Uniformly hyperbolic $SL(2, \mathbf{R})$ cocycles

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We will always assume that F is invertible.

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- F is the tangent map of f.

Uniform hyperbolicity

Definition A linear cocycle $F : E \to E$ is *uniformly hyperbolic* if there is an F-invariant continuous splitting $E = E^s \oplus E^u$ and constants $C > 0$, $0 < \lambda < 1$ such that, for all $n \geq 0$ and $x \in X$

$$\begin{aligned} ||F^n_{|E^s_x}|| &\leq C\lambda^n, \\ ||F^{-n}_{|E^u_x}|| &\leq C\lambda^n. \end{aligned}$$

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In this case, one can always find an *adapted norm* on E such that $C = 1$.

The conefield criterion

A linear cocycle $F : E \to E$ is uniformly hyperbolic iff there are constants $C > 0$, $\lambda > 1$ and, for each $x \in X$, a splitting $E_x = E_x^1 \oplus E_x^2$ and norms $|.|_1$, $|.|_2$ on E_x^1, E_x^2 respectively such that, writing $F_x(v_1 + v_2) = w_1 + w_2$ with $v_i \in E_x^i$, $w_i \in E_{f(x)}^i$

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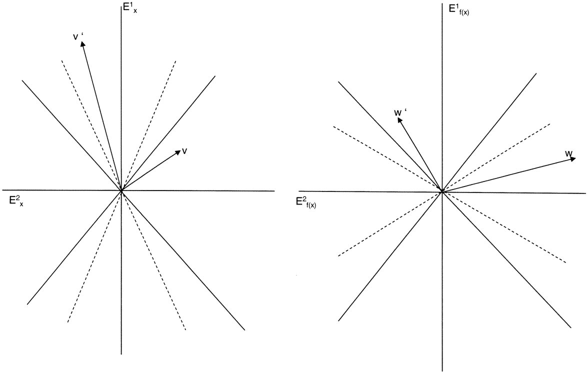
It follows immediately from this criterion that uniform hyperbolicity is an open property.



$$E_x^s = \lim_{n \to +\infty} F^{-n}(E^1_{f^n(x)}), \; E_x^u = \lim_{n \to +\infty} F^{n}(E^2_{f^{-n}(x)}).$$

Lyapunov exponents

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Theorem(Oseledets) There exist $r \geq 1$, $\lambda_1 > \cdots > \lambda_r$ and, for μ-a.e $x \in X$, a splitting

$$E_x \;=\; E_x^1 \oplus \cdots \oplus \; E_x^r$$

which is F-invariant and depends measurably on x, such that, for $1 \leq i \leq r$, $v \in E_x^i$, $v \neq 0$, one has

$$\lim_{n \to \pm\infty} \frac{1}{n} \log ||F_x^n(v)|| = \lambda_i .$$

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The converse is not true.

$SL(2, \mathbf{R})$ cocycles

From now on, we will have

$$E = X \times \mathbf{R}^2, \quad F(x, v) = (f(x), A(x)v),$$

with $A \in C^0(X, SL(2, \mathbf{R}))$.

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with $A \in C^0(X, SL(2,\mathbf{R}))$.
Then, we have $F^n(x,v) = (f^n(x), A^{(n)}(x)(v))$, with

$$\begin{aligned} A^{(n)}(x) &= A(f^{n-1}(x)) \cdots A(x), \\ A^{(-n)}(x) &= A(f^{-n}(x))^{-1} \cdots A(f^{-1}(x))^{-1}, \end{aligned}$$

for $n \geq 0$.

A criterion of uniform hyperbolicity for $SL(2,\mathbf{R})$ cocycles

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Let $F(x, v) = (f(x), A(x)v)$ be a $SL(2, \mathbf{R})$ cocycle.

Proposition The cocycle F is uniformly hyperbolic iff there exist $C > 0$, $\lambda > 1$ such that

$$||A^{(n)}(x)|| \geq \ C\, \lambda^n,$$

for all $x \in X$, $n \geq 0$.

Quasiperiodic $SL(2,\mathbf{R})$ cocycles

The case where X is a torus $\mathbf{T}^d$ and f is a translation $x \to x + \alpha$ has been much studied, in relation to 1-d discrete Schrdinger operators with quasiperiodic potentials.

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Let $V \in C(\mathbf{T}^d, \mathbf{R})$, $\alpha, \theta \in \mathbf{T}^d$. Define $H = H_{V,\alpha,\theta} : \ell^2(\mathbf{Z}) \rightarrow \ell^2(\mathbf{Z})$ by

$$(Hu)_n = u_{n+1} + u_{n-1} + V(\theta + n\alpha)u_n.$$

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Observe that u is an eigenvector with eigenvalue λ iff, for all $n \in \mathbf{Z}$

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = A_{\lambda,V}(\theta + n\alpha) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix},$$

with $A_{\lambda,V}(\theta) = \begin{pmatrix} \lambda - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}$.

Spectral vs. dynamical properties

The spectral properties of the one-parameter family of operators $(H_{V,\alpha,\theta})_{\theta\in\mathbf{T}}$ and the dynamical properties of the one-parameter families of cocycles over $\theta \to \theta + \alpha$ defined by $A_{\lambda,V}$, $\lambda \in \mathbf{R}$ are strongly correlated.

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For instance, the spectrum of $H_{V,\alpha,\theta}$ is independent of θ. A real number λ belongs to the spectrum iff the cocycle over $\theta \to \theta + \alpha$ defined by $A_{\lambda,V}$ is uniformly hyperbolic.

Locally constant $SL(2, \mathbf{R})$-cocycles over subshifts of finite type

From now on we will consider linear cocycles over chaotic (rather than quasiperiodic) dynamics in the base.

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Let $\mathcal{A}$ be a finite alphabet with $N \geq 2$ letters. Let $\sigma : \mathcal{A}^{\mathbf{Z}} \to \mathcal{A}^{\mathbf{Z}}$ be the *shift map* defined by $(\sigma x)_n = x_{n+1}$.

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We will take as base dynamics (X, f) the *full shift on N symbols* $(\mathcal{A}^{\mathbf{Z}}, \sigma)$ or more generally a *subshift of finite type*:

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We will only consider $SL(2, \mathbf{R})$-cocycles defined by functions $A : \Sigma \to SL(2, \mathbf{R})$ depending only on the letter x_0 in position 0.

The hyperbolicity locus

The parameter space for the class of cocycles under consideration is therefore the finite dimensional manifold $SL(2,\mathbf{R})^{\mathcal{A}} = SL(2,\mathbf{R})^N$.

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The results below were obtained in collaboration with **Artur Avila** (Clay Institute, CNRS Paris, IMPA Rio de Janeiro), and **Jairo Bochi** (PUC, Rio de Janeiro). They will appear soon in Commentarii Helvetici and are available on arXiv.

Induced projective action and multicones

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The conefield criterion for uniform hyperbolicity can be stated as follows: There exists a family of intervals $I_x \subset \mathbf{P}^1(\mathbf{R})$ with $A_{x_0} I_x \subset\subset I_{\sigma x}$ and the distances between the endpoints of $A_{x_0} I_x$, $I_{\sigma x}$ uniformly bounded from below.

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Definition A *multicone* is a non empty open subset of $\mathbf{P}^1 = \mathbf{P}^1(\mathbf{R})$ with finitely many connected components with disjoint closures.

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  Theorem [A-B-Y] The cocycle defined by (A_α) over the full shift $(\mathcal{A}^{\mathbf{Z}}, \sigma)$ is uniformly hyperbolic iff there exists a multicone M such that $A_\alpha M \subset\subset M$ for all $\alpha \in \mathcal{A}$.

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- The general case of subshifts of finite type.

  Theorem [A-B-Y] The cocycle defined by (A_α) over a subshift of finite type (Σ, σ) is uniformly hyperbolic iff there exist multicones M_α such that $A_\beta M_\alpha \subset\subset M_\beta$ for all allowed transitions $\alpha \to \beta$.

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Definition An invariant multicone M is *tight* if

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An invariant multicone M is *tight* iff its number of components is minimal (amongst invariant multicones).

Combinatorial invariants

The number of components of any tight invariant multicone M and the way that these components of M are sent by the A_α into each other is invariant under deformation in the hyperbolicity locus $\mathcal{H}$ and are therefore combinatorial invariants of the components of $\mathcal{H}$.

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Definition A matrix $A \in SL(2, \mathbf{R})$ is *hyperbolic* (resp. *elliptic*, resp. *parabolic*) if $|\, tr\, A\,| > 2$ (resp.< 2, resp.$= 2$).

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Remarks

- Let $\epsilon_\alpha \in \{-1, +1\}, A_\alpha \in SL(2, \mathbf{R})$. Then the cocycle defined by $(A_\alpha)_{\alpha \in \mathcal{A}}$ is hyperbolic iff the cocycle defined by $(\epsilon_\alpha A_\alpha)$ is.
- If A is hyperbolic, $(\epsilon_\alpha A)_{\alpha \in \mathcal{A}} \in \mathcal{H}$ for all $\epsilon_\alpha \in \{-1, +1\}$. Over the full shift on N symbols, the 2^N components of $\mathcal{H}$ containing such elements are called the *principal* components of $\mathcal{H}$.

Characterization of principal components

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Proposition[Y] The parameter (A_α) belongs to a principal component iff it has a connected invariant multicone.

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Theorem [A-B-Y] One has $0 < p < q$ and $p \wedge q = 1$. A component of M intersects $A(M)$ iff it does not intersect $B(M)$.

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Theorem [A-B-Y] One has $0 < p < q$ and $p \wedge q = 1$. A component of M intersects $A(M)$ iff it does not intersect $B(M)$. Conversely, for every $0 < p < q$ with $p \wedge q = 1$, there are exactly 8 nonprincipal components of $\mathcal{H}$ with these data.

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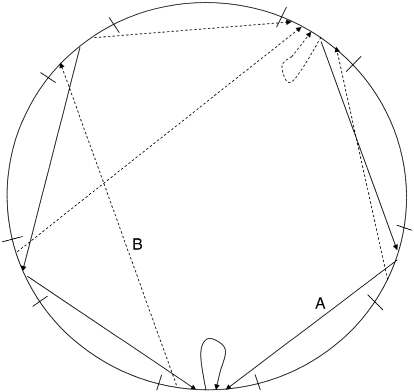
Let M be a tight invariant multicone for (A, B). Let q be the number of components of M; let p be the number of components of M which intersect $B(M)$. Assume that $q > 1$, i.e (A, B) does not belong to a principal component of $\mathcal{H}$.

Theorem [A-B-Y] One has $0 < p < q$ and $p \wedge q = 1$. A component of M intersects $A(M)$ iff it does not intersect $B(M)$. Conversely, for every $0 < p < q$ with $p \wedge q = 1$, there are exactly 8 nonprincipal components of $\mathcal{H}$ with these data. They are deduced from each other by changes of sign of A and B and conjugacy by an element of $GL(2, \mathbf{R})$ of determinant -1.

The case $p/q = 2/5$



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- Let x,y be periodic points in Σ, of periods k,l, such that $A^{(k)}(x),A^{(l)}(y)$ are hyperbolic. A point $z\in W^u_{loc}(x)\cap\sigma^{-n}W^s_{loc}(y)$ (for some $n\geq 0$) defines a heteroclinic connexion from x to y if

$$A^{(n)}(z)u(A^{(k)}(x))=s(A^{(l)}(y)).$$

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Corollary Each component H of $\mathcal{H}$, and the boundary of H, is semialgebraic.

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