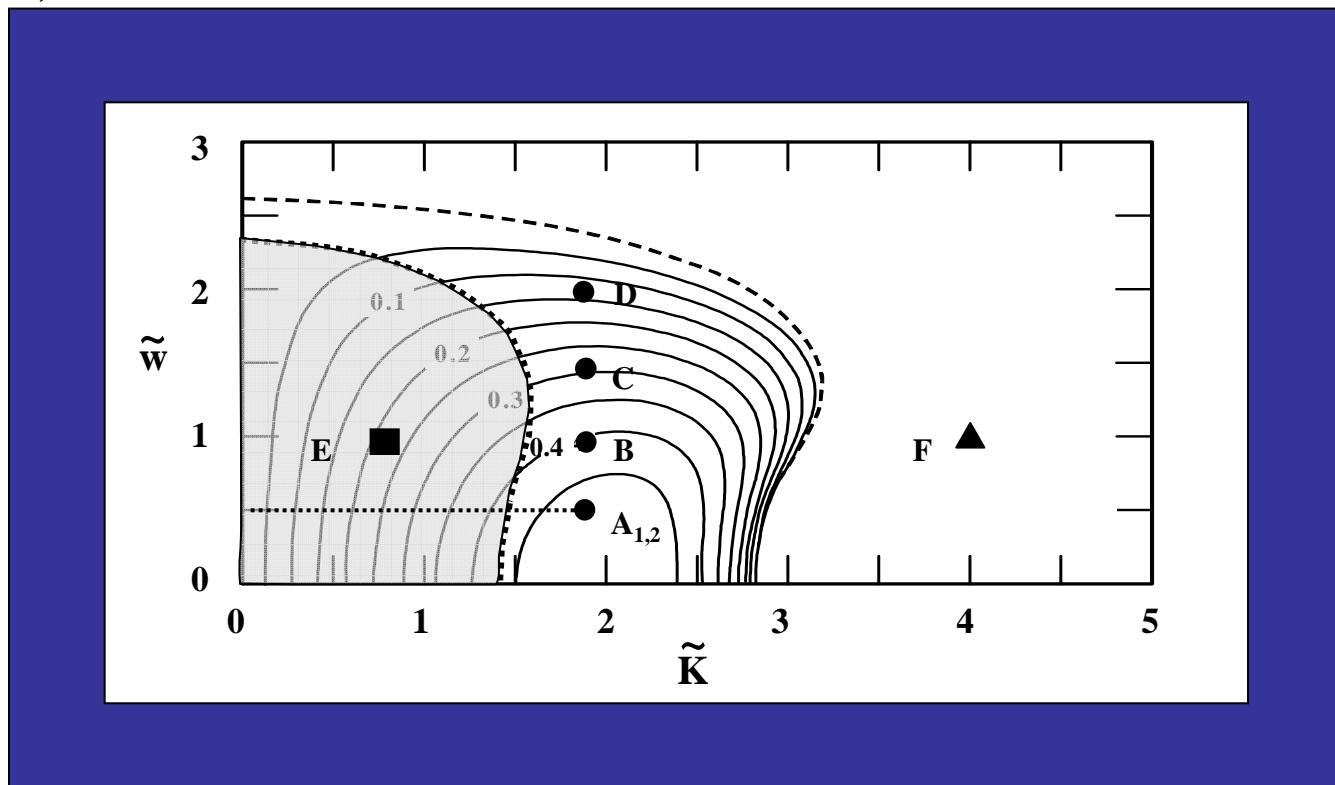
$$R(r) = \rho_h \left(\frac{K}{k_0} \cos(Kr/2) + i \sin(Kr/2) \right); \quad \delta = 2 a_l k_0$$

The initial homogeneous correlation is:

$$\rho_h(r) = \frac{1.133\sqrt{\pi}}{4} e^{-((rw)^2/6.45)}$$

Stability Diagram

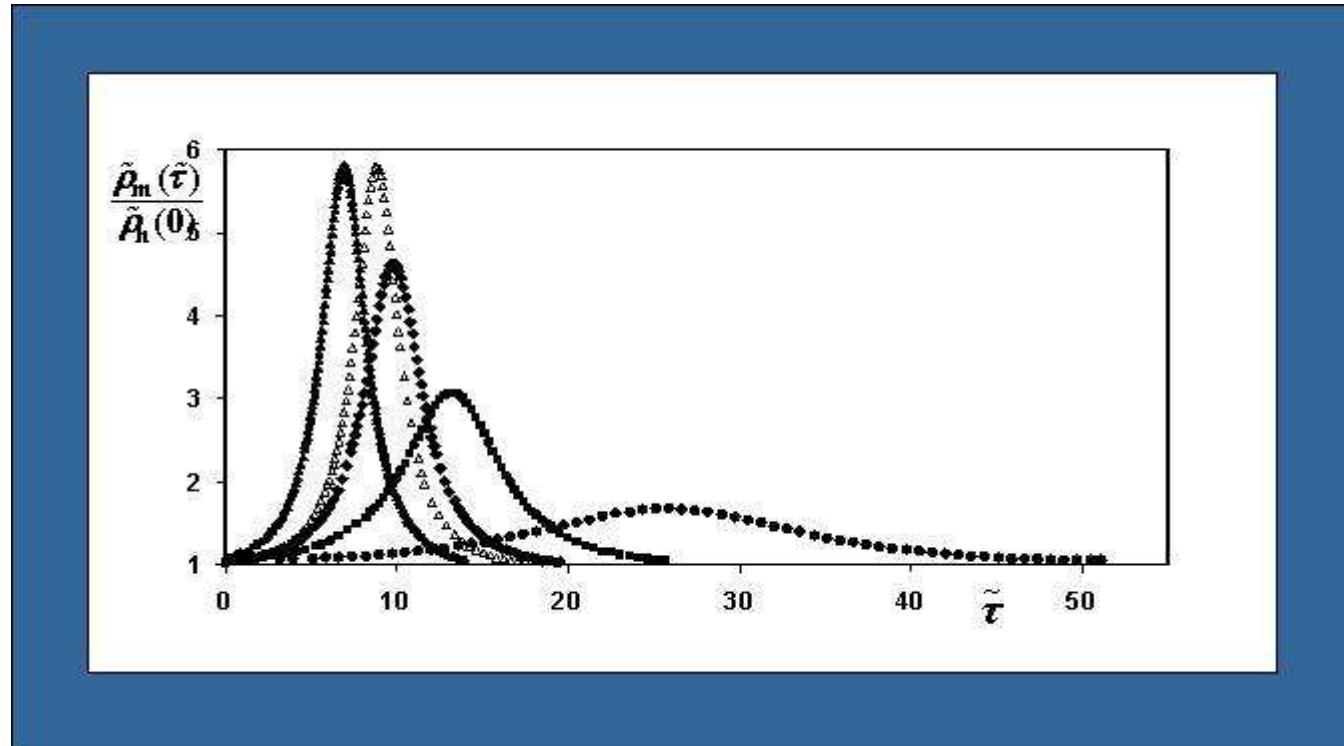
Stability Diagram: Isolines of the non-dimensional growth-rate for a Gaussian spectrum, $T_s=8$ sec. (triangle), $T_s=10$ sec. (circles), $T_s=12$ sec. (square).



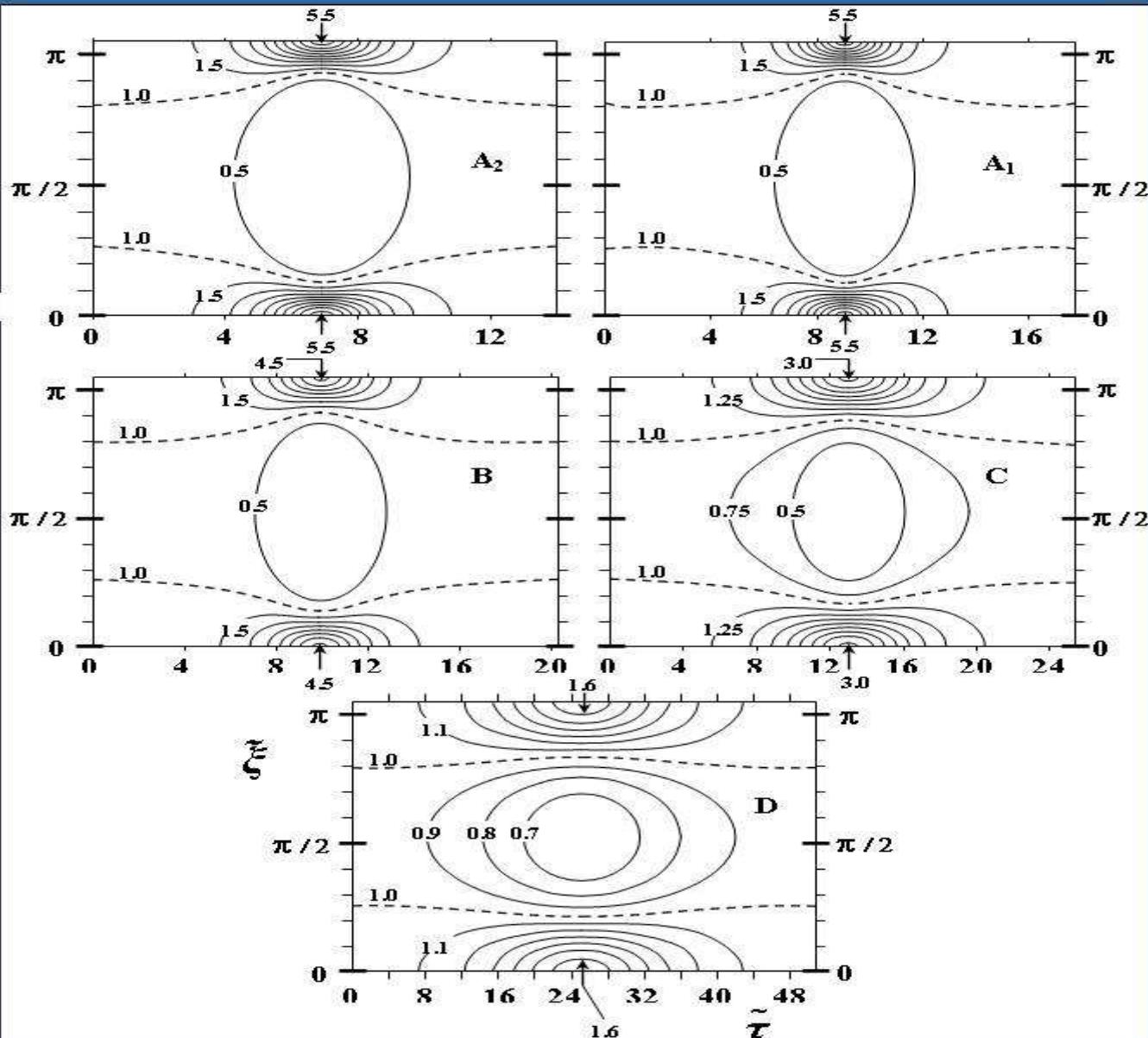
Recurrent Solutions

A typical cycle of the long time recurring evolution of $\tilde{\rho}_m(\tilde{\tau})/\tilde{\rho}_h(0)$,
for:

A2▲ ▲ ; A1 △ △; B◆◆ ; C ■■ ; D ● ●.

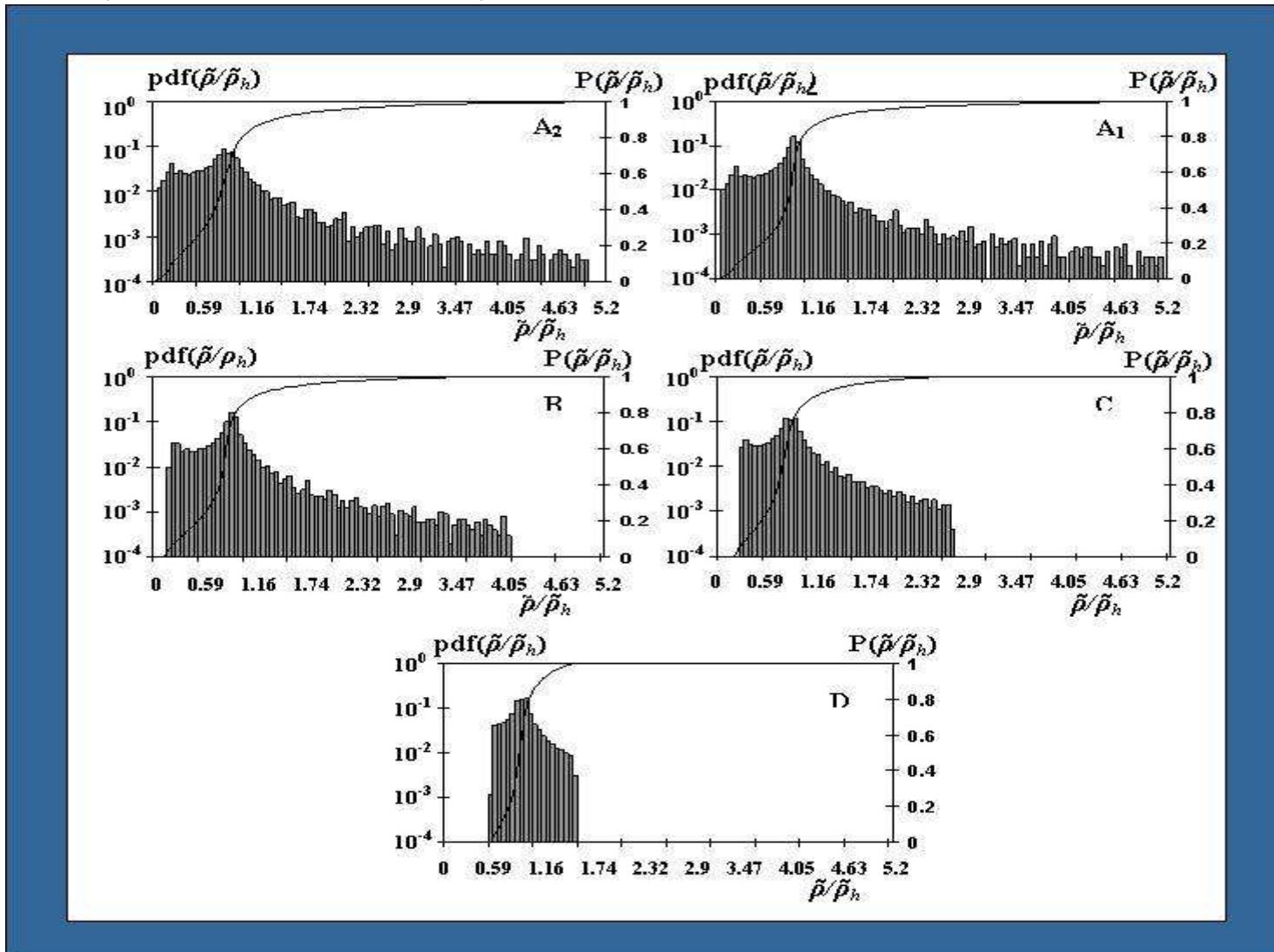


Iso-Lines of $\tilde{\rho}(\xi, 0, \tilde{\tau})/\tilde{\rho}_h(0)$.



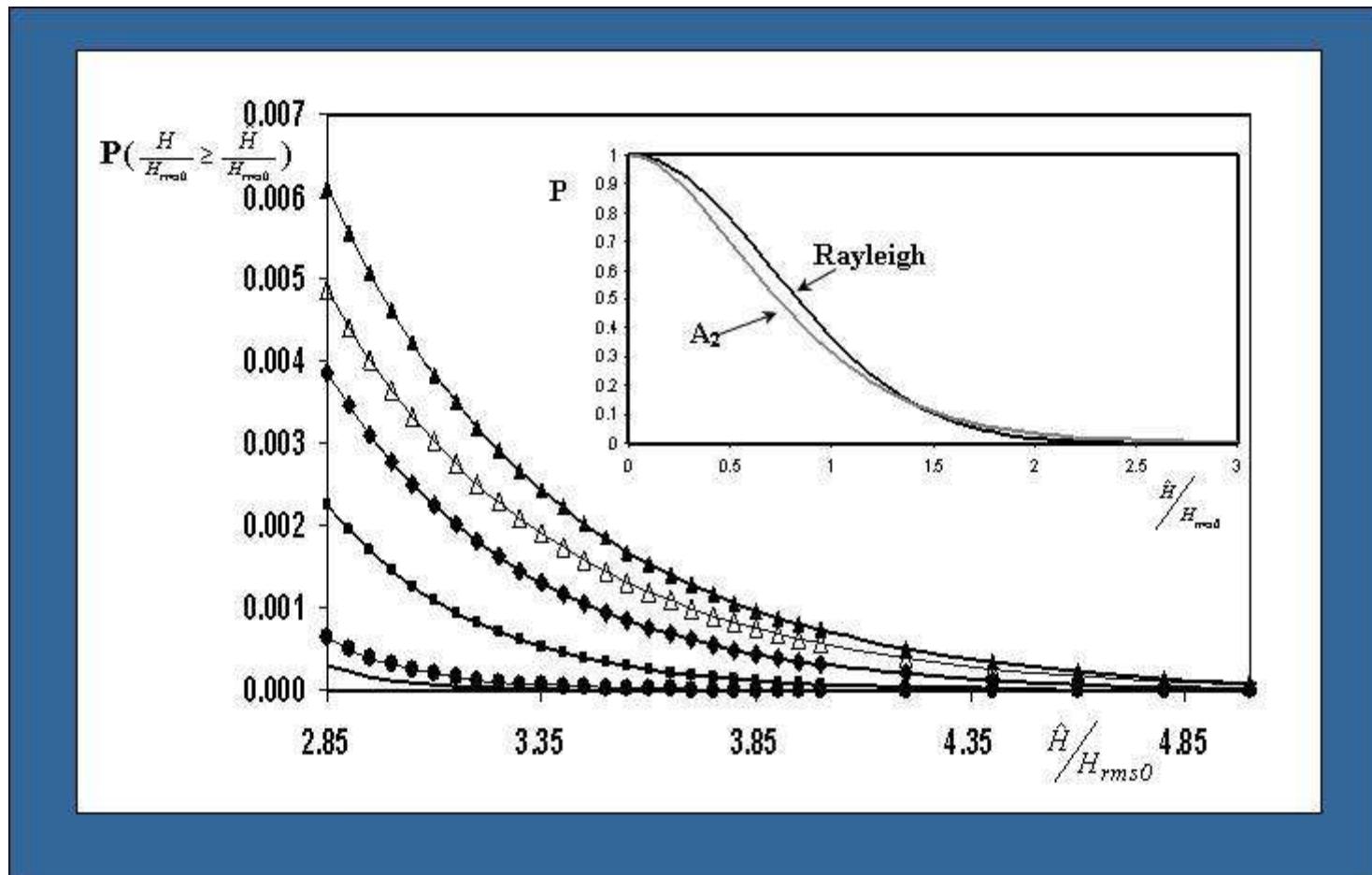
Probabilities

Probability density function, $\text{pdf}(\tilde{p}/\tilde{p}_h)$, and probability function, $P(\tilde{p}/\tilde{p}_h)$, as functions of \tilde{p}/\tilde{p}_h .



Probabilities

The probability of freak-waves ($\hat{H}/H_{rms0} \geq 2.85$) for the Rayleigh distribution — , and the probability obtained from Alber's equation, Case A2 ▲▲ ; Case A1 ▲△ ; Case B ♦♦ ; Case C □□ ; Case D ●● . The insert shows the probability function for the Rayleigh distribution — — and case A2 — — .



“Monte-Carlo” Simulations

Coupled NLS

$$i \left(\frac{\partial A_s}{\partial t} + \frac{1}{2} \sqrt{\frac{g}{k_0}} \frac{\partial A_s}{\partial x} \right) - \frac{g^{1/2}}{8k_0^{3/2}} \frac{\partial^2 A_s}{\partial x^2} = \frac{g^{1/2} k_0^{5/2}}{2} |A_s|^2 A_s + g^{1/2} k_0 K^{3/2} |A_l|^2 A_s$$

$$i \left(\frac{\partial A_l}{\partial t} + \frac{1}{2} \sqrt{\frac{g}{K}} \frac{\partial A_l}{\partial x} \right) - \frac{g^{1/2}}{8K^{3/2}} \frac{\partial^2 A_l}{\partial x^2} = \frac{g^{1/2} K^{5/2}}{2} |A_l|^2 A_l + g^{1/2} k_0^{1/2} K^2 |A_s|^2 A_l$$

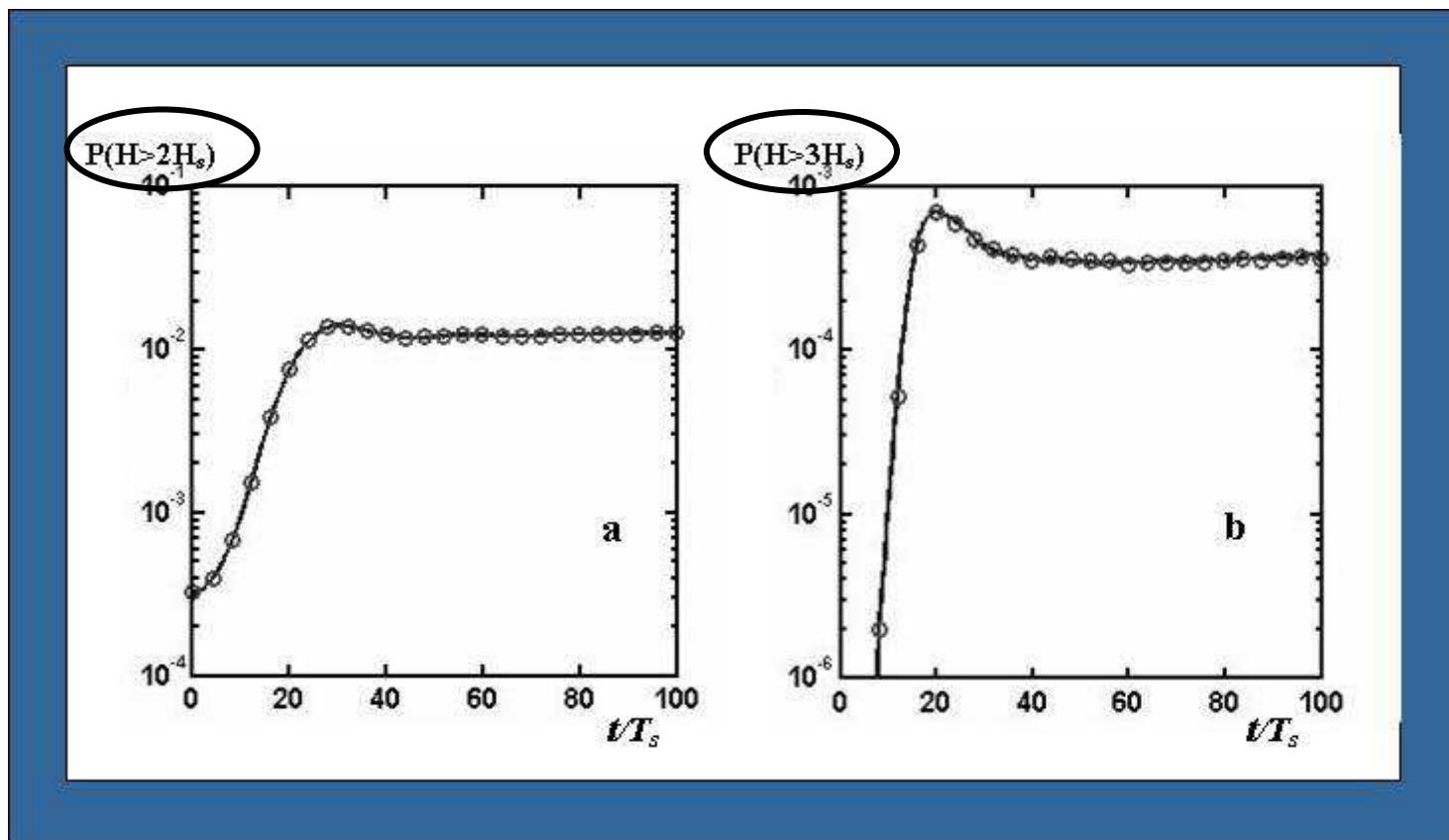
Single NLS

$$i \left(\frac{\partial A_s}{\partial t} + \frac{1}{2} \sqrt{\frac{g}{k_0}} \frac{\partial A_s}{\partial x} \right) - \frac{g^{1/2}}{8k_0^{3/2}} \frac{\partial^2 A_s}{\partial x^2} = \frac{g^{1/2} k_0^{5/2}}{2} |A_s|^2 A_s$$

2000 Realizations

“Monte-Carlo” Simulations

Probability of freak-waves as a function of time using a single nonlinear Schrödinger equation (Full line), and a coupled nonlinear Schrödinger equations (circles) for a) $H = 2H_s$, and b) exceptionally high freak-waves $H = 3H_s$.



Summary

- Alber's equation was used to study the statistics of freak waves in an unidirectional inhomogeneous sea. The inhomogeneity arises due to the interaction of a deterministic, long swell with a stochastic, short sea.
- The probability of freak waves increased up to 20 times (compared to the reference, Rayleigh distribution) as the spectral width of the sea decreases and the amplitude of the swell increases. The probability for exceptionally high freak-waves was increased by a factor of about 30,000.
- The results were confirmed by "Monte-Carlo" simulations of a system of two coupled nonlinear Schrödinger equations, and also of a single Schrödinger equation.

References

- Alber I. E. 1978. The effects of randomness on the stability of two-dimensional surface wavetrains. *Proc. R. Soc. Lond. A* **363**, 525–546.
- Gramstad O. and Trulsen K., 2007. Influence of crest and group length on the occurrence of freak-waves. *J. Fluid Mech.* **583**, 463-472.
- Janssen P.A.E.M. 2003. Nonlinear four-wave interactions and freak-waves. *J. Physical Oceanography* **36**, 863-884.
- Kharif C. and Pelinovsky E., 2003, Physical mechanism of the rogue wave phenomenon. *European J. of Mech. B/Fluids*, **22**, 603-634.
- Lo E. and Mei C.C., 1985, A numerical study of water-wave modulation based on a higher-order nonlinear Schrödinger equation. *J. Fluid Mech.*, **150**, 395-416.
- Longuet-Higgins M.S., 1952, On the statistical distribution of the heights of sea waves, *J. Marine Res.* **11**, 1245–1266.
- Mei C.C., Stiassnie M., and Yue D.K-P, 2005, Theory and applications of ocean surface waves, Advanced Series on Ocean Engineering, **23**, World scientific.
- Mori N., Onorato M., Janssen P.A., Osborne A.R., Serio M., 2007, On the extreme statistics of long-crested deep water waves: Theory and experiments, *J. Geophys. Res.*, **112**(C9), C09011.
- Onorato M., Osborne A. R., Serio M., Cavalieri L., Brandini C. and Stansberg C.T., 2004, Observation of strongly non-Gaussian statistics for random sea surface gravity waves in wave flume experiments, *Phys. Rev. E.*, **70**, 067302.
- Onorato, M, Osborne, A.R. and Serio M. 2006, Modulational instability in crossing sea state: a possible mechanism for the formation of freak waves. *PRL* 96, 014503, 4 p.
- Rasmussen, J. and Stiassnie, M. Discretization of Zakharov's equation, 1999, *European J. of Mechanics – B/Fluids*, **18**, 353 – 364.
- Shemer L., Kit E. and Jiao H.-Y., 2002, An experimental and numerical study of the spatial evolution of unidirectional nonlinear water-wave groups. *Phys. Fluids*, **14**, 3380-3390.
- Stiassnie M., Regev A. and Agnon Y., 2008, Recurrent solutions of Alber's equation for random water-wave fields, *J. Fluid Mech.* **598**, 245-266.
- Tayfun M.A. 1981, Distribution of crest-to-trough wave height. *J. water-way, port, Costal and Ocean Eng.*, **107**, 149-158.