The Evolution of Dispersal

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Talk Outline

1. Evolution of dispersal
2. An eigenvalue problem
3. Applications to evolution of dispersal
4. Proof: behavior of principal eigenvalue
5. Some recent progress

Motivation

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- How did organisms adopt their dispersal behaviors?
- What are potential impact of dispersals?
- How will these dispersal behaviors evolve?

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- Dilemma: To disperse or not to disperse?
- Resource, resource, resource
- Species often compete for the same/similar resource
- Can competition be a driving force in the selection of dispersal strategies?

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&v_t = \nu \Delta v + vf(u+v, x) \quad \text{in } \Omega \times (0, \infty), \qquad (1) \\
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- $u(x,t), v(x,t)$: densities of species
- $\mu, \nu > 0$: random dispersal rates
- Ω: bounded domain in R^d with C^2 boundary; n: outward unit normal vector on $\partial\Omega$.

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Then $(\tilde{u}, 0)$ is globally asymptotically stable, where $\tilde{u}$ is the unique positive steady-state of

$$\tilde{u}_t = \mu\Delta\tilde{u} + \tilde{u}(m - \tilde{u}) \quad \text{in} \;\; \Omega \times (0,\infty),$$

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- α: the strength of advection

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- *Non-convex domains*: conditional dispersal may not evolve. There exist some non-convex domains and $m(x)$ such that with $\mu = \nu$ and α positive small, species v is always the winner.

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- Chen and L. (Indiana Univ. Math. J, 08): for large α, if m has a unique local maximum, u concentrates around this maximum.

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- Any difference with the case $\beta = 0$?

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$$\nabla \cdot [\mu \nabla \varphi - \alpha \varphi \nabla m] + \varphi(m - \theta) = -\lambda \varphi \quad \text{in } \Omega,$$

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- What is the behavior of $\lambda(\alpha)$ for large α?

Principal eigenvalue

Consider the principal eigenvalue, denoted by $\lambda(\alpha)$, of the problem

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where $\mathcal{I} = \{\varphi \in H^1(\Omega) : \varphi \neq 0, \mathbf{v}\cdot\nabla\varphi = 0 \ in\ \Omega\}$.

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- Consider the case when $\mathbf{v} = -\nabla m$, i.e.,

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where $\mathcal{M}$ is the set of points of local maximum of m.

Previous work

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- Frame extiction/propagation in fluid: Constantin, Kiselev, Ryzhik and Zalatos

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- Recall

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- It follows from previous result of Chen and L. that

$$\lim_{\alpha\to\infty}\lambda(\alpha) = \min_{\mathcal{M}}(\theta - m),$$

where $\mathcal{M}$=the set of points of local maximum of $m(x)$.

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Theorem

Suppose that $|\nabla m| > 0$ *a.e. in* Ω*. If* $\beta \le \nu / \max_{\overline{\Omega}} m$*, system* (3) *has one stable positive steady state.*

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Theorem

Suppose that $|\nabla m| > 0$ a.e. in Ω. If $\beta \leq \nu / \max_{\overline{\Omega}} m$, system (3) has one stable positive steady state.

- For large α, the species *u* concentrates at places of locally most favorable environments, leaving the other species to utilize other resources in the habitat.

Advection-induced extinction: $\beta \geq \nu / \min_{\overline{\Omega}} m$

(A1) $\partial m/\partial n < 0$ on $\partial\Omega$, m has only one critical point x_0 in $\overline{\Omega}$, with $x_0 \in \Omega$ and $D^2 m(x_0) < 0$.

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- Strong biased movement of both species can induce the extinction of the species with stronger biased movement

Upper bound of principal eigenvalue

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- Recall that

$$-\Delta\varphi - \alpha\nabla m \cdot \nabla\varphi + c(x)\varphi = \lambda(\alpha)\varphi \quad \text{in } \Omega, \quad \nabla\varphi \cdot n|_{\partial\Omega} = 0.$$

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- Fix $z \in \mathcal{M}$. For $\delta > 0$, Choose $\epsilon < \delta$ such that

$$\min_{\bar{B}(z,\epsilon)} m =: M_1 > M_2 := \max_{\bar{B}(z,2\delta)\setminus B(z,\delta)} m.$$

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$$\varphi(x) = \begin{cases} 1 & \text{if } \ x \in B(z,\delta), \\ (2\delta - |x|)/\delta & \text{if } \ x \in B(z,2\delta) \setminus B(z,\delta), \\ 0 & \text{if } \ x \notin B(z,2\delta). \end{cases}$$

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- Sending $\alpha \to \infty$ then $\delta \to 0$ we obtain $\overline{\lim}_{\alpha\to\infty} \lambda(\alpha) \leqslant c(z)$.

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Adaptive dynamics approach

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(Hambrock and L., BMB in revision)

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- The stronger advector wins if advection rates are not large; The weaker advector wins if advection rates are large.

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- The slower diffuser wins when advection rates are not large, and the faster diffuser wins if advection rates are large.

Predator and prey dispersal strategies

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- Patch models for mutual behaviorial adjustments of predator and perys: Iwasa 1982; Sih 1984, 1988; Schwinning and Rosenzweig 1990; Abrams 1992; Hugie and Dill 1994; Alonzo 2002; Abrams 2007; Abrams, Cressman, and Krivan; etc

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- Reaction-diffusion model: Kareiva and Odell 1987

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$$P_{i,t} = \nabla \cdot [d_i \nabla P_i - \alpha_i P_i \nabla \mathbf{f_i}(\mathbf{R}, \mathbf{V})] + P_i(-k_i + a_i V), \ i = 1, 2,$$

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- The resource tracking predator can invade before the prey tracking predator when both are rare; When one predator has invaded and the other is rare, the resource tracking predator can invade the prey tracking predator, but not vice versa.

Acknowledgement

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